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SINGULARITY OF THE SECOND OSTROGRADSKIĬ RANDOM SERIES

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Abstract. We study properties of the distribution of the second Ostrogradskiĭ series, for which the differences of terms are independent identically distributed random variables. We completely describe the Lebesgue structure of this distribution. In particular, we prove that it cannot be absolutely continuous. We also develop ergodic theory for the second Ostrogradskiĭ expansion. One of the results is that, for almost all (in the sense of Lebesgue measure) real numbers of the unit interval, an arbitrary symbol of an alphabet occurs finitely often in the corresponding Ostrogradskiĭ difference expansion. We also study properties of the dynamical system generated by the one-sided shift transformations T of the Ostrogradskiĭ difference representation. It is shown that there is no probability measure that is invariant and ergodic with respect to T and absolutely continuous with respect to Lebesgue measure.

1. Introduction

In the early 1860s, M. V. Ostrogradskiĭ considered two algorithms for expansions of positive real numbers into sign-alternating series:

$$\sum_k \frac{(-1)^{k-1}}{q_1 q_2 \cdots q_k}, \quad \text{where } q_k \in \mathbb{N},\ q_{k+1} > q_k, \tag{1}$$

$$\sum_k \frac{(-1)^{k-1}}{q_k}, \quad \text{where } q_k \in \mathbb{N},\ q_{k+1} \geq q_k(q_k+1) \tag{2}$$

(called the first and second Ostrogradskiĭ series, respectively). His brief notes on the matter were discovered in The Manuscript Depository of the National Academy of Sciences of Ukraine and explained by E. Ya. Remez in 1951 (see [13], where it is shown that Ostrogradskiĭ's algorithms give better approximations than continued fractions in certain cases).

In a footnote in the Khinchin book [14], B. V. Gnedenko wrote "…Unfortunately, no detailed study of these algorithms, even for computational purposes, has as yet been made." It is worth mentioning that nowadays there is a number of researches (see [1, 2, 7, 8, 9, 11, 12] and further references therein) devoted to expansions of real numbers into the sign-alternating first Ostrogradskiĭ series (also known as Pierce series) and to probability distributions related to these series.

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The second Ostrogradskiĭ expansions are much less studied ([10, 11, 13]). In particular, it is known that every real number $x \in (0,1)$ can be represented as a finite or infinite expression of the form (1). If a number x is irrational, then this representation is unique and the series in (2) is infinite, while if x is rational, then there are two different representations of the form (2) and both of them are finite. The Ostrogradskiĭ series converges quickly and this allows one to approximate irrational numbers effectively by partial sums of their Ostrogradskiĭ series. Note that the partial sums are rational numbers.

The following representation of a number x:

$$x = \sum_k \frac{(-1)^{k-1}}{q_k(x)}, \quad \text{where } q_k(x) \in \mathbb{N},\ q_{k+1}(x) \geq q_k(x)(q_k(x)+1), \tag{3}$$

is called the O^2-*representation* of x and is denoted by $O^2(q_1(x), \dots, q_n(x), \dots)$.

If the initial k terms are fixed, the $(k+1)$th term of the O^2-representation cannot equal $1, 2, 3, 4, 5, \dots, q_k(q_k+1)-1$, which shows that the terms of an O^2-representation have "unequal rights". Let

$$d_1 = q_1 \quad \text{and} \quad d_{k+1} = q_{k+1} - q_k(q_k+1) + 1 \quad \text{for all } k \in \mathbb{N}.$$

Then the latter series can be rewritten as follows:

$$x = \sum_k \frac{(-1)^{k+1}}{q_{k-1}(x)(q_{k-1}(x)+1) - 1 + d_k(x)} =: \overline{O}^2\big(d_1(x), d_2(x), \dots, d_k(x), \dots\big). \tag{4}$$

The expression on the right hand side of (4) is called the $\overline{O}^2$-expansion (other names are the second Ostrogradskiĭ difference expansion and the second Ostrogradskiĭ expansion with independent increments), and the number $d_k = d_k(x)$ is called the kth $\overline{O}^2$-symbol of the number x. In a $\overline{O}^2$-representation, any symbol, independently of the values of the preceding one, may attain every natural value.

The main goals in this paper are:

1) To develop an ergodic theory of $\overline{O}^2$-expansions of real numbers expressed in terms of asymptotic frequencies $\nu_i(x, \overline{O}^2)$ of $\overline{O}^2$-*symbols* ($i \in \mathbb{N}$), where

$$\nu_i(x, \overline{O}^2) = \lim_{n\to\infty} \frac{N_i(x,n)}{n}$$

and where $N_i(x,n)$ is the number of i's in the $\overline{O}^2$-representation of a number x up to the nth position; in particular, to find normal properties of real numbers (in other words, those properties that hold for almost all real numbers with respect to Lebesgue measure).

2) To investigate properties of the dynamical system generated by the following one-sided shift transformation T in the $\overline{O}^2$-representation:

$$T(x) = T\big(\overline{O}^2(d_1(x), d_2(x), \dots, d_n(x), \dots)\big) = \overline{O}^2\big(d_2(x), d_3(x), \dots, d_n(x), \dots\big)$$
$$\text{for all } x = \overline{O}^2\big(d_1(x), d_2(x), \dots, d_n(x), \dots\big) \in [0,1].$$

3) To study properties of the random variable

$$\eta = \overline{O}^2(\eta_1, \eta_2, \dots, \eta_k, \dots),$$

whose $\overline{O}^2$-symbols η_k are independent identically distributed random variables assuming values $1, 2, \dots, m, \dots$ with probabilities $p_1, p_2, \dots, p_m, \dots$, respectively, where the sequence $\{p_m\}$ is such that $p_m \geq 0$ and $\sum_{m=1}^{\infty} p_m = 1$.

2. Geometry of an O^2-representation and of a $\overline{O}^2$-representation of real numbers

Let $(c_1, c_2, \dots, c_n)$ be a given family of positive integers. The set of numbers $x \in (0,1]$ of the form

$$\Delta^{O^2}_{c_1 c_2 \dots c_n} = \big\{x\colon x = O^2(q_1, q_2, \dots, q_n, \dots),\ q_i = c_i,\ i = 1, \dots, n,\ q_{n+j} \in \mathbb{N}\big\}$$

is called the *cylinder of rank n with base $c_1 c_2 \dots c_n$*.

Below we list some properties of cylinder sets.

1. $\Delta^{O^2}_{c_1 \dots c_n} = \bigcup_{i=c_n(c_n+1)}^{\infty} \Delta^{O^2}_{c_1 \dots c_n i}$.
2.

$$\begin{aligned}
\inf \Delta^{O^2}_{c_1 \dots 2m-1} &= \sum_{k=1}^{2m-1} \frac{(-1)^{k-1}}{c_k} - \frac{1}{c_{2m-1}(c_{2m-1}+1)} \\
&= O^2(c_1, \dots, c_{2m-1}, c_{2m-1}(c_{2m-1}+1)) \\
&= O^2(c_1, \dots, c_{2m-2}, c_{2m-1}+1) \in \Delta^{O^2}_{c_1 \dots c_{2m-1}};
\end{aligned}$$

$$\sup \Delta^{O^2}_{c_1 \dots 2m-1} = \sum_{k=1}^{2m-1} \frac{(-1)^{k-1}}{c_k} = O^2(c_1, \dots, c_{2m-1}) \in \Delta^{O^2}_{c_1 \dots c_{2m-1}};$$

$$\inf \Delta^{O^2}_{c_1 \dots 2m} = \sum_{k=1}^{2m} \frac{(-1)^{k-1}}{c_k} = O^2(c_1, \dots, c_{2m}) \in \Delta^{O^2}_{c_1 \dots c_{2m}};$$

$$\begin{aligned}
\sup \Delta^{O^2}_{c_1 \dots 2m} &= \sum_{k=1}^{2m} \frac{(-1)^{k-1}}{c_k} + \frac{1}{c_{2m}(c_{2m}+1)} = O^2(c_1, \dots, c_{2m}, c_{2m}(c_{2m}+1)) \\
&= O^2(c_1, \dots, c_{2m-1}, c_{2m}+1) \in \Delta^{O^2}_{c_1 \dots c_{2m}}.
\end{aligned}$$

3.

$$\sup \Delta^{O^2}_{c_1 \dots c_{2m-1} i} = \inf \Delta^{O^2}_{c_1 \dots c_{2m-1}(i+1)};$$
$$\inf \Delta^{O^2}_{c_1 \dots c_{2m} i} = \sup \Delta^{O^2}_{c_1 \dots c_{2m}(i+1)}.$$

4. The length of a cylinder interval of rank n satisfies

$$\left|\Delta^{O^2}_{c_1 \dots c_n}\right| = \frac{1}{c_n(c_n+1)} \to 0, \qquad n \to \infty.$$

5. The length of a cylinder depends on the last symbol of the basis only, that is,

$$\left|\Delta^{O^2}_{c_1 \dots c_n i}\right| = \frac{1}{i(i+1)} = \left|\Delta^{O^2}_{s_1 \dots s_m i}\right|.$$

6.

$$\frac{\left|\Delta^{O^2}_{c_1 \dots c_n c_{n+1}}\right|}{\left|\Delta^{O^2}_{c_1 \dots c_n}\right|} = \frac{c_n(c_n+1)}{c_{n+1}(c_{n+1}+1)} \le \frac{1}{2^{2^{n-1}}} \quad \text{for all } n \in \mathbb{N}.$$

Any of the cylinders of an O^2-representation can be rewritten in terms of a $\overline{O}^2$-representation as follows:

$$\Delta^{O^2}_{c_1 \dots c_n} \equiv \bar{\Delta}^{O^2}_{a_1 \dots a_n},$$

where $a_1 = c_1$, $a_k = c_k + 1 - c_{k-1}(c_{k-1}+1)$, $1 < k < n$.

Properties of the cylinders corresponding to an O^2-representation are analogous to those of cylinders corresponding to a $\overline{O}^2$-representation. We mention only one of those properties:

$$\left|\Delta^{\overline{O}^2}_{a_1 \dots a_n a_{n+1}}\right| \le \frac{1}{2^{2^{n-1}}} \left|\Delta^{\overline{O}^2}_{a_1 \dots a_n}\right| \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

3. Normal properties of numbers and their $\overline{O}^2$-representations

A property of real numbers is called normal if it holds for almost all (in the sense of Lebesgue measure) real numbers. Examples of normal properties are given by the properties "to be irrational" or "to be transcendental," which do not depend on the chosen number system (in other words, on the way the real numbers are represented). If a representation of real numbers is fixed, their normal properties can be conveniently expressed in terms of symbols (digits) in their representation. For example, the following are normal properties if one uses the decimal expansions of numbers: "to contain infinitely many zeros in the decimal expansion", "to be aperiodic", "to contain every digit with frequency 10^{-1}," and so on. If one uses the representation in the form of continued fractions, then the following are also examples of normal properties: "to have infinitely many elements in the representation", "to contain a symbol i with asymptotic frequency $\frac{1}{\ln 2}\ln\frac{(i+1)^2}{i(i+2)}$," and so on.

The main goal in this section is to find some normal properties of real numbers expressed in terms of the sequence $\{d_k(x)\}$ for the second Ostrogradskiĭ difference expansion.

Theorem 3.1. *Almost all (in the sense of Lebesgue measure) numbers of the interval* $(0,1]$ *contain every* $\overline{O}^2$*-symbol finitely many times in their* $\overline{O}^2$*-representation.*

Proof. Let $N_i(x)$ be the number of occurrences of a symbol i in the $\overline{O}^2$-representation of a number x. We prove that the Lebesgue measure of the set $A_i = \{x\colon N_i(x)=\infty\}$ is equal to zero for all $i \in N$.

Consider the set

$$\bar{\Delta}_i^k = \left\{x\colon x = \overline{O}^2(d_1,\dots,d_{k-1},i,d_{k+1},\dots),\ d_j \in \mathbb{N} \text{ for all } j \neq k\right\}$$

of numbers of the interval $(0,1]$ for which the symbol i occurs at the position k of the $\overline{O}^2$-representation, that is, $d_k(x) = i$.

Lemma 3.1. *For all positive integers i and k, we have*

$$\lambda\left(\bar{\Delta}_i^1\right) = \frac{1}{i(i+1)} \le \frac{1}{2}, \qquad \lambda(\bar{\Delta}_i^k) \le \frac{1}{2^{2^{k-2}}} \quad \text{for } k>1.$$

Proof. Since

$$\bar{\Delta}_i^1 = \Delta_i^{\overline{O}^2} = \left[\frac{1}{i+1}, \frac{1}{i}\right],$$

we obtain $\lambda\left(\bar{\Delta}_i^1\right) = (i(i+1))^{-1} \le \frac{1}{2}$.

Properties 1 and 6 of cylinder sets listed above and the definition of the set $\bar{\Delta}_i^k$ imply that

$$\bar{\Delta}_i^k = \bigcup_{a_1=1}^{\infty}\dots\bigcup_{a_{k-1}=1}^{\infty}\Delta^{\overline{O}^2}_{a_1\dots a_{k-1}i} \quad\text{and}\quad \frac{\left|\Delta^{\overline{O}^2}_{a_1\dots a_{k-1}i}\right|}{\left|\Delta^{\overline{O}^2}_{a_1\dots a_{k-1}}\right|} \le \frac{1}{2^{2^{k-2}}}.$$

Thus

$$\begin{aligned}\lambda\left(\bar{\Delta}_i^k\right) &= \sum_{a_1=1}^{\infty}\dots\sum_{a_{k-1}=1}^{\infty}\left|\Delta^{\overline{O}^2}_{a_1\dots a_{k-1}i}\right| \le \frac{1}{2^{2^{k-2}}}\left(\sum_{a_1=1}^{\infty}\dots\sum_{a_{k-1}=1}^{\infty}\left|\Delta^{\overline{O}^2}_{a_1\dots a_{k-1}}\right|\right)\\ &= \frac{1}{2^{2^{k-2}}}.\end{aligned}$$

□

It is obvious that A_i is the upper limit of the sequence of sets $\{\bar{\Delta}_i^k\}$, that is,

$$A_i = \limsup_{k\to\infty} \bar{\Delta}_i^k = \bigcap_{m=1}^{\infty} \left(\bigcup_{k=m}^{\infty} \bar{\Delta}_i^k \right).$$

Since

$$\sum_{k=1}^{\infty} \lambda\left(\bar{\Delta}_i^k\right) \le \sum_{k=1}^{\infty} \frac{1}{2^{2^{k-2}}} < +\infty,$$

the Borel–Cantelli lemma implies that

$$\lambda(A_i) = 0 \quad \text{for all } i \in \mathbb{N}.$$

Hence

$$\lambda(\bar{A}_i) = 1 \quad \text{for all } i \in \mathbb{N}.$$

Let

$$\bar{A} = \bigcap_{i=1}^{\infty} \bar{A}_i.$$

It is obvious that $\lambda(\bar{A}) = 1$. The theorem is proved. □

Let $N_i(x,k)$ be the number of occurrences of the symbol i up to the position k in the $\overline{O}^2$-representation of a number x. If the limit $\lim_{k\to\infty} N_i(x,k)/k$ exists, it is called the asymptotic frequency of the digit (symbol) i in the $\overline{O}^2$-representation of a number x. The asymptotic frequency is denoted by $\nu_i(x, \overline{O}^2)$ or simply $\nu_i(x)$, if it is clear from the context what it stays for.

Corollary 3.1. *For λ-almost all x and for all $i \in \mathbb{N}$,*

$$\nu_i(x) = 0.$$

Corollary 3.2. *For an arbitrary random vector $\overrightarrow{p} = (p_1, p_2, \dots, p_k, \dots)$, the set*

$$I_{\overrightarrow{p}} = \left\{x \colon x = \overline{O}^2(d_1(x), \dots, d_k(x), \dots),\ \nu_i(x) = p_i \text{ for all } i \in \mathbb{N}\right\}$$

has zero Lebesgue measure.

Corollary 3.3. *Let $V = \{1, 2, \dots, n\}$. Then, for an arbitrary set $V \subset N$, the set $C[\overline{O}^2, V] = \left\{x \colon x = \overline{O}^2(d_1, d_2, \dots, d_n, \dots), d_j(x) \in V\right\}$ has zero Lebesgue measure.*

Let $B(\overline{O}^2)$ be the set of all real numbers with bounded $\overline{O}^2$-symbols. In other words, $x \in B(\overline{O}^2)$ if and only if there exists a number $K(x)$ such that $d_k(x) \le K(x)$ for all $k \in \mathbb{N}$.

Corollary 3.4. *The set $B(\overline{O}^2)$ of all numbers with bounded $\overline{O}^2$-symbols has zero Lebesgue measure.*

4. Singularity of the distributions of random variables represented by the second Ostrogradskiĭ series with independent increments and properties of the corresponding symbolic dynamics

Consider the following random variable:

$$\eta = \overline{O}^2(\eta_1, \eta_2, \dots, \eta_k, \dots) \tag{6}$$

whose $\overline{O}^2$-symbols η_k are independent identically distributed random variables assuming values $1, 2, \dots, m, \dots$ with probabilities $p_{1k}, p_{2k}, \dots, p_{mk}, \dots$, respectively. This means that

$$\mathsf{P}\{\eta_k = m\} = p_{mk}, \qquad p_{mk} \ge 0, \quad \sum_{m=1}^{\infty} p_{mk} = 1 \quad \text{for all } k \in \mathbb{N}.$$

It is obvious that the distribution of the random variable η is completely determined by the matrix $P = \|p_{mk}\|$. A criterion for the distribution of the random variable η to be purely discrete is given in [11]. Sufficient conditions for this distribution to be singular in Cantor's sense are also presented in [11]. The main goal in this section is to discover the Lebesgue structure of the distribution of the random variable η in the case where the members of the sequence $\{\eta_k\}$ are independent and identically distributed.

Consider the dynamical system generated by the following one-sided shift transformation T in the $\overline{O}^2$-representation:

$$T(x) = T\left(\overline{O}^2(d_1(x), d_2(x), \dots, d_n(x), \dots)\right) = \overline{O}^2\big(d_2(x), d_3(x), \dots, d_n(x), \dots\big)$$
$$\text{for all } x = \overline{O}^2\big(d_1(x), d_2(x), \dots, d_n(x), \dots\big) \in [0,1].$$

Recall that a set A is called invariant or fixed with respect to a transformation T if $A = T^{-1}A$. A measure μ is called ergodic with respect to the transformation T if an arbitrary invariant measure $A \in \mathfrak{B}$ is of either zero or complete measure. A measure μ is called invariant with respect to the transformation T if $\mu(T^{-1}E) = \mu(E)$ for an arbitrary set $E \in \mathfrak{B}$.

Lemma 4.1. *The measure μ_η is invariant and ergodic with respect to the shift transformation T defined above.*

Proof. 1) Let A be a set fixed with respect to the transformation T, that is,

$$T^{-1}A = A, \qquad A \in \mathfrak{B}.$$

Then $T(T^{-1}A) = T(A)$, whence $A = TA$. Thus $A = T^{-1}A = T^{-1}(TA)$.

If $x = \overline{O}^2(d_1(x), d_2(x), \dots, d_k(x), \dots)$ and $x \in A$, then

$$T^{-1}(Tx) = \big\{x\colon x = \overline{O}^2(c_1, d_2(x), \dots, d_k(x), \dots), c_1 \in \mathbb{N}\big\} \subset A.$$

Hence a point x belongs to the invariant set A independently of the first $\overline{O}^2$-symbol of x. Similarly we prove that a point x belongs to the fixed set A independently of the initial n of the $\overline{O}^2$-symbols of x whatever $n \geq 1$ is. Thus the Kolmogorov 0-1 law implies that either $\mu_\eta(A) = 0$ or $\mu_\eta(A) = 1$. Hence μ_η is ergodic with respect to the transformation T.

2) Since the Borel σ-algebra $\mathcal{B}$ is generated by the system of cylinders of the $\overline{O}^2$-representation, that is, by the system of sets of the form $\Delta^{\overline{O}^2}_{c_1 c_2 \dots c_n}$, it suffices to show that the measure μ_η is invariant on these cylinders. It is obvious that

$$\mu_\eta\left(\Delta^{\overline{O}^2}_{c_1 c_2 \dots c_n}\right) = p_{c_1} \cdot p_{c_2} \cdots p_{c_n}.$$

Since

$$T^{-1}\left(\Delta^{\overline{O}^2}_{c_1 c_2 \dots c_n}\right) = \Delta^{\overline{O}^2}_{i\ c_1 c_2 \dots c_n}, \qquad i \in \mathbb{N},$$

we derive that

$$\mu_\eta\left(T^{-1}\left(\Delta^{\overline{O}^2}_{c_1 c_2 \dots c_n}\right)\right) = \sum_{i=1}^{\infty} \mu_\eta\left(\Delta^{\overline{O}^2}_{i\ c_1 c_2 \dots c_n}\right) = p_{c_1} \cdot p_{c_2} \cdots p_{c_n} \sum_{i=1}^{\infty} p_i$$
$$= p_{c_1} \cdot p_{c_2} \cdots p_{c_n} = \mu_\eta\left(\Delta^{\overline{O}^2}_{c_1 c_2 \dots c_n}\right),$$

and this is what was to be proved. □

Lemma 4.2. *For μ_η-almost all $x \in [0,1]$, we have*

$$\nu_i\left(x, \overline{O}^2\right) = p_i \quad \text{for all } i \in \mathbb{N}. \tag{7}$$

Proof. Let $T^j(x)$ denote the jth iteration of the above shift transformation T. Since the measure μ_η is ergodic and invariant with respect to T, the Birkhoff ergodic theorem implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(T^j(x)\right) = \int_0^1 \varphi(x)\, d(\mu_\eta(x))$$

for μ_η-almost all $x \in [0,1]$ and for an arbitrary function $\varphi \in L^1([0,1], d\mu_\eta)$.

Fix a positive integer i. Let $\Delta_i^{\overline{O}^2}$ be the corresponding cylinder of the first rank of the $\overline{O}^2$-representation. We take for a function φ the indicator of the set $\Delta_i^{\overline{O}^2}$, that is, $\varphi(x) = 1$ for $x \in \Delta_i^{\overline{O}^2}$ and $\varphi(x) = 0$ otherwise.

Then

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(T^j(x)\right) = \frac{N_i(x,n)}{n},$$

$$\int_0^1 \varphi(x)\, d(\mu_\eta(x)) = \int_{\Delta_i^{\overline{O}^2}} d(\mu_\eta(x)) = p_i.$$

Hence $\nu_i(x, \overline{O}^2) = p_i$ for μ_η-almost all numbers $x \in [0,1]$. Let

$$M_{(p_1,p_2,\dots,p_k,\dots)} = \{x \colon x \in [0,1],\ \nu_i(x,\overline{O}^2) = p_i \text{ for all } i \in \mathbb{N}\}.$$

The μ_η-measure of $M_{(p_1,p_2,\dots,p_k,\dots)}$ is equal to 1, since this set is the intersection of a countable number of the sets

$$M_i = \left\{x \colon x \in [0,1],\ \nu_i\left(x, \overline{O}^2\right) = p_i\right\}$$

and each of them is of a complete μ_η-measure. □

The following theorem completely answers the question on the Lebesgue structure of the distribution of the random variable η for the case where the members of the sequence $\{\eta_k\}$ are identically distributed.

Theorem 4.1. *Let $\{\eta_k\}$ be a sequence of independent identically distributed random variables assuming values $1, 2, 3, \dots$ with probabilities $p_1, p_2, p_3, \dots$ and let $\sum_{i=1}^\infty p_i = 1$. Then the distribution of the random variable η of the second Ostrogradskiĭ representation with independent increments (that is, the random variable defined by equality* (6)*) is*

1) *either a degenerate discrete distribution (if $p_i = 1$ for some $i \in \mathbb{N}$)*
2) *or a singularly continuous distribution (otherwise).*

Proof. 1) Statement 1) follows directly from the fact that the random variables $\{\eta_k\}$ are identically distributed and from a general criterion for the discreteness of the distribution of the random variable η (see [11]); namely, the distribution of the random variable η is purely discrete if and only if $\prod_{k=1}^\infty \max_i p_{ik} > 0$.

2) We prove that if the distribution of the random variable η is continuous, then it does not contain an absolutely continuous component. We choose a positive integer i_0 such that $p_{i_0} > 0$ (at least one number i_0 with this property exists) and consider the set

$$M_{i_0} = \left\{x \colon x \in [0,1],\ \nu_{i_0}(x, \overline{O}^2) = p_{i_0} > 0\right\}.$$

By Lemma 4.2, the set M_{i_0} is of full μ_η-measure.

We also consider the set $L_{i_0}^* = \left\{x \colon x \in [0,1], \nu_{i_0}(x, \overline{O}^2) = 0\right\}$. Formula (5) implies that $\lambda(L_{i_0}^*) = 1$. The sets M_{i_0} and $L_{i_0}^*$ are disjoint. The first of them is the support of the probability measure μ_η, while the second is the support of Lebesgue measure on the unit interval. Thus the measure μ_η is singular with respect to Lebesgue measure, and this is what was to be proved. □

Corollary 4.1. *The distribution of the first Ostrogradskiĭ series with independent identically distributed increments is pure. Moreover, this distribution cannot be absolutely continuous.*

Corollary 4.2. *All probability measures corresponding to the second Ostrogradskiĭ series with independent identically distributed increments are invariant and ergodic with respect to the one-sided shift transformation T in the $\overline{O}^2$-representation. However, none of these measures is absolutely continuous with respect to Lebesgue measure.*

As is known, the studies of metric (ergodic) theory of representations of real numbers will significantly simplify if one can find a measure that is invariant and ergodic with respect to the shift transformation in that representation and, at the same time, is absolutely continuous with respect to Lebesgue measure (see [7]). For example, since the Gauss measure (that is, an absolutely continuous probability measure with density $f(x) = \frac{1}{\ln 2}\frac{1}{1+x}$ on the unit interval) is invariant and ergodic with respect to the shift transformation in the representation by continued fractions, one can derive the main ergodic properties of continued fractions (see, for example, [14, 7]).

In conclusion, we show that Corollary 4.2 holds true even if the measure does not necessarily belong to the class of probability measures corresponding to the Ostrogradskiĭ series with independent identically distributed $\overline{O}^2$-symbols.

Theorem 4.2. *There is no probability measure that is invariant and ergodic with respect to the shift transformation T in the $\overline{O}^2$-representation and, at the same time, is absolutely continuous with respect to Lebesgue measure.*

Proof. We prove this by contradiction. Assume that there exists an absolutely continuous probability measure ν that is invariant and ergodic with respect to T. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(T^j(x)\right) = \int_0^1 \varphi(x)\, d(\nu(x)) = \int_0^1 \varphi(x) f_\nu(x)\, dx$$

for ν-almost all $x \in [0,1]$ (thus for all sets of positive Lebesgue measure) and for an arbitrary function $\varphi \in L^1([0,1], d\nu)$, where $f_\nu(x)$ is the density of the measure ν.

Let $\varphi_i(x) = 1$ if $x \in \Delta_i^{\overline{O}^2}$ and $\varphi_i(x) = 0$ otherwise. Then

$$\int_0^1 \varphi_i(x) f_\nu(x)\, dx = \int_{x\in\Delta_i^{\overline{O}^2}} f_\nu(x)\, dx > 0$$

for at least one $i \in \mathbb{N}$, say, i_0.

On the other hand, (5) implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_{i_0}\left(T^j(x)\right) = \lim_{n\to\infty} \frac{N_{i_0}(x,n)}{n} = 0$$

for λ-almost all $x \in [0,1]$. Therefore,

$$\lim_{n\to\infty} \frac{N_{i_0}(x,n)}{n} = 0$$

for λ-almost all $x \in [0,1]$ and, at the same time,

$$\lim_{n\to\infty} \frac{N_{i_0}(x,n)}{n} > 0$$

for a set of positive Lebesgue measure. This contradiction completes the proof of the theorem. ☐

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