

A MINIMAL UNIFORM RENEWAL THEOREM AND TRANSITION PHENOMENA FOR A NONHOMOGENEOUS PERTURBATION OF THE RENEWAL EQUATION

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ABSTRACT. We consider the classical continuous time nonhomogeneous renewal equation on the half-line, that is, the integral Volterra equation with a nonnegative bounded kernel. It is assumed that the kernel can be approximated by a convolution kernel generated by a strong substochastic distribution in the large time scale. We study the scheme of series (transition phenomena) where the corresponding defect tends to zero and the time scale tends to infinity. We find the limit of a solution of the renewal equation under certain assumptions. This result is based on a minimal renewal theorem in the scheme of series.

Some applications are also considered.

1. INTRODUCTION

Risk processes are considered in the papers [2]–[11] for the case of a nonhomogeneous environment.

Similarly to the author's preceding paper [3], we study a time nonhomogeneous generalized classical renewal equation on the half-line with continuous time. The equation can be reduced to the integral Volterra equation with a nonnegative bounded (or substochastic) kernel. We assume that, in a large scale, this kernel is approximated in variation by a convolution kernel generated by a strictly substochastic distribution on the positive half-line.

We consider a scheme of series where the total mass of the limit substochastic distribution approaches as the time scale increases (this model corresponds to the well-known scheme of transition phenomena in renewal theory; see [17]).

We prove the existence of a solution of the nonhomogeneous renewal equation and find its limit as the time scale increases.

The proof below uses a minimal renewal theorem in the scheme of series (see Section 2) and is based on some of the methods developed in [12, 13].

2. A MINIMAL RENEWAL THEOREM IN THE SCHEME OF SERIES

Consider the half-line $\mathbb{R}_+ = [0, \infty)$ equipped with the σ -algebra $\mathfrak{B}_+ = \mathfrak{B}(\mathbb{R}_+)$. Introduce the following class of functions:

$$\mathcal{B}_0 \equiv \left\{ x: \mathbb{R}_+ \rightarrow \mathbb{R}, \text{ where } x \text{ is a Borel function such that } \sup_{s \leq t} |x(s)| < \infty, \forall t \geq 0 \right\}.$$

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To estimate the rate of convergence, we fix a real positive nondecreasing function φ defined on \mathbb{R}_+ and such that

$$(1) \quad \varphi(t+s) \leq \varphi(t)\varphi(s), \quad \lim_{t \rightarrow \infty} \frac{\varphi(t+s)}{\varphi(t)} = 1, \quad t, s \geq 0.$$

In particular, the functions $1, (1 + \ln(1+t))^\alpha$ for $\alpha > 0, (1+t)^\beta$ for $\beta > 0,$ and $\exp(t^\gamma)$ for $0 < \gamma < 1,$ as well as their multiplicative combinations satisfy all of the above conditions.

Definition 1. We say that a family of measures $(G_\alpha, \alpha > 0)$ belongs to the class $L(\nu, m)$ if there exist an absolutely continuous nonzero measure ν and a number $m \geq 1$ such that

$$G_\alpha^{*m}(B) \geq \nu(B) \quad \text{for all } \alpha > 0 \quad \text{and } B \in \mathfrak{B}_+.$$

Note that the latter condition implies that the distribution G_α is of the absolutely continuous type.

Some equivalent conditions of the absolute continuity are presented in [12, 13].

Definition 2. We say that a family of measures $(G_\alpha, \alpha > 0)$ belongs to the class $M(l, \varphi)$ if there exists a function $l(t) \rightarrow 0, t \rightarrow \infty,$ such that

$$\sup_\alpha \int_t^\infty \varphi(s) G_\alpha([s, \infty)) ds \leq l(t), \quad t \geq 0.$$

Definition 3. Let $t_\alpha \rightarrow \infty$ as $\alpha \rightarrow 0.$ We say that a family of positive Borel functions $\tau = (\tau_\alpha(t))$ belongs to the class $\Upsilon_\infty(\{t_\alpha\})$ if $\tau_\alpha(t) \leq t$ and

$$\inf(\tau_\alpha(t) : \tau_\alpha(t) < t \leq t_\alpha) \rightarrow \infty, \quad \alpha \rightarrow 0.$$

Definition 4. For the class of Borel functions $y = (y_\alpha, \alpha > 0)$ and for nonnegative functions $\tau = (\tau_\alpha(t))$ such that $v_\alpha \leq t_\alpha,$ we introduce the following indices:

$$(2) \quad \begin{aligned} \sigma_{y,\varphi,\tau}(\alpha, t) &\equiv \sup_{0 \leq s < \tau_\alpha(t)} \frac{|y_\alpha(t-s)|}{\varphi(s)}, \\ \sigma_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) &\equiv \sup_{v_\alpha \leq t \leq t_\alpha} \sigma_{y,\varphi,\tau}(\alpha, t), \\ \lambda_{y,\varphi,\tau}(\alpha, t) &\equiv \int_{0 \leq s < \tau_\alpha(t)} \frac{|y_\alpha(t-s)|}{\varphi(s)} ds, \\ \lambda_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) &\equiv \sup_{v_\alpha \leq t \leq t_\alpha} \lambda_{y,\varphi,\tau}(\alpha, t). \end{aligned}$$

Definition 5. For a function φ satisfying (1) and for functions $0 \leq v_\alpha \leq t_\alpha \rightarrow \infty, \alpha \rightarrow 0,$ we define the following classes of Borel functions:

$$(3) \quad B_0^\varphi(\{v_\alpha\}, \{t_\alpha\}) = \left\{ \begin{array}{l} y = (y_\alpha(t)) : \exists \tau = (\tau_\alpha(t)) \in \Upsilon_\infty(\{t_\alpha\}) : \\ \sigma_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) = o(1), \sigma_{y,\varphi,t}(\alpha, 0, t_\alpha) = O(1), \alpha \rightarrow 0 \end{array} \right\}$$

and

$$(4) \quad L_0^\varphi(\{v_\alpha\}, \{t_\alpha\}) = \left\{ \begin{array}{l} y = (y_\alpha(t)) : \exists \tau = (\tau_\alpha(t)) \in \Upsilon_\infty(\{t_\alpha\}) : \\ \lambda_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) = o(1), \lambda_{y,\varphi,t}(\alpha, 0, t_\alpha) = O(1), \alpha \rightarrow 0 \end{array} \right\}.$$

Theorem 1. Let, for each $\alpha > 0,$ a function $x_\alpha \in B_0$ satisfy the renewal equation with $y_\alpha \in B_0,$ namely

$$(5) \quad x_\alpha(t) = y_\alpha(t) + x_\alpha * F_\alpha(t) \equiv y_\alpha(t) + \int_{[0, t)} x_\alpha(t-s) F_\alpha(ds), \quad t \geq 0,$$

where the members of the family of probability distributions (F_α) have the mean values $m_\alpha \equiv \int_0^\infty s F_\alpha(ds)$ and are such that

$$(F_\alpha) \subset L(\nu, m) \cap M(l, \varphi)$$

(the classes $L(\nu, m)$ and $M(l, \varphi)$ are introduced in Definitions 1 and 2, respectively).

If the functions are such that $0 \leq v_\alpha \leq t_\alpha \rightarrow \infty$ as $\alpha \rightarrow 0$ and

$$(y_\alpha) \subset B_0^\varphi(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^\varphi(\{v_\alpha\}, \{t_\alpha\}),$$

then

$$(6) \quad \sup_{v_\alpha \leq t \leq t_\alpha} \left| x_\alpha(t) - m_\alpha^{-1} \int_0^t y_\alpha(s) ds \right| \\ = O(\sigma_{y, \varphi, \tau}(\alpha, v_\alpha, t_\alpha)) + o(\sigma_{y, \varphi, t}(\alpha, 0, t_\alpha)) \\ + O(\lambda_{y, \varphi, \tau}(\alpha, v_\alpha, t_\alpha)) + o(\lambda_{y, \varphi, t}(\alpha, 0, t_\alpha)) \rightarrow 0, \quad \alpha \rightarrow 0,$$

according to definitions (3) and (4).

Remark 1. The first condition in definition (4) follows from the first condition in definition (3) if $v_\alpha \equiv 0$.

We need the following definition to study the rate of convergence in (6).

Definition 6. Assume that a function φ satisfies (1) and that the functions v_α are such that $0 \leq v_\alpha \leq t_\alpha \rightarrow \infty$ as $\alpha \rightarrow 0$. We introduce the classes of Borel functions $\bar{B}_0^\varphi(\{v_\alpha\}, \{t_\alpha\})$ and $\bar{L}_0^\varphi(\{v_\alpha\}, \{t_\alpha\})$ according to relations (3) and (4), respectively, where the indices (2) are changed by

$$(7) \quad \bar{\sigma}_{y, \varphi, \tau}(\alpha, t) \equiv \sup_{0 \leq s < \tau_\alpha(t)} |y_\alpha(t-s)| \varphi(t-s)$$

and

$$\bar{\lambda}_{y, \varphi, \tau}(\alpha, t) \equiv \int_{0 \leq s < \tau_\alpha(t)} |y_\alpha(t-s)| \varphi(t-s) ds,$$

respectively.

Theorem 2. Let the assumptions of Theorem 1 hold with \bar{B}_0^φ and \bar{L}_0^φ instead of B_0^φ and L_0^φ , respectively. Then

$$(8) \quad \sup_{v_\alpha \leq t \leq t_\alpha} \varphi(t) \left| x_\alpha(t) - m_\alpha^{-1} \int_0^t y_\alpha(s) ds \right| \rightarrow 0, \quad \alpha \rightarrow 0.$$

3. PERTURBED TRANSITION RENEWAL EQUATION

Let $(G_\alpha, \alpha > 0)$ be a family of strictly substochastic measures on \mathfrak{B}_+ .

The integral equation

$$(9) \quad x_\alpha(t) = y_\alpha(t) + \int_{[0, t)} x_\alpha(t-s) G_\alpha(ds), \quad t \geq 0,$$

with respect to an unknown function $x_\alpha \in B_0$ is called an improper renewal equation on \mathbb{R}_+ generated by a family of strictly substochastic measures (G_α) .

Let $(F_\alpha(t, B), t \geq 0, B \in \mathfrak{B}_+, \alpha > 0)$ be a family of bounded nonnegative kernels.

Definition 7. The integral Volterra equation

$$(10) \quad x_\alpha(t) = y_\alpha(t) + \int_{[0, t)} x_\alpha(t-s) F_\alpha(t, ds), \quad t \geq 0,$$

with respect to an unknown function $x_\alpha \in B_0$ and a given function y_α is called a nonhomogeneous perturbation of the renewal equation (9).

The kernel F_α in equation (10) approximates the measure G_α in variation:

$$(11) \quad \Delta_\alpha(t, B) \equiv F_\alpha(t, B) - G_\alpha(B) \rightarrow 0, \quad t \rightarrow \infty,$$

uniformly with respect to α and $B \in \mathfrak{B}_+ \cap [0, t)$. This means that the perturbation function $\delta_\alpha(t)$ is such that

$$(12) \quad \delta_\alpha(t) \equiv |\Delta_\alpha|(t, [0, t)) = \sup_{|x| \leq 1} \left| \int_0^t x(s) \Delta_\alpha(t, ds) \right| \rightarrow 0, \quad t \rightarrow \infty.$$

Assume that the following extended Cramér condition holds:

$$(13) \quad \forall \alpha > 0 \exists \delta > 0: \quad \widehat{g}_\alpha(\delta) \equiv \int_0^\infty \exp(\delta x) G_\alpha(dx) \in (1, \infty).$$

The Lundberg index

$$\beta = \beta(\alpha) \equiv \sup(s \geq 0: \widehat{g}_\alpha(s) \leq 1)$$

is finite under the Cramér condition (13). At the same time, the Lundberg index is a unique positive root of the equation

$$(14) \quad \widehat{g}_\alpha(\beta) = 1$$

and moreover

$$\mu_\alpha \equiv \widetilde{g}_\alpha(\beta) = \int_0^\infty x \exp(\beta x) G_\alpha(dx) < \infty.$$

This implies that the corresponding exponential transform of G_α is a probability measure; namely,

$$(15) \quad \overline{G}_\alpha(B) \equiv \int_B \exp(\beta s) G_\alpha(ds), \quad B \in \mathfrak{B}_+,$$

where $\overline{G}_\alpha(\mathbb{R}_+) = 1$.

The transition phenomenon in the renewal theory [17] means that $G_\alpha(\mathbb{R}_+) \rightarrow 1$ as $\alpha \rightarrow 0$. This assumption implies the following convergences:

$$(16) \quad \beta \rightarrow 0, \quad \frac{\beta \mu_\alpha}{(1 - G_\alpha(\mathbb{R}_+))} \rightarrow 1, \quad \alpha \rightarrow 0.$$

In what follows we consider a scheme with the function φ in (1) and (2) being equal to 1 for all arguments.

Theorem 3. *Let a family $(x_\alpha) \subset B_0$ be a solution of the nonhomogeneous perturbation (10) of the improper renewal equation (9). Further let a family of kernels $(F_\alpha(t, B))$ be a perturbation (11) of strictly substochastic measures $(G_\alpha(B))$ and let the Cramér condition (13) hold. Assume that the generated probabilities (15) are such that*

$$(17) \quad (\overline{G}_\alpha) \subset L(\nu, m) \cap M(l, 1).$$

Let $\alpha \rightarrow 0$ and let the functions $0 \leq v_\alpha \leq t_\alpha \rightarrow \infty$ be such that $\beta t_\alpha \rightarrow u > 0$, where the index β is defined in (14). Assume also that

$$(18) \quad \begin{aligned} (y_\alpha) &\subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\}), \\ (\delta_\alpha) &\subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\}), \end{aligned}$$

and that the solutions (x_α) are locally bounded in the sense that

$$(19) \quad \sup_{\alpha > 0} \sup_{t \leq t_\alpha} |x_\alpha(t)| < \infty.$$

Finally we assume that

$$(20) \quad \rho_\alpha \equiv \int_0^{t_\alpha} |\Delta_\alpha|(t, [t - v_\alpha, t)) dt \rightarrow 0, \quad \alpha \rightarrow 0,$$

$$(21) \quad \varepsilon_\alpha \equiv \int_0^{t_\alpha} \delta_\alpha(s) ds \int_s^{t_\alpha} |\Delta_\alpha|(t, [0, t - s)) dt \rightarrow 0, \quad \alpha \rightarrow 0.$$

Then the representation

$$(22) \quad e^{\beta t} x_\alpha(t) - \mu_\alpha^{-1} \int_0^t \left[e^{\beta s} y_\alpha(s) + \mu_\alpha^{-1} \int_0^s e^{\beta u} y_\alpha(u) \int_0^{s-u} e^{\beta v} \Delta_\alpha(s, dv) du \right] ds \\ = O(\gamma_\alpha + \rho_\alpha + \varepsilon_\alpha) + o(\Gamma_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0,$$

holds uniformly with respect to $t \in [v_\alpha, t_\alpha]$, where

$$(23) \quad \gamma_\alpha \equiv \sigma_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) + \lambda_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) \\ + \sigma_{\delta,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) + \lambda_{\delta,\varphi,\tau}(\alpha, v_\alpha, t_\alpha) = o(1), \quad \alpha \rightarrow 0, \\ \Gamma_\alpha \equiv \sigma_{y,\varphi,t}(\alpha, 0, t_\alpha) + \lambda_{y,\varphi,t}(\alpha, 0, t_\alpha) \\ + \sigma_{\delta,\varphi,t}(\alpha, 0, t_\alpha) + \lambda_{\delta,\varphi,t}(\alpha, 0, t_\alpha) = O(1), \quad \alpha \rightarrow 0,$$

by (3) and (4). All other symbols are defined in (2).

Remark 2. Condition (19) holds if $x_\alpha(t)$ is the probability of a certain random event. Moreover, condition (19) holds if

$$(24) \quad \sup_{\alpha > 0, t \geq 0} \frac{|y_\alpha(t)|}{1 - F_\alpha(t, [0, t])} < \infty.$$

Corollary 1. Let a family $(x_\alpha) \subset B_0$ be a solution of an improper renewal equation (9) with strictly substochastic measures (G_α) . Assume that Cramér's condition (13) holds and that the generated probabilities (15) are such that

$$(\overline{G}_\alpha) \subset L(v, m) \cap M(l, 1).$$

Let $\alpha \rightarrow 0$ and the functions $0 \leq v_\alpha \leq t_\alpha \rightarrow \infty$ be such that $\beta t_\alpha \rightarrow u > 0$, where β is defined by (14). We also assume that

$$(y_\alpha) \subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\}).$$

If the solutions (x_α) are locally bounded, that is,

$$\sup_{\alpha > 0} \sup_{t \leq t_\alpha} |x_\alpha(t)| < \infty,$$

then

$$x_\alpha(t) - \mu_\alpha^{-1} \int_0^t e^{-\beta(t-s)} y_\alpha(s) ds \rightarrow 0, \quad \alpha \rightarrow 0,$$

uniformly with respect to $t \in [v_\alpha, t_\alpha]$.

4. AN APPLICATION TO THE CLAIM PROCESSES

In the paper [3], we considered a risk process modeled by the underlying ruin probabilities at sequential claim moments. Below we describe an application of the limit theorem for the perturbed transition renewal equation to the asymptotic analysis of this model of risk processes.

Consider an insurance company with an initial capital $t > 0$ and assume that the claims $(\xi_k, k \geq 1)$ are independent and have a common distribution function $G(t) = P(\xi_1 < t)$ of an absolutely continuous type. Put $S_n = \xi_1 + \dots + \xi_n$, $S_0 = 0$, and let $\mu = E \xi_1$.

Assume that the insurance company ruins with probability $\pi_\alpha(t - S_{k-1})$ between the $(k-1)$ -th and k -th payments (the company does continue functioning with probability $1 - \pi_\alpha(t - S_{k-1})$). Note that the probability $\pi_\alpha(t - S_{k-1})$ depends on the current capital of the company available between the $(k-1)$ -th and k -th payments. Here $\alpha > 0$ stands for the parameter of the scheme of series.

We assume that the ruin is independent of the sequence of claims given the rest of the capital of the company at a corresponding claim moment.

Denote by $\nu(t)$ the total number of claims that occurred prior to the ruin of the insurance company. Then

$$\mathbb{P}_\alpha(\nu(t) = n \mid S_0, \dots, S_n) = \left[\prod_{k=0}^{n-1} (1 - \pi_\alpha(t - S_k)) \right] \pi_\alpha(t - S_n) \mathbf{1}_{\{S_n < t\}}, \quad n \geq 0.$$

The distribution function of the rest of the capital of the insurance company at the moment of ruin is given by

$$x_\alpha(t, b) = \mathbb{P}_\alpha(t - b \leq S_{\nu(t)} < t) = \mathbb{P}_\alpha(0 < t - S_{\nu(t)} \leq b)$$

for $b < t$.

This distribution function is a solution of the nonhomogeneous transition renewal equation

$$(25) \quad x_\alpha(t, b) = \pi_\alpha(t) \mathbf{1}_{t \leq b} + (1 - \pi_\alpha(t)) \int_0^t x_\alpha(t - s, b) G(ds).$$

Assume that the limit

$$(26) \quad \theta_\alpha \equiv \lim_{t \rightarrow \infty} \pi_\alpha(t) \in (0, 1)$$

exists for all $\alpha > 0$.

Equation (25) coincides with (10) if

$$(27) \quad F_\alpha(t, ds) \equiv (1 - \pi_\alpha(t)) G(ds) \mathbf{1}_{s < t}, \quad y_\alpha(t) = \pi_\alpha(t) \mathbf{1}_{t \leq b}.$$

The kernel (F_α) satisfies condition (11) with the measure

$$(28) \quad G_\alpha = (1 - \theta_\alpha) G,$$

where the perturbation (12) is of the form

$$(29) \quad \Delta_\alpha(t, ds) = -\nu_\alpha(t) G(ds) \mathbf{1}_{s < t}, \quad \delta_\alpha(t) = |\nu_\alpha(t)| G(t),$$

with the perturbation function

$$(30) \quad \nu_\alpha(t) \equiv \pi_\alpha(t) - \theta_\alpha.$$

We assume that

$$(31) \quad \theta_\alpha \rightarrow 0, \quad \alpha \rightarrow 0,$$

in the scheme of transition phenomena.

Let the following Cramér condition hold for the amount of a single claim; namely, let a random variable ξ_1 be such that

$$(32) \quad \exists \delta > 0: \quad \widehat{G}(\delta) \equiv \mathbb{E} \exp(\delta \xi_1) \in (1, \infty).$$

The Lundberg index is defined by

$$(33) \quad \beta = \beta(\alpha) \equiv \sup \left(s \geq 0: (1 - \theta_\alpha) \widehat{G}(s) \leq 1 \right).$$

Note that the Lundberg index is a unique positive solution of the equation

$$(34) \quad (1 - \theta_\alpha) \widehat{G}(\beta) = 1.$$

Note also that $\beta \rightarrow 0$ as $\alpha \rightarrow 0$ and moreover

$$(35) \quad \frac{\theta_\alpha}{\beta} \rightarrow \mu = \mathbf{E} \xi_1, \quad \alpha \rightarrow 0,$$

under the condition (31).

Theorem 4. *Let the distribution function G be of the absolutely continuous type and satisfy Cramér's condition (32).*

Let the limit relations (26) and (31) hold for the ruin probability and moreover let $\alpha \rightarrow 0$ and $b_\alpha \rightarrow \infty$ as $t_\alpha \rightarrow \infty$ in such a way that $\beta t_\alpha \rightarrow u > 0$ and $\beta b_\alpha \rightarrow v \in (0, u)$. We also assume that the perturbation (30) is such that

$$(36) \quad \psi_\alpha \equiv \sup_{0 \leq t < t_\alpha} |\nu_\alpha(t)| = o(1), \quad \alpha \rightarrow 0,$$

$$(37) \quad \phi_\alpha \equiv \int_0^{t_\alpha} |\nu_\alpha(s)| ds = o(1), \quad \alpha \rightarrow 0.$$

Then

$$(38) \quad \begin{aligned} x_\alpha(t_\alpha, b_\alpha) - e^{-u+v} + e^{-u} + e^{-u} \mu^{-1} \int_0^u \nu_\alpha(t/\beta) \beta^{-1} [e^t \mathbf{1}_{t \leq v} - e^v + 1] dt \\ = o(\theta_\alpha) + O(\phi_\alpha) + O(\psi_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0. \end{aligned}$$

Example 1. Let $\pi_\alpha(t) = \theta_\alpha(1 + l(t))$, where the function $l(t) > -1$ is integrable and bounded. Then conditions (36) and (37) hold and

$$x_\alpha(t_\alpha, b_\alpha) \rightarrow \exp(-u + v) - \exp(-u), \quad \alpha \rightarrow 0.$$

Example 2. Let $\pi_\alpha(t) = \theta_\alpha + \varkappa_\alpha \beta l(\beta t)$, where $0 < \varkappa_\alpha \rightarrow 0$ is such that $\theta_\alpha = o(\varkappa_\alpha)$ as $\alpha \rightarrow 0$, while the function $l(t) > -\inf_{\alpha > 0} \theta_\alpha / \varkappa_\alpha \beta$ is integrable and bounded. Then conditions (36) and (37) hold and

$$x_\alpha(t_\alpha, b_\alpha) - e^{-u+v} + e^{-u} + \varkappa_\alpha e^{-u} \mu^{-1} \int_0^u l(t) [e^t \mathbf{1}_{t \leq v} - e^v + 1] dt = o(\varkappa_\alpha), \quad \alpha \rightarrow 0.$$

5. PROOFS

Proof of Theorem 1. Let $x_\alpha \in \mathbf{B}_0$ be a solution of equation (9). This solution is unique and is given by

$$x_\alpha(t) = y_\alpha * U_\alpha(t) \equiv \int_0^t y_\alpha(t-s) U_\alpha(ds)$$

(see [14]–[16]), where U_α is the renewal measure,

$$U_\alpha(B) = \sum_{n \geq 0} F_\alpha^{*n}(B), \quad B \in \mathfrak{B}_+.$$

The inclusion $(F_\alpha) \subset \mathbf{L}(\nu, m) \cap \mathbf{M}(l, \varphi)$ implies that the assumptions of Theorem 2 in [13] hold for the class (F_α) . Thus we obtain the Stone representation $U_\alpha = V_\alpha + H_\alpha$, and moreover,

$$(39) \quad \sup_\alpha \int_0^\infty \varphi(s) V_\alpha(ds) < \infty, \quad \lim_{t \rightarrow \infty} \sup_\alpha \int_t^\infty \varphi(s) V_\alpha([s, \infty)) ds = 0.$$

The measure H_α possesses the density h_α such that

$$(40) \quad \sup_{\alpha, t} \varphi(t) |h_\alpha(t) - m_\alpha^{-1}| < \infty, \quad \lim_{t \rightarrow \infty} \varphi(t) \sup_\alpha |h_\alpha(t) - m_\alpha^{-1}| = 0,$$

where the family of the mean values

$$(41) \quad m_\alpha \equiv \int_0^\infty t F_\alpha(dt) \geq m^{-1} \int_0^\infty t \nu(dt) > 0$$

is bounded away from zero by Definition 1.

In particular,

$$(42) \quad x_\alpha(t) - m_\alpha^{-1} \int_0^t y_\alpha(s) ds = y_\alpha * V_\alpha(t) + y_\alpha * (h_\alpha - m_\alpha^{-1})(t).$$

Put $\tau(\alpha) \equiv \inf(\tau_\alpha(t) : \tau_\alpha(t) < t \leq t_\alpha)$. Then

$$(43) \quad \tau(\alpha) \rightarrow \infty, \quad \alpha \rightarrow 0,$$

in view of the inclusion $\tau = (\tau_\alpha(t)) \in \Upsilon_\infty(\{t_\alpha\})$.

Lemma 1. *Let a family of measures (V_α) satisfy conditions (39) and*

$$0 \leq v_\alpha \leq t_\alpha \rightarrow \infty \quad \text{as } \alpha \rightarrow 0.$$

(a) *If $(y_\alpha) \subset B_0^\varphi(\{v_\alpha\}, \{t_\alpha\})$, then*

$$\sup_{v_\alpha \leq t \leq t_\alpha} |y_\alpha * V_\alpha(t)| = O(\sigma_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha)) + o(\sigma_{y,\varphi,t}(\alpha, 0, t_\alpha)), \quad \alpha \rightarrow 0.$$

(b) *If $(y_\alpha) \subset \overline{B}_0^\varphi(\{v_\alpha\}, \{t_\alpha\})$, then*

$$\sup_{v_\alpha \leq t \leq t_\alpha} \varphi(t) |y_\alpha * V_\alpha(t)| \rightarrow 0, \quad \alpha \rightarrow 0.$$

Proof. For statement (a), we have

$$\begin{aligned} \sup_{v_\alpha \leq t \leq t_\alpha} |y_\alpha * V_\alpha(t)| &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \int_0^{\tau_\alpha(t)} |y_\alpha(t-s)/\varphi(s)| \varphi(s) V_\alpha(ds) \\ &\quad + \sup_{t \leq t_\alpha} \int_{\tau_\alpha(t)}^t |y_\alpha(t-s)/\varphi(s)| \varphi(s) V_\alpha(ds) \\ &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \sigma_{y,\varphi,\tau}(\alpha, t) \sup_{v_\alpha \leq t \leq t_\alpha} \int_0^t \varphi(s) V_\alpha(ds) \\ &\quad + \sup_{t \leq t_\alpha} \sigma_{y,\varphi,t}(\alpha, t) \int_{\tau(\alpha)}^{t_\alpha} \varphi(s) V_\alpha(ds) \rightarrow 0, \quad \alpha \rightarrow 0, \end{aligned}$$

in view of (2), (3), and (43).

For statement (b), we use properties (1) and obtain

$$\begin{aligned} \sup_{v_\alpha \leq t \leq t_\alpha} \varphi(t) |y_\alpha * V_\alpha(t)| &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \int_0^{\tau_\alpha(t)} |y_\alpha(t-s)\varphi(t-s)| \varphi(s) V_\alpha(ds) \\ &\quad + \sup_{t \leq t_\alpha} \int_{\tau_\alpha(t)}^t |y_\alpha(t-s)\varphi(t-s)| \varphi(s) V_\alpha(ds) \\ &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \overline{\sigma}_{y,\varphi,\tau}(\alpha, t) \sup_{v_\alpha \leq t \leq t_\alpha} \int_0^{\tau_\alpha(t)} \varphi(s) V_\alpha(ds) \\ &\quad + \sup_{t \leq t_\alpha} \overline{\sigma}_{y,\varphi,t}(\alpha, t) \int_{\tau(\alpha)}^{t_\alpha} \varphi(s) V_\alpha(ds) \rightarrow 0, \quad \alpha \rightarrow 0, \end{aligned}$$

in view of (7) and (43). □

Lemma 2. *Let the densities (h_α) satisfy conditions (40) and let $0 \leq v_\alpha \leq t_\alpha \rightarrow \infty$ as $\alpha \rightarrow 0$.*

(a) *If $(y_\alpha) \in L_0^\varphi(\{v_\alpha\}, \{t_\alpha\})$, then*

$$\sup_{v_\alpha \leq t \leq t_\alpha} |y_\alpha * (h_\alpha - m_\alpha^{-1})(t)| = O(\lambda_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha)) + o(\lambda_{y,\varphi,t}(\alpha, 0, t_\alpha)), \quad \alpha \rightarrow 0.$$

(b) *If $(y_\alpha) \in \bar{L}_0^\varphi(\{v_\alpha\}, \{t_\alpha\})$, then*

$$\sup_{v_\alpha \leq t \leq t_\alpha} \varphi(t) |y_\alpha * (h_\alpha - m_\alpha^{-1})(t)| \rightarrow 0, \quad \alpha \rightarrow 0.$$

Proof. In the case of (a), definition (4) implies that

$$\begin{aligned} & \sup_{v_\alpha \leq t \leq t_\alpha} |y_\alpha * (h_\alpha - m_\alpha^{-1})(t)| \\ &= \sup_{v_\alpha \leq t \leq t_\alpha} \left| \int_0^t y_\alpha(t-s) [h_\alpha(s) - m_\alpha^{-1}] ds \right| \\ &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \int_0^{\tau_\alpha(t)} |y_\alpha(t-s)/\varphi(s)| \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| ds \\ &\leq \sup_{t \leq t_\alpha} \int_{\tau_\alpha(t)}^t |y_\alpha(t-s)/\varphi(s)| \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| ds \\ &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \lambda_{y,\varphi,\tau}(\alpha, t) \sup_{s \geq 0} \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| \\ &\quad + \sup_{t \leq t_\alpha} \lambda_{y,\varphi,t}(\alpha, t) \sup_{s \geq \tau(\alpha)} \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| \rightarrow 0, \quad \alpha \rightarrow 0, \end{aligned}$$

in view of (43).

In the case of (b), we derive from (1),

$$\begin{aligned} & \sup_{v_\alpha \leq t \leq t_\alpha} \varphi(t) |y_\alpha * (h_\alpha - m_\alpha^{-1})(t)| \\ &= \sup_{v_\alpha \leq t \leq t_\alpha} \varphi(t) \left| \int_0^t y_\alpha(t-s) [h_\alpha(s) - m_\alpha^{-1}] ds \right| \\ &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \int_0^{\tau_\alpha(t)} |y_\alpha(t-s)\varphi(t-s)| \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| ds \\ &\quad + \sup_{t \leq t_\alpha} \int_{\tau_\alpha(t)}^t |y_\alpha(t-s)\varphi(t-s)| \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| ds \\ &\leq \sup_{v_\alpha \leq t \leq t_\alpha} \bar{\lambda}_{y,\varphi,\tau}(\alpha, t) \sup_{s \geq 0} \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| \\ &\quad + \sup_{t \leq t_\alpha} \bar{\lambda}_{y,\varphi,t}(\alpha, t) \sup_{s \geq \tau(\alpha)} \varphi(s) |h_\alpha(s) - m_\alpha^{-1}| \rightarrow 0, \quad \alpha \rightarrow 0, \end{aligned}$$

in view of (43). □

Applying Lemmas 1(a) and 2(a) to (42) and taking into account equality (41), we complete the proof of Theorem 1. □

To prove Remark 1 we choose

$$\tau_\alpha(t) = \min \left\{ \tau_\alpha(t), \left(\sup_{0 \leq t \leq t_\alpha} \sigma_{y,\varphi,\tau}(\alpha, t) \right)^{-1/2} \right\}$$

in definition (4), where the function $\tau = (\tau_\alpha(t))$ on the right hand side is defined by (3).

Now we estimate the integrand on the right hand side of (2), defining $\lambda_{y,\varphi,\tau}(\alpha, t)$ by its supremum over the interval $[0, \tau_\alpha(t))$. Then we use the result in definition (4) of $\lambda_{y,\varphi,\tau}(\alpha, v_\alpha, t_\alpha)$.

Proof of Theorem 2. The proof is completely similar to the preceding one and easily follows from representation (42) and Lemmas 1(b) and 2(b). \square

Proof of Theorem 3. Equality (14) follows from (13), since the function $\widehat{g}_\alpha(s)$ is continuous and bounded for $s \in [0, \delta]$.

The equivalence of functions in (16) follows from the mean value theorem and equality (14):

$$\frac{\beta\mu_\alpha}{1 - G_\alpha(\mathbb{R}_+)} = \frac{\beta\mu_\alpha}{\widehat{g}_\alpha(\beta) - \widehat{g}_\alpha(0)} = \frac{\beta\mu_\alpha}{\beta\widehat{g}'_\alpha(\widetilde{\beta})} \rightarrow 1, \quad \alpha \rightarrow 0.$$

The convergence $\gamma_\alpha \rightarrow 0$ as $\alpha \rightarrow 0$ in relation (22) follows from relations (18), (3), and (4).

For a kernel F , we introduce the following linear bounded operator on B_0 :

$$F[x](t) = \int_0^t F(t, ds)x(t-s), \quad t \geq 0.$$

Similarly to [2, 3] we derive the equation

$$(44) \quad x_\alpha(t) = y_\alpha(t) + \Delta_\alpha[x_\alpha](t) + x_\alpha * G_\alpha(t)$$

from (10) by substituting $F_\alpha(t, ds) = \Delta_\alpha(t, ds) + G_\alpha(ds)$.

Now we introduce the following functions, kernel, and measure:

$$(45) \quad \begin{aligned} \bar{x}_\alpha(t) &= \exp(\beta t)x_\alpha(t), & \bar{y}_\alpha(t) &= \exp(\beta t)y_\alpha(t), \\ \bar{\Delta}_\alpha(t, ds) &= \exp(\beta s)\Delta_\alpha(t, ds), & \bar{G}_\alpha(ds) &= \exp(\beta s)G_\alpha(ds). \end{aligned}$$

According to Remark 1 of [3], we derive the equation

$$(46) \quad \bar{x}_\alpha(t) = \bar{y}_\alpha(t) + \bar{\Delta}_\alpha[\bar{x}_\alpha](t) + \bar{x}_\alpha * \bar{G}_\alpha(t)$$

from (44) by multiplying it by the function $\exp(\beta t)$.

In view of definitions (13) and (14), \bar{G}_α is a probability measure. Thus one can apply Theorem 1 for equation (46) with the distributions $F_\alpha = \bar{G}_\alpha$.

The inclusion $(\bar{G}_\alpha) \subset L(\nu, m) \cap M(l, 1)$ follows by condition (17).

The mean values are such that

$$m_\alpha = \int_0^\infty s \bar{G}_\alpha(ds) = \int_0^\infty e^{\beta s} s G_\alpha(ds) = \widehat{g}'_\alpha(\beta) = \mu_\alpha.$$

In particular,

$$(47) \quad \inf_\alpha m_\alpha = \inf_\alpha \mu_\alpha > 0$$

by condition (17) and according to (41).

Finally, the assumption of Theorem 1 imposed on (y_α) is reduced to the inclusion

$$(48) \quad (|\bar{y}_\alpha|) + (|\bar{\Delta}_\alpha[\bar{x}_\alpha]|) \subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\}).$$

Note that the values $y_\alpha(t)$ on the right hand side of (48) are considered only for $t \leq t_\alpha$ by definitions (3) and (4). Since $\exp(\beta t) \leq \exp(\beta t_\alpha) \rightarrow \exp(u) < \infty$ and

$$(49) \quad \begin{aligned} |\bar{y}_\alpha(t)| &\leq \exp(\beta t_\alpha) |y_\alpha(t)|, \\ |\bar{\Delta}_\alpha[\bar{x}_\alpha](t)| &\leq \exp(\beta t_\alpha) \delta_\alpha(t) \sup_{\alpha > 0, t \geq 0} |x_\alpha(t)|, \end{aligned}$$

inclusion (48) follows from (18), (19), (3), and (4).

Thus Theorem 1 yields the convergence

$$(50) \quad \sup_{v_\alpha \leq t \leq t_\alpha} |\omega_\alpha(t)| = O(\gamma_\alpha) + o(\Gamma_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0,$$

in view of (48), (49), and (23), where ω_α equals the argument of the absolute value in (6) for equation (46):

$$(51) \quad \begin{aligned} \omega_\alpha(t) &\equiv \bar{x}_\alpha(t) - u_\alpha(t) - v_\alpha(t), \\ u_\alpha(t) &\equiv \mu_\alpha^{-1} \int_0^t \bar{y}_\alpha(s) ds, \\ v_\alpha(t) &\equiv \mu_\alpha^{-1} \int_0^t \bar{\Delta}_\alpha[\bar{x}_\alpha](s) ds. \end{aligned}$$

Using (51) we express \bar{x}_α via ω_α and substitute the result into (46) under the sign of the operator $\bar{\Delta}_\alpha$:

$$(52) \quad \bar{x}_\alpha(t) = \bar{y}_\alpha(t) + z_\alpha(t) + q_\alpha(t) + r_\alpha(t) + \bar{x}_\alpha * \bar{G}_\alpha(t),$$

where

$$z_\alpha(t) \equiv \bar{\Delta}_\alpha[u_\alpha](t), \quad q_\alpha(t) \equiv \bar{\Delta}_\alpha[v_\alpha](t), \quad r_\alpha(t) \equiv \bar{\Delta}_\alpha[\omega_\alpha](t).$$

Now we apply Theorem 1 to equation (52).

The inclusions $(\bar{G}_\alpha) \subset L(\nu, m) \cap M(l, 1)$ hold by condition (17), while the inclusion

$$(\bar{y}_\alpha) \subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\})$$

is proved in (48).

Now we obtain from (52) and (45) that

$$\begin{aligned} z_\alpha(t) &= \mu_\alpha^{-1} \int_0^t \exp(\beta s) \Delta_\alpha(t, ds) \int_0^{t-s} \exp(\beta u) y_\alpha(u) du \\ &= \mu_\alpha^{-1} \int_0^t \exp(\beta u) y_\alpha(u) \left[\int_0^{t-u} \exp(\beta s) \Delta_\alpha(t, ds) \right] du. \end{aligned}$$

Thus

$$(53) \quad \begin{aligned} |z_\alpha(t)| &\leq \exp(\beta t_\alpha) \int_0^t |y_\alpha(u)| |\Delta_\alpha|(t, [0, t-u]) du \\ &\leq \exp(\beta t_\alpha) \delta_\alpha(t) \int_0^{t_\alpha} |y_\alpha(u)| du \end{aligned}$$

for $0 \leq t \leq t_\alpha$.

Since $(y_\alpha) \subset L_0^1(\{v_\alpha\}, \{t_\alpha\})$, the integral on the right hand side of (53) is bounded by Definition 5. Hence the inclusion $(z_\alpha) \subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\})$ follows from inequality (53) and inclusions (18).

Further,

$$(54) \quad \begin{aligned} |q_\alpha(t)| &= \mu_\alpha^{-1} \left| \int_0^t e^{\beta s} \Delta_\alpha(t, ds) \int_0^{t-s} \bar{\Delta}_\alpha[\bar{x}_\alpha](u) du \right| \\ &= \mu_\alpha^{-1} \left| \int_0^t e^{\beta s} \Delta_\alpha(t, ds) \int_0^{t-s} \int_0^u \Delta_\alpha(u, dv) e^{\beta v} x_\alpha(u-v) e^{\beta(u-v)} du \right| \\ &\leq \mu_\alpha^{-1} \exp(\beta t_\alpha) \int_0^t |\Delta_\alpha|(t, ds) \int_0^{t-s} \delta_\alpha(u) du \sup_{\alpha, t \leq t_\alpha} |x_\alpha(t)| \\ &\leq \mu_\alpha^{-1} \exp(\beta t_\alpha) \delta_\alpha(t) \sup_{\alpha} \int_0^{t_\alpha} \delta_\alpha(s) ds \sup_{\alpha, t \leq t_\alpha} |x_\alpha(t)| \end{aligned}$$

for $0 \leq t \leq t_\alpha$.

The integral in the last line of (54) is finite, since $(\delta_\alpha) \subset L_0^1(\{v_\alpha\}, \{t_\alpha\})$. Thus inequality (54) and inclusions (18) imply that $(q_\alpha) \subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\})$.

Finally,

$$(55) \quad |r_\alpha(t)| = \mu_\alpha^{-1} \left| \int_0^t e^{\beta s} \Delta_\alpha(t, ds) \omega_\alpha(t-s) \right| \leq \mu_\alpha^{-1} \exp(\beta t_\alpha) \delta_\alpha(t) \sup_{\alpha, s \leq t_\alpha} \omega_\alpha(s),$$

whence (in a manner as above) we deduce that $(r_\alpha) \subset B_0^1(\{v_\alpha\}, \{t_\alpha\}) \cap L_0^1(\{v_\alpha\}, \{t_\alpha\})$. Thus all the assumptions of Theorem 1 hold for equation (52).

Therefore

$$(56) \quad x_\alpha(t) - \mu_\alpha^{-1} \int_0^t (\bar{y}_\alpha(s) + z_\alpha(s) + q_\alpha(s) + r_\alpha(s)) ds = O(\gamma_\alpha) + o(\Gamma_\alpha) \rightarrow 0$$

as $\alpha \rightarrow 0$ uniformly with respect to $v_\alpha \leq t \leq t_\alpha$. Condition (21) together with the first inequality in (54) implies that

$$\begin{aligned} \left| \int_0^t q_\alpha(t) dt \right| &\leq \int_0^{t_\alpha} |q_\alpha(t)| dt \leq C \int_0^{t_\alpha} \int_0^t |\Delta_\alpha|(t, ds) \int_0^{t-s} \delta_\alpha(u) du dt \\ &= C \int_0^{t_\alpha} \delta_\alpha(u) du \int_u^{t_\alpha} |\Delta_\alpha|(t, [0, t-u]) dt = O(\varepsilon_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0, \end{aligned}$$

for $v_\alpha \leq t \leq t_\alpha$.

Similarly, (55) and condition (20) imply that

$$\begin{aligned} \int_0^{t_\alpha} |r_\alpha(t)| dt &\leq \mu_\alpha^{-1} \int_0^{t_\alpha} \left| \int_0^{t-v_\alpha} e^{\beta s} \Delta_\alpha(t, ds) \omega_\alpha(t-s) \right| dt \\ &\quad + \mu_\alpha^{-1} \int_0^{t_\alpha} \left| \int_{t-v_\alpha}^t e^{\beta s} \Delta_\alpha(t, ds) \omega_\alpha(t-s) \right| dt \\ &\leq \mu_\alpha^{-1} \exp(\beta t_\alpha) \int_0^{t_\alpha} \delta_\alpha(t) dt \sup_{v_\alpha \leq s \leq t_\alpha} \omega_\alpha(s) \\ &\quad + \mu_\alpha^{-1} \exp(\beta t_\alpha) \int_0^{t_\alpha} |\Delta_\alpha|([t-v_\alpha, t]) dt \sup_{\alpha, s \leq t_\alpha} \omega_\alpha(s) \\ &= O(\gamma_\alpha) + o(\Gamma_\alpha) + O(\rho_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0. \end{aligned}$$

Taking into account (53) and (50) we derive (22) from (56). \square

To prove Remark 2 one can use relation (4) of Lemma 1 in [3] with the constant function $w(t)$ being equal to the left hand side of (24).

Proof of Corollary 1. The proof is reduced to the substitutions

$$\Delta_\alpha(t, B) \equiv 0 \quad \text{and} \quad \delta_\alpha(t) \equiv 0$$

in Theorem 3. \square

Proof of Theorem 4. We apply Theorem 3 to equation (25) and take into account (29) and (30).

Cramér's condition (13) follows from (28), (34), and (32):

$$\int_0^\infty \exp(\delta x) \bar{G}_\alpha(dx) = (1 - \theta_\alpha) \widehat{G}(\beta + \delta) > (1 - \theta_\alpha) \widehat{G}(\beta) = 1.$$

The inclusion $(\bar{G}_\alpha, \alpha > 0) \subset L(\nu, m)$ follows from Definition 1, since

$$\bar{G}_\alpha^{*m}(B) = \int_B \exp(\beta s) G_\alpha^{*m}(ds) \geq G_\alpha^{*m}(B) \geq \nu(B).$$

Further, the inclusion $(\overline{G}_\alpha, \alpha > 0) \subset M(l, 1)$ is a consequence of Cramér’s condition (32), since $\overline{G}_\alpha(ds) \leq \exp(\beta s)G(ds)$, where $\beta \rightarrow 0$.

We use Theorem 3 with the functions

$$(57) \quad v_\alpha \equiv 0.$$

Then we prove the inclusion

$$(58) \quad (\nu_\alpha(t)) \subset B_0^1(\{0\}, \{t_\alpha\}) \cap L_0^1(\{0\}, \{t_\alpha\}).$$

Indeed, the first condition in (3) follows from (36), while the second condition there holds, since $\nu_\alpha(t)$ is bounded, namely $|\nu_\alpha(t)| \leq 1$.

Further, the first condition in (4) follows from Remark 1 with functions v_α as in (57), while the second condition is equivalent to (37).

By definitions (27) and (30),

$$(59) \quad y_\alpha(t) = (\pi_\alpha(t) - \theta_\alpha)1_{t \leq b_\alpha} + \theta_\alpha 1_{t \leq b_\alpha} = \nu_\alpha(t)1_{t \leq b_\alpha} + \theta_\alpha 1_{t \leq b_\alpha}.$$

Hence the inclusion

$$(y_\alpha) \subset B_0^1(\{0\}, \{t_\alpha\}) \cap L_0^1(\{0\}, \{t_\alpha\})$$

follows from (57), (58), the equality $\theta_\alpha 1_{t \leq b_\alpha} = 0$ for $t > b_\alpha$, and from the convergence $\beta b_\alpha \rightarrow v < u < \infty$ as $\alpha \rightarrow 0$. Here $\tau_\alpha(t) \equiv \min(t_\alpha - b_\alpha, t)$. The inclusion

$$\tau = (\tau_\alpha(t)) \in \Upsilon_\infty(\{t_\alpha\})$$

holds, since

$$\tau_\alpha(t) = t_\alpha - b_\alpha \rightarrow \infty, \quad \alpha \rightarrow 0,$$

as $\tau_\alpha(t) < t$ by the strict inequality $v < u$.

The inclusion $(\delta_\alpha) \subset B_0^1(\{0\}, \{t_\alpha\}) \cap L_0^1(\{0\}, \{t_\alpha\})$ is an obvious corollary of (58) and (29). The corresponding functions in (23) are such that $\gamma_\alpha = O(\psi_\alpha)$ and $\Gamma_\alpha = O(\phi_\alpha)$ as $\alpha \rightarrow 0$ in view of (29), (37), and (36).

Condition (20) follows from (57), since $|\Delta_\alpha|(t, [t, t]) = 0$.

Similarly, condition (21) holds by (37) in view of (29):

$$\begin{aligned} \varepsilon_\alpha &= \int_0^{t_\alpha} |\nu_\alpha(s)| G(s) ds \int_s^{t_\alpha} |\nu_\alpha(t)| G(t-s) dt \\ &\leq \int_0^{t_\alpha} |\nu_\alpha(t)| \int_0^t |\nu_\alpha(s)| ds dt = \frac{1}{2} \left(\int_0^{t_\alpha} |\nu_\alpha(t)| dt \right)^2 \\ &= O(\phi_\alpha^2), \quad \alpha \rightarrow 0. \end{aligned}$$

Finally, $x_\alpha(t, b)$ is the probability of a random event, that is,

$$x_\alpha(t, b) \in [0, 1],$$

whence $0 \leq \overline{x}_\alpha(t) \leq \exp(\beta t) \leq \exp(\beta t_\alpha) \rightarrow \exp(u) < \infty$ and boundedness (19) follows.

Further $\mu_\alpha \rightarrow \mu = \mathbf{E} \xi_1$ by (47). Now we obtain from (22) that

$$(60) \quad e^{\beta t_\alpha} x_\alpha(t_\alpha, b_\alpha) - h_\alpha = O(\psi_\alpha + \phi_\alpha^2) + O(\phi_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0,$$

where, by (27), (29), (30), and (59),

$$\begin{aligned}
(61) \quad h_\alpha &\equiv \mu_\alpha^{-1} \int_0^{t_\alpha} \left[e^{\beta t} y_\alpha(t) + \mu_\alpha^{-1} \int_0^t e^{\beta s} y_\alpha(s) \int_0^{t-s} e^{\beta r} \Delta_\alpha(t, dr) ds \right] dt \\
&= \mu_\alpha^{-1} \int_0^{t_\alpha} \left[e^{\beta t} \pi_\alpha(t) 1_{t \leq b_\alpha} + \mu_\alpha^{-1} \int_0^t e^{\beta s} \pi_\alpha(s) 1_{s \leq b_\alpha} \right. \\
&\quad \left. \times \int_0^{t-s} e^{\beta r} (\theta_\alpha - \pi_\alpha(t)) G(dr) ds \right] dt \\
&= \mu_\alpha^{-1} (e^{\beta b_\alpha} - 1) \beta^{-1} \theta_\alpha + \mu_\alpha^{-1} \int_0^{t_\alpha} \nu_\alpha(t) e^{\beta t} 1_{t \leq b_\alpha} dt \\
&\quad - \mu_\alpha^{-2} \int_0^{t_\alpha} \left[\int_0^t e^{\beta s} \theta_\alpha 1_{s \leq b_\alpha} \int_0^{t-s} e^{\beta r} \nu_\alpha(t) G(dr) ds \right] dt \\
&\quad - \mu^{-2} \theta_\alpha \beta^{-1} \int_0^u \left[\int_0^t e^s 1_{s \leq v} \int_0^{(t-s)/\beta} e^{\beta r} \nu_\alpha(t/\beta) \beta^{-1} G(dr) ds \right] dt.
\end{aligned}$$

The first term on the right hand side is equivalent to $\mu^{-1}(\exp(v) - 1)\mu + O(\beta)$ by convergence (35). Since $\beta t_\alpha \rightarrow u > 0$, $\beta b_\alpha \rightarrow v > 0$, and since the integrand is bounded, we scale the variables by β^{-1} and prove that the right hand side of (61) is asymptotically equivalent to

$$\begin{aligned}
&e^v - 1 + \mu^{-1} \int_0^u \nu_\alpha(t/\beta) \beta^{-1} e^t 1_{t \leq v} dt \\
&\quad - \mu^{-2} \theta_\alpha \beta^{-1} \int_0^u \left[\int_0^t e^s 1_{s \leq v} \int_0^{(t-s)/\beta} e^{\beta r} \nu_\alpha(t/\beta) \beta^{-1} G(dr) ds \right] dt \\
&\quad + O(\beta) + O(\phi_\alpha^2)
\end{aligned}$$

in view of (37). Here we used the following upper bound for the last term on the right hand side of (61):

$$\mu_\alpha^{-2} \exp(\beta t_\alpha) \int_0^{t_\alpha} |\nu_\alpha(t)| \int_0^t |\nu_\alpha(s)| ds dt = O(\phi_\alpha^2), \quad \alpha \rightarrow 0.$$

Applying the dominated convergence theorem we conclude that

$$\int_0^{(t-s)/\beta} e^{\beta r} G(dr) - 1 = O(\beta), \quad \beta \rightarrow 0,$$

whence

$$\begin{aligned}
h_\alpha - e^v + 1 - \mu^{-1} \int_0^v \nu_\alpha(t/\beta) \beta^{-1} e^t dt + \mu^{-1} \int_0^u \nu_\alpha(t/\beta) \beta^{-1} \int_0^t e^s 1_{s \leq v} ds dt \\
= O(\beta) + O(\phi_\alpha^2)
\end{aligned}$$

as $\alpha \rightarrow 0$. This proves (38). □

Proof of Example 1. The expressions in (36) and (37) do not exceed

$$\psi_\alpha = \theta_\alpha \sup_{s \geq 0} |l(s)| = O(\theta_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0,$$

and, respectively,

$$\phi_\alpha = \theta_\alpha \int_0^{t_\alpha} |l(s)| ds = O(\theta_\alpha) \rightarrow 0, \quad \alpha \rightarrow 0. \quad \square$$

Proof of Example 2. The expressions in (36) and (37) do not exceed

$$\psi_\alpha = \varkappa_\alpha \beta \sup_{t \geq 0} |l(\beta t)| = o(\varkappa_\alpha), \quad \beta \rightarrow 0,$$

and, respectively,

$$\phi_\alpha = x_\alpha \beta \int_0^{t_\alpha} |l(\beta s)| ds = O(\varkappa_\alpha), \quad \beta \rightarrow 0. \quad \square$$

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