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STOCHASTIC PROCESSES IN THE SPACES $D_{V,W}$

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Abstract. We introduce the spaces of random variables $D_{V,W}$. We study the conditions for the convergence of series and the distribution of the supremum of stochastic processes in these spaces.

1. Introduction

We introduce the spaces of random variables $D_{V,W}$ and study their properties. We investigate the convergence of certain series in these spaces. We also study the distribution of the supremum of stochastic processes belonging to the spaces $D_{V,W}$.

Let $\{\Omega, \mathcal{B}, \mathsf{P}\}$ be a standard probability space, $L_0(\Omega)$ the space of random variables defined on $\{\Omega, \mathcal{B}, \mathsf{P}\}$, and $\mathcal{M} \subset L_0(\Omega)$ some linear space.

Definition 1.1 ([1])**.** A function

$$\Theta = (\Theta(\xi), \xi \in \mathcal{M})$$

is called a prenorm if, for all random variables $\xi \in \mathcal{M}$,

1. $\Theta(\xi) \in [0, \infty)$;
2. $\Theta(0) = 0$;
3. $\Theta(-\xi) = \Theta(\xi)$.

Definition 1.2 ([1])**.** We say that $\mathcal{M}$ is a pre-Banach space if $\mathcal{M}$ is complete with respect to a prenorm Θ.

Definition 1.3. A pre-Banach space $\mathcal{M}$ is called a pre-K_σ-space if it has the following three properties:

a1) $\max(\xi, \eta) \in \mathcal{M}$ and $\min(\xi, \eta) \in \mathcal{M}$ for all $\xi, \eta \in \mathcal{M}$. As a consequence, $|\xi| \in \mathcal{M}$ for all $\xi \in \mathcal{M}$.
a2) $|\xi| \in \mathcal{M}$ if $|\xi| \le |\eta|$ almost surely for some $\eta \in \mathcal{M}$.
a3) $\sup_{n\ge 1} |\xi_n| \in \mathcal{M}$ if $\{\xi_n, n \ge 1\}$ is a sequence of random variables belonging to $\mathcal{M}$ and such that $\sup_{n\ge 1} |\xi_n| \le \eta$ for some $\eta \in \mathcal{M}$.

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Definition 1.4 ([3])**.** Let a functional $\|\cdot\|$ assign a nonnegative number $\|\xi\|$ to each random variable $\xi \in \mathcal{M}$ such that

1) $\|\xi\| = 0 \Leftrightarrow \xi = 0$ with probability one;
2) $\|\xi + \eta\| \le \|\xi\| + \|\eta\|$;
3) $\|\lambda\xi\| \le \|\xi\|$ if $|\lambda| \le 1$.

Then $\|\cdot\|$ is called a quasinorm.

Definition 1.5. A space $\mathcal{M}$ is called quasi-Banach if $\mathcal{M}$ is complete with respect to a quasinorm $\|\cdot\|$.

Remark 1.1. Every quasinorm is a prenorm. Moreover, if one assumes that

$$\|\lambda\xi\| \le |\lambda| \cdot \|\xi\|$$

for all λ instead of condition 3) in Definition 1.4, then the quasinorm $\|\cdot\|$ is a norm.

Definition 1.6 ([2])**.** A positive nondecreasing sequence $\mu(n)$, $n \ge 1$, is called a characteristic of a pre-Banach K_σ-space $\mathcal{M}$ if

$$\Theta\left(\max_{1\le k\le n} |\xi_k|\right) \le \mu(n) \max_{1\le k\le n} \Theta(\xi_k)$$

for all $\xi_k \in \mathcal{M}$, $k = 1, 2, \dots, n$.

The definition of a characteristic is introduced in [7]–[8] for Orlicz spaces, in [4] for K_σ-spaces, and in [5] for quasi-Banach K_σ-spaces.

Definition 1.7 ([2])**.** Let $J = J(\lambda)$ be a nondecreasing function such that $J(\lambda) \ge 0$ and $J(\lambda) \to 0$ as $\lambda \to 0$. We say that a prenorm $\Theta(\cdot)$ in a space $\mathcal{M}$ is subordinate to a function J if $\Theta(\lambda\xi) \le J(|\lambda|)\Theta(\xi)$.

Examples of quasi-Banach spaces are considered in the paper [2]. In particular, Orlicz spaces, $D(\Omega)$-spaces, and $\Gamma_\psi(\Omega)$-spaces introduced in [6] are special cases of quasi-Banach spaces. Below we introduce and study the quasi-Banach K_σ-spaces $D_{V,W}(\Omega)$ that are a generalization of the $D(\Omega)$-spaces studied in [2].

2. Spaces $D_{V,W}$

Definition 2.1 ([1])**.** A continuous even convex function $U = (U(x), x \in \mathbf{R})$ is called a C-function if $U(0) = 0$ and $U(x)$ increases for $x > 0$.

Lemma 2.1 ([1])**.** *Every C-function U has the following properties:*

a) *$U(\alpha x) \le \alpha U(x)$ for all $x \in \mathbf{R}$ if $0 \le \alpha \le 1$;*
b) *$U(\alpha x) \ge \alpha U(x)$ for all $x \in \mathbf{R}$ if $\alpha > 1$;*
c) *$U(|x| + |y|) \ge U(x) + U(y)$ for all $x, y \in \mathbf{R}$;*
d) *there exists a constant $c > 0$ such that $U(x) \ge cx$ for $x > 1$;*
e) *$U(x)/x$ decreases for $x > 0$.*

Lemma 2.2 ([1])**.** *Let $U^{-1}(x)$, $x \ge 0$, be the inverse function to a C-function $U(x)$, $x \ge 0$. Then*

a) *$U^{-1}(x)$, $x \ge 0$, is an increasing concave positive function such that $U^{-1}(x) \to \infty$ as $x \to \infty$;*
b) *$U^{-1}(\alpha x) \le \alpha U^{-1}(x)$ for $x \ge 0$ if $\alpha > 1$;*
c) *$U^{-1}(\alpha x) \ge \alpha U^{-1}(x)$ for $x \ge 0$ if $0 \le \alpha \le 1$;*
d) *$U^{-1}(x + y) \le U^{-1}(x) + U^{-1}(y)$ for all $x, y \ge 0$;*
e) *there exists a constant $c > 0$ such that $U^{-1}(x) \le cx$ for $x > 1$;*
f) *$U^{-1}(x)/x$ is nonincreasing for $x > 0$.*

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Now we are ready to give the definition of the spaces $D_{V,W}(\Omega)$.

Definition 2.2. Let $W=\{W(x), x\in\mathbf{R}\}$ and $V=\{V(x), x\in\mathbf{R}\}$ be two even increasing continuous functions for $x>0$ such that $W(0)=0$, $W(x)>0$, and $V(x)>0$ for $x\neq 0$. Assume that there exists a constant $C>0$ such that

$$W^{(-1)}(x+y)\le C\left(W^{(-1)}(x)+W^{(-1)}(y)\right)$$

and there exists a continuous function $Z=\{Z(x), x>0\}$ such that $0<Z(x)<\infty$ for $|x|<\infty$ and $V(ax)\le Z(a)V(x)$ for all $a>0$ and $x>0$. We say that a random variable ξ belongs to the space $D_{V,W}(\Omega)$ if

$$\sup_{x\ge 0} V(x)W^{(-1)}(\mathsf{P}\{|\xi|>x\})<\infty. \tag{1}$$

The functions $W(x)=|x|^a$ or $W(x)=\exp\{|x|^a\}-1$ and $V(x)=|x|^b$ if $a>0$, $b>0$ may serve as examples of functions that fit the conditions of Definition 2.2.

Theorem 2.1. *Every space $D_{V,W}(\Omega)$ is a pre-K_σ-space with respect to the prenorm*

$$\|\xi\|_{V,W}=\left(\sup_{x>0} V(x)W^{-1}(\mathsf{P}\{|\xi|>x\})\right)^{1/2}.$$

If $\|\xi_n-\xi_m\|_{V,W}\to\infty$ as $n,m\to\infty$ and $\sup_n\|\xi_n\|_{V,W}<\infty$, then there exists a random variable $\xi\in D_{V,W}(\Omega)$ such that $\|\xi_n-\xi\|_{V,W}\to 0$, $n\to\infty$. Moreover, the prenorm $\|\cdot\|_{V,W}$ is subordinate to the function $J(\lambda)=(Z(\lambda))^{1/2}$.

Further let

(B1) *$W(x)$ be a C-Orlicz function and $V(x)$ the inverse function to a C-Orlicz function.*

Then the functional $\|\cdot\|$ is a quasinorm and the space $D_{V,W}(\Omega)$ is complete with respect to this quasinorm.

Finally,

$$\mathsf{P}\{|\xi|>x\}\le W\left(\frac{\|\xi\|^2_{V,W}}{V(x)}\right) \tag{2}$$

for all $x>0$.

Proof. It is clear that conditions a1)–a3) of Definition 1.3 hold for the space $D_{V,W}(\Omega)$; thus $D_{V,W}(\Omega)$ is a K_σ-space. Moreover $\|\xi\|_{V,W}=0$ if and only if $\xi=0$ with probability one.

Now we prove that the space $D_{V,W}(\Omega)$ is linear. To prove this property we need to show that if ξ and η belong to $D_{V,W}$, then $a\xi\in D_{V,W}(\Omega)$ and $\xi+\eta\in D_{V,W}(\Omega)$. Indeed,

$$\begin{aligned}\|a\xi\|_{V,W}&=\left(\sup_{x>0} V(x)W^{(-1)}(\mathsf{P}\{|a\xi|>x\})\right)^{1/2}\\&=\left(\sup_{x>0} V(x)W^{(-1)}\left(\mathsf{P}\left\{|\xi|>\frac{x}{|a|}\right\}\right)\right)^{1/2}\\&=\left(\sup_{x>0} V\left(x\frac{|a|}{|a|}\right)W^{(-1)}\left(\mathsf{P}\left\{|\xi|>\frac{x}{|a|}\right\}\right)\right)^{1/2}.\end{aligned}$$

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Putting $y := x/|a|$, we get

$$\begin{aligned}\|a\xi\|_{V,W} &= \left(\sup_{y>0} V(|a|y)W^{(-1)}(\mathsf{P}\{|\xi| > y\})\right)^{1/2}\\ &\le \left(Z(|a|)\cdot \sup_{y>0} V(y)W^{(-1)}(\mathsf{P}\{|\xi| > y\})\right)^{1/2} = Z(|a|)^{1/2}\|\xi\|_{V,W} < \infty.\end{aligned}$$

As a by-product of the above reasoning, we obtain that the prenorm $\|\cdot\|_{V,W}$ is subordinate to the function $J(\lambda) = (Z(\lambda))^{1/2}$.

Now we prove that the sum of two elements of the space $D_{V,W}$ also belongs to the same space:

$$\begin{aligned}\|\xi+\eta\|_{V,W} &= \left(\sup_{x>0} V(x)W^{(-1)}(\mathsf{P}\{|\xi+\eta| > x\})\right)^{1/2}\\ &\le \left(\sup_{x>0} V(x)W^{(-1)}(\mathsf{P}\{|\xi| > x/2\} + \mathsf{P}\{|\eta| > x/2\})\right)^{1/2}\\ &\le \left(\sup_{x>0} V(x)\cdot C\cdot \left(W^{(-1)}(\mathsf{P}\{|\xi| > x/2\}) + W^{(-1)}(\mathsf{P}\{|\eta| > x/2\})\right)\right)^{1/2}\\ &\le \Bigg(C\sup_{x>0} V(x)\cdot \left(W^{(-1)}(\mathsf{P}\{|\xi| > x/2\})\right)\\ &\qquad + \sup_{x>0} V(x)\cdot \left(W^{(-1)}(\mathsf{P}\{|\eta| > x/2\})\right)\Bigg)^{1/2}\\ &\le (CZ(2)(\|\xi\|_{V,W} + \|\eta\|_{V,W}))^{1/2} < \infty.\end{aligned} \tag{3}$$

Let $\xi_n \in D_{V,W}(\Omega)$ be a sequence such that $\|\xi_n - \xi_m\|_{V,W} \to 0$ as $n,m\to\infty$ and $\sup_n \|\xi_n\|_{V,W} < \infty$. Inequality (2) implies that

$$\mathsf{P}\{|\xi_n - \xi_m| > \varepsilon\} \le W\left(\frac{\|\xi_n - \xi_m\|^2_{V,W}}{V(\varepsilon)}\right) \to 0.$$

This means that there exists a random variable ξ such that $\xi_n \to \xi$ in probability, whence $\mathsf{P}\{|\xi_n| > x\} \to \mathsf{P}\{|\xi| > x\}$ at the points of continuity of the function $\mathsf{P}\{|\xi| > x\}$. Then

$$\mathsf{P}\{|\xi| > x\} \le \sup_{n\ge1} \mathsf{P}\{|\xi_n| > x\} \le \sup_{n\ge1} W\left(\frac{\|\xi_n\|^2_{V,W}}{V(x)}\right) \le W\left(\frac{\sup_{n\ge1}\|\xi_n\|^2_{V,W}}{V(x)}\right)$$

at the points of continuity of the function $\mathsf{P}\{|\xi| > x\}$.

Since the function $W(a/V(x))$ is continuous with respect to x (where $a > 0$ is a certain constant) and $\mathsf{P}\{|\xi| > x\}$ is nonincreasing, the latter inequality holds for all $x > 0$. Hence $\xi \in D_{V,W}$ and $\|\xi\|_{V,W} \le \sup_{n\ge1}\|\xi_n\|_{V,W}$.

Now we consider $\xi_n - \xi_m$ instead of the sequence ξ_n. As above,

$$\|\xi - \xi_m\|_{V,W} \le \sup_{n\ge m} \|\xi_n - \xi_m\|_{V,W},$$

that is, $\|\xi_m - \xi\|_{V,W} \to 0$ as $m\to\infty$.

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Let $0<\alpha<1$ and let condition (B1) hold. Then

$$\begin{aligned}
\|\xi+\eta\|_{V,W}^2 &= \sup_{x>0} V(x)W^{-1}(\mathsf{P}\{|\xi+\eta|>x\}) \\
&\le \sup_{x>0} \big(V(x)W^{-1}(\mathsf{P}\{|\xi|+|\eta|>x\})\big) \\
&\le \sup_{x>0} \big(V(x)W^{-1}(\mathsf{P}\{|\xi|>\alpha x\}+\mathsf{P}\{|\eta|>(1-\alpha)x\})\big) \\
&\le \sup_{x>0} V\left(\frac{\alpha x}{\alpha}\right) W^{-1}(\mathsf{P}\{|\xi|>\alpha x\}) \\
&\qquad + \sup_{x>0} V\left(\frac{(1-\alpha)x}{(1-\alpha)}\right) W^{-1}(\mathsf{P}\{|\eta|>(1-\alpha)x\}) \\
&\le \frac{1}{\alpha}\|\xi\|_{V,W}^2 + \frac{1}{1-\alpha}\|\eta\|_{V,W}^2.
\end{aligned}$$

Putting

$$\alpha = \frac{\|\xi_{V,W}\|}{(\|\xi\|_{V,W}+\|\eta\|_{V,W})},$$

we derive the triangle inequality from the latter result.

Now we show that condition (B1) implies that the space $D_{V,W}(\Omega)$ is complete. Since $\big|\|\xi_n\|_{V,W}-\|\xi_m\|_{V,W}\big| \le \|\xi_n-\xi_m\|_{V,W}\to 0$ as $n,m\to\infty$, the limit

$$\sigma = \lim_{n\to\infty}\|\xi_n\|_{V,W}$$

exists.

Passing to the limit in the inequality

$$\mathsf{P}\{|\xi_n|>x\} \le W\left(\frac{\|\xi_n\|_{V,W}^2}{V(x)}\right)$$

as $n\to\infty$ (one can restrict the consideration for the points of continuity of the distribution function of the random variable ξ), we get

$$\mathsf{P}\{|\xi|>x\} \le W\left(\frac{\sigma^2}{V(x)}\right).$$

Thus the random variable ξ belongs to the space $D_{V,W}(\Omega)$ and $\|\xi\|_{V,W}\le\sigma$.

Now we consider $\xi_m-\xi_n$, $m>n$, instead of the sequence ξ_n. As in the preceding case, we get

$$\|\xi-\xi_n\|_{V,W} \le \lim_{m\to\infty}\|\xi_m-\xi_n\|_{V,W}.$$

Therefore $\|\xi-\xi_n\|_{V,W}\to 0$ as $n\to\infty$ and thus the space $D_{V,W}$ is complete. □

In what follows we write $\|\cdot\|$ instead of $\|\cdot\|_{V,W}$.

Our current goal is to determine the majorizing characteristic of the space $D_{V,W}(\Omega)$.

Theorem 2.2. *The sequence*

$$\mu(n) = \sup_{0<t<1/n}\left(\frac{W^{(-1)}(tn)}{W^{(-1)}(t)}\right)^{1/2}$$

is a majorizing characteristics of the space $D_{V,W}(\Omega)$.

Proof. Let ξ_i, $i=1,\dots,n$, be random variables of the space $D_{V,W}(\Omega)$ and let

$$a = \max_{1\le i\le n}\|\xi_i\|_{V,W}.$$

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Then

$$\begin{aligned}\left\|\max_{1\le i\le n}|\xi_i|\right\|_{V,W}^2 &= \sup_{x>0}\left(V(x)W^{(-1)}\left(\mathsf{P}\left\{\max_{1\le i\le n}|\xi_i|>x\right\}\right)\right)\\ &\le \sup_{x>0}\left(V(x)W^{(-1)}\left(\min\left\{1,\sum_{i=1}^n \mathsf{P}\{|\xi_i|>x\}\right\}\right)\right)\\ &\le \sup_{x>0}\left(V(x)W^{(-1)}\left(\min\left\{1,\sum_{i=1}^n W\left(\frac{\|\xi_i\|_{V,W}^2}{V(x)}\right)\right\}\right)\right)\\ &\le \sup_{x>0}\left(V(x)W^{(-1)}\left(\min\left\{1,nW\left(\frac{a^2}{V(x)}\right)\right\}\right)\right).\end{aligned} \tag{4}$$

Put $t=W(a^2/V(x))$. Thus $V(x)=a^2/W^{(-1)}(t)$. Hence (4) implies that

$$\left\|\max_{1\le i\le n}|\xi_i|\right\|_{V,W}^2 \le \sup_{t>0}\left(\frac{a^2}{W^{(-1)}(t)}W^{(-1)}(\min\{1,nt\})\right) = a^2\sup_{0<t\le\frac{1}{n}}\left(\frac{W^{(-1)}(nt)}{W^{(-1)}(t)}\right). \quad \square$$

3. The convergence of series in the spaces $D_{V,W}$

Theorem 3.1. *Let ξ_k be random variables belonging to the space $D_{V,W}(\Omega)$. Let $\|\cdot\|$ be a prenorm and $\|\xi_k\|>0$. Put $f(x)=xV(W(x))$, $x>0$, and let $f^{(-1)}(x)$ be the inverse function to $f(x)$. The series*

$$\sum_{k=1}^\infty \xi_k \tag{5}$$

converges in probability if the series

$$\sum_{k=1}^\infty \alpha_k^* \tag{6}$$

converges, where $\alpha_k^*=V^{(-1)}\left(\|\xi_k\|^2/f^{(-1)}(\|\xi_k\|^2)\right)$. Moreover,

$$\mathsf{P}\left\{\left|\sum_{k=1}^\infty \xi_k\right|\ge x\right\}\le\sum_{k=1}^\infty W\left(\frac{\|\xi_k\|^2}{V(\alpha_k^* x/\mu)}\right) \tag{7}$$

for

$$x\ge\mu=\sum_{k=1}^\infty V^{(-1)}\left(\frac{\|\xi_k\|^2}{f^{(-1)}(\|\xi_k\|^2)}\right)$$

and the series on the right hand side of (7) converges for $x\ge\mu$.

Remark 3.1. The function $x/f^{(-1)}(x)$ increases, since so does the function

$$f(x)/x=V(W(x)).$$

Proof. First we prove that the convergence of series (6) implies the convergence of the series on the right hand side of (7). It is clear that if the series

$$\sum_{k=1}^\infty W\left(\frac{\|\xi_k\|^2}{V(\alpha_k^*)}\right) \tag{8}$$

converges, then the series (7) converges, too, since the functions W and V are increasing.

In what follows we need the following equality:

$$W\left(\frac{\|\xi_k\|^2}{V(\alpha_k^*)}\right)=V^{(-1)}\left(\frac{\|\xi_k\|^2}{f^{(-1)}(\|\xi_k\|^2)}\right).$$

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Below is the proof:

$$\begin{aligned}\frac{\|\xi_k\|^2}{f^{(-1)}(\|\xi_k\|^2)} &= \frac{f^{(-1)}(\|\xi_k\|^2)V\left(W\left(f^{(-1)}\left(\|\xi_k\|^2\right)\right)\right)}{f^{(-1)}(\|\xi_k\|^2)}\\ &= V\left(W\left(\frac{\|\xi_k\|^2 f^{(-1)}(\|\xi_k\|^2)}{\|\xi_k\|^2}\right)\right)\\ &= V\left(W\left(\frac{\|\xi_k\|^2}{V\left(V^{(-1)}\left(\|\xi_k\|^2/f^{(-1)}\left(\|\xi_k\|^2\right)\right)\right)}\right)\right)\\ &= V\left(W\left(\frac{\|\xi_k\|^2}{V(\alpha_k^*)}\right)\right).\end{aligned}$$

The latter equality implies the result desired.

Now we estimate the probability $\mathsf{P}\{|\sum_{k=l}^m \xi_k| > x\}$. Fix $x > 0$. Then

$$\mathsf{P}\left\{\left|\sum_{k=l}^m \xi_k\right| > x\right\} \le \sum_{k=l}^m \mathsf{P}\left\{|\xi_k| > \alpha_k x\right\},$$

where $\sum_{k=l}^m \alpha_k = 1$, $\alpha_k > 0$. According to Theorem 2.1,

$$\mathsf{P}\{|\xi| > x\} \le W\left(\frac{\|\xi\|^2}{V(x)}\right)$$

for all $\xi \in D_{V,W}(\Omega)$. Therefore

$$\sum_{k=l}^m \mathsf{P}\{|\xi_k| > \alpha_k x\} \le \sum_{k=l}^m W\left(\frac{\|\xi_k\|^2}{V(\alpha_k x)}\right).$$

Put $\alpha_k = \alpha_k^*/\mu_{lm}$, where $\mu_{lm} = \sum_{k=l}^m V^{(-1)}\left(\|\xi_k\|^2/f^{(-1)}(\|\xi_k\|^2)\right)$. Then

$$\sum_{k=l}^m W\left(\frac{\|\xi_k\|^2}{V(\alpha_k)}\right) = \sum_{k=l}^m W\left(\frac{\|\xi_k\|^2}{V(\alpha_k^* x/\mu_{lm})}\right).$$

Since the series (8) converges, $\mu_{lm} < x$ for sufficiently large l and m. Thus

$$\mathsf{P}\left\{\left|\sum_{k=l}^m \xi_k\right| > x\right\} \le \sum_{k=l}^m W\left(\frac{\|\xi_k\|^2}{V(\alpha_k^* x/\mu_{lm})}\right). \tag{9}$$

The convergence of the series (8) implies that $\mathsf{P}\{|\sum_{k=l}^m \xi_k| > x\} \to 0$ as $l, m \to \infty$. This means that the series (5) converges in probability.

Finally, inequality (7) follows from (9) by putting $l := 1$ and passing to the limit as $m \to \infty$. □

Note that one cannot improve the above conditions and inequalities in general. This follows from the next result.

Theorem 3.2. *Let $W(x) = |x|^a$ and $V(x) = |x|^b$, $a > 0$, $b > 0$. Then series* (5) *converges in probability if the series*

$$\mu = \sum_{k=1}^\infty \|\xi_k\|^{2a/(ab+1)}$$

converges. Moreover

$$\mathsf{P}\left\{\left|\sum_{k=1}^\infty \xi_k\right| > x\right\} \le \frac{1}{x^{ab}}\left(\sum_{k=1}^\infty \|\xi_k\|^{2a/(ab+1)}\right)^{ab+1}$$

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for $x \geq \mu$, that is, $\sum_{k=1}^{\infty} \xi_k$ belongs to the space $D_{V,W}$ and

$$\left\| \sum_{k=1}^{\infty} \xi_k \right\| \leq \left(\sum_{k=1}^{\infty} \|\xi_k\|^{2a/(ab+1)} \right)^{(ab+1)/(2a)}.$$

Proof. Note that

$$\mathsf{P}\left\{ \left| \sum_{k=1}^{\infty} \xi_k \right| > x \right\} \leq \sum_{k=1}^{\infty} \mathsf{P}\{|\xi_k| > \alpha_k x\},$$

where $\sum_{k=1}^{\infty} \alpha_k = 1$. Since

$$\mathsf{P}\{|\xi| > x\} \leq W\left(\|\xi\|^2 / V(x)\right)$$

by Theorem 2.1 for all $\xi \in D_{V,W}(\Omega)$, we have

$$\sum_{k=1}^{\infty} \mathsf{P}\{|\xi_k| > \alpha_k x\} \leq \sum_{k=1}^{\infty} W\left(\frac{\|\xi_k\|^2}{V(\alpha_k x)} \right).$$

Using the explicit form of the functions W and V, we get

$$\sum_{k=1}^{\infty} W\left(\frac{\|\xi_k\|^2}{V(\alpha_k x)} \right) = \frac{1}{x^{ab}} \sum_{k=1}^{\infty} \frac{\|\xi_k\|^{2a}}{\alpha_k^{ab}}.$$

The minimum of the latter expression is attained at the following values of members of the sequence $\{\alpha_k\}$:

$$\alpha_k = \frac{1}{d} \|\xi_k\|^{2a/(ab+1)}, \qquad d = \sum_{i=1}^{\infty} \|\xi_i\|^{2a/(ab+1)}.$$

For these α_k,

$$\begin{aligned} \frac{1}{x^{ab}} \sum_{k=1}^{\infty} \frac{\|\xi_k\|^{2a}}{\alpha_k^{ab}} &= \frac{1}{x^{ab}} \sum_{k=1}^{\infty} \frac{\|\xi_k\|^{2a}}{\frac{\|\xi_k\|^{2a^2b/(ab+1)}}{\left(\sum_{i=1}^{\infty} \|\xi_i\|^{2a/(ab+1)}\right)^{ab}}} \\ &= \frac{1}{x^{ab}} \left(\sum_{k=1}^{\infty} \|\xi_k\|^{2a/(ab+1)} \right)^{ab} \sum_{i=1}^{\infty} \|\xi_i\|^{2a/(ab+1)} \\ &= \frac{1}{x^{ab}} \left(\sum_{k=1}^{\infty} \|\xi_k\|^{2a/(ab+1)} \right)^{ab+1}. \end{aligned}$$

The latter series converges under the assumptions of the theorem.

Now we use the precise form of the functions U and V in (6):

$$\sum_{k=1}^{\infty} V^{(-1)} \left(\frac{\|\xi_k\|^2}{f^{(-1)}(\|\xi_k\|^2)} \right) = \sum_{k=1}^{\infty} \frac{\|\xi_k\|^{2/b}}{\|\xi_k\|^{2/(b(ab+1))}} = \sum_{k=1}^{\infty} \|\xi_k\|^{2a/(ab+1)} < \infty. \qquad \square$$

We see that the results of Theorems 3.1 and 3.2 coincide, that is, one cannot improve the assumptions of Theorem 3.1.

If a prenorm $\|\cdot\|$ is a quasinorm, Theorem 21.1 implies the following result.

Theorem 3.3. *Let the random variables ξ_k belong to the space $D_{V,W}$, W be a C-function, and let V be the inverse function to some C-function. Assume that the series*

$$\sum_{k=1}^{\infty} \|\xi_k\|$$

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converges. Then the series

$$\sum_{k=1}^{\infty} \xi_k$$

converges in probability and its sum belongs to the space $D_{V,W}$. Moreover

$$\mathsf{P}\left\{\left|\sum_{k=1}^{\infty} \xi_k\right| \geq x\right\} \leq W\left(\frac{\left(\sum_{k=1}^{\infty} \xi_k\right)^2}{V(x)}\right).$$

Remark 3.2. Note that W is a C-function if $W(x) = |x|^a$, $a \geq 1$, and that $V(x) = |x|^b$, $b \in (0,1]$, is the inverse to a C-function. Then $\frac{2a}{ab+1} \geq 1$. In particular, the assumption of Theorem 3.3 may hold even though the assumption of Theorem 3.2 fails.

Definition 3.1. We say that a stochastic process $X(t) = \{X(t), t \in T\}$ belongs to the space $D_{V,W}$ if $X(t) \in D_{V,W}$ for all t.

Stochastic processes represented in the form of the series

$$\xi(t) = \sum_{k=1}^{\infty} \xi_k \phi_k(t), \qquad t \in T, \tag{10}$$

serve as an example of processes belonging to the space $D_{V,W}$ if $\xi_k \in D_{V,W}$ and if series (10) converges in the space $D_{V,W}$.

Theorem 3.4. *Let $W(x) = |x|^a$ and $V(x) = |x|^b$, $a > 0$, $b > 0$. Then series* (10) *converges in probability if the series*

$$\mu = \sum_{k=1}^{\infty} \varphi_k^{ab/(ab+1)}(t) \|\xi_k\|^{2a/(ab+1)}$$

converges. Moreover,

$$\mathsf{P}\left\{\left|\sum_{k=1}^{\infty} \phi_k(t)\xi_k\right| \geq x\right\} \leq \frac{1}{x^{ab}}\left(\sum_{k=1}^{\infty} \phi_k^{ab/(ab+1)}(t)\|\xi_k\|^{2a/(ab+1)}\right)^{ab+1}$$

for $x \geq \mu$, that is, $\phi(t) \in D_{V,W}$.

Proof. The result follows from Theorems 3.2 and 2.1. □

4. The boundedness of the supremum of stochastic processes in the space $D_{V,W}$

Let $X = \{X(t), t \in T\}$ be a stochastic process belonging to the space $D_{V,W}$ and let $\rho_X(t,s) = \|X(s) - X(t)\|$ be the premetric generated by the process X (a function $\rho(t,s)$, $t, s \in T$, is called a premetric if $\rho(t,s) \in [0,\infty)$, $\rho(t,t) = 0$, and $\rho(t,s) = \rho(s,t)$; see [1]).

Assume that

(A1)

$$\sup_{t \in T} \|X(t)\| < \infty;$$

(A2) the space (T, ρ_X) is separable and the process X is separable in (T, ρ_X).

Put $\varepsilon_0 = \sup_{t,s \in T} \rho_X(t,s)$. Condition (A1) implies that $\varepsilon_0 < \infty$. Let $\varepsilon_k = \varepsilon_0 \theta^k$ for some $\theta \in (0,1)$. We also let $N(\varepsilon)$ be the metric capacity of the space (T, ρ), that is, $N(\varepsilon)$ is the minimal number of closed balls covering (T, ρ).

The following theorem contains conditions for $\sup_{t \in T} X(t) < \infty$ almost surely. It also provides some bounds for this supremum.

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Theorem 4.1. *Let a process X satisfy conditions (A1) and (A2). If the series*

$$\sum_{n=1}^{\infty} V^{(-1)}\left(\frac{\mu(N(\varepsilon_n))^2\varepsilon_{n-1}^2}{f^{(-1)}\left(\mu(N(\varepsilon_n))^2\varepsilon_{n-1}^2\right)}\right) \tag{11}$$

converges, where the function f is the same as in Theorem 3.1, *then*

$$\mathsf{P}\left\{\sup_{t\in T}|X(t)|\ge x\right\}\le W\left(\frac{\inf_{t\in T}\|X(t)\|^2}{V(\psi_0 x)}\right)+\sum_{k=1}^{\infty}W\left(\frac{\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2}{V(\psi_k x)}\right) \tag{12}$$

for $x>\Psi$, where

$$\psi_0=\frac{1}{\Psi}V^{(-1)}\left(\frac{\inf_{t\in T}\|X(t)\|^2}{f^{(-1)}(\inf_{t\in T}\|X(t)\|^2)}\right),\qquad \psi_k=\frac{1}{\Psi}V^{(-1)}\left(\frac{\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2}{f^{(-1)}\left(\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2\right)}\right),$$

$$\Psi=V^{(-1)}\left(\frac{\inf_{t\in T}\|X(t)\|^2}{f^{(-1)}\left(\inf_{t\in T}\|X(t)\|^2\right)}\right)+\sum_{k=0}^{\infty}V^{(-1)}\left(\frac{\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2}{f^{(-1)}\left(\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2\right)}\right).$$

Proof. Denote by S_n the minimal ε_n-net of the set T with respect to the pseudometric ρ_x and let $S=\bigcup_{n=0}^{\infty}S_n$. The set S_0 consists of a single point denoted by t_0. Further, the set S is countable and everywhere dense in T with respect to the pseudometric ρ_X. Since the process X is separable and continuous in probability, every countable and everywhere dense set in (T,ρ_X) is a set of separability of the process X. Then

$$\sup_{t\in T}|X(t)|=\sup_{t\in S}|X(t)|$$

almost surely.

A family of mappings $\alpha_k(t)$, $k=0,1,\dots$, is called an α-procedure if every point of S corresponds to a unique point α_k of S_k such that $\rho(t,\alpha_k(t))\le\varepsilon_k$.

Applying the α-procedure we choose points of the set S_n and obtain a collection of points $t=t_m$ such that $t_{m-1}=\alpha_{m-1}(t_m),\dots,t_1=\alpha_1(t_2),t_0=\alpha_0(t_1)$, and $t_n\in S_n$, $n=0,1,\dots,m$, and $S_0=t_0$. Since

$$X(t)=X(t_0)+\sum_{n=1}^{m}(X(t_n)-X(t_{n-1})),$$

we obtain

$$\sup_{t\in T}|X(t)|\le|X(t_0)|+\sum_{k=1}^{\infty}\max_{s\in S_k}|X(s)-X(\alpha_{k-1}(s))|, \tag{13}$$

whence

$$\mathsf{P}\left\{\sup_{t\in T}|X(t)|\ge x\right\}\le\mathsf{P}\{|X(t_0)|\ge\psi_0 x\}+\sum_{k=1}^{\infty}\mathsf{P}\left\{\max_{s\in S_k}|X(s)-X(\alpha_{k-1}(s))|\ge\psi_k x\right\},$$

where the ψ_k are the numbers such that $\psi_k>0$, $\sum_{k=0}^{\infty}\psi_k=1$. Using the inequality of Theorem 2.1, we get

$$\begin{aligned}&\mathsf{P}\{|X(t_0)|\ge\psi_0 x\}+\sum_{k=1}^{\infty}\mathsf{P}\left\{\max_{s\in S_k}|X(s)-X(\alpha_{k-1}(s))|\ge\psi_k x\right\}\\&\quad\le W\left(\frac{\|X(t_0)\|^2}{V(\psi_0 x)}\right)+\sum_{k=1}^{\infty}W\left(\frac{\left\|\max_{s\in S_k}|X(s)-X(\alpha_{k-1}(s))|\right\|^2}{V(\psi_k x)}\right).\end{aligned} \tag{14}$$

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Now we estimate $\left\|\max_{s\in S_k}|X(s)-X(\alpha_{k-1}(s))|\right\|$. First,

$$\left\|\max_{s\in S_k}|X(s)-X(\alpha_{k-1}(s))|\right\| \le \mu(N(\varepsilon_k))\max_{s\in S_k}\|X(s)-X(\alpha_{k-1}(s))\|.$$

Since $\rho_X(s,\alpha_{k-1}(s))=\|X(s)-X(\alpha_{k-1}(s))\|\le\varepsilon_{k-1}$, we get

$$\mu(N(\varepsilon_k))\max_{s\in S_k}\|X(s)-X(\alpha_{k-1}(s))\|\le\mu(N(\varepsilon_k))\varepsilon_{k-1}.$$

Substituting this bound into inequality (14), we obtain

$$\mathsf{P}\left\{\sup_{t\in T}|X(t)|\ge x\right\}\le W\left(\frac{\|X(t_0)\|^2}{V(\psi_0 x)}\right)+\sum_{k=1}^{\infty}W\left(\frac{\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2}{V(\psi_k x)}\right).$$

Since t_0 is arbitrary,

$$\mathsf{P}\left\{\sup_{t\in T}|X(t)|\ge x\right\}\le W\left(\frac{\inf_{t\in T}\|X(t)\|^2}{V(\psi_0 x)}\right)+\sum_{k=1}^{\infty}W\left(\frac{\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2}{V(\psi_k x)}\right). \tag{15}$$

This series converges in view of Theorem 3.1 if ψ_0 and ψ_k are chosen as follows:

$$\psi_0=\frac{1}{\Psi}V^{(-1)}\left(\frac{\inf_{t\in T}\|X(t)\|^2}{f^{(-1)}(\inf_{t\in T}\|X(t)\|^2)}\right),$$
$$\psi_k=\frac{1}{\Psi}V^{(-1)}\left(\frac{\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2}{f^{(-1)}(\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2)}\right),$$

where

$$\Psi=V^{(-1)}\left(\frac{\inf_{t\in T}\|X(t)\|^2}{f^{(-1)}(\inf_{t\in T}\|X(t)\|^2)}\right)+\sum_{k=0}^{\infty}V^{(-1)}\left(\frac{\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2}{f^{(-1)}(\mu(N(\varepsilon_k))^2\varepsilon_{k-1}^2)}\right)$$

and where $f^{(-1)}$ is defined in Theorem 3.1.

Since the majorizing series in (15) converges and $\mathsf{P}\{\sup_{t\in T}|X(t)|\ge x\}\to 0$ as $x\to\infty$, the supremum $\sup_{t\in T}|X(t)|$ is bounded almost surely. □

Theorem 4.2. *Let a process $X=\{X(t),t\in T\}$ be such that $X\in D_{V,W}$ for $W=|x|^a$ and $V=|x|^b$, $a>0$ and $b>0$. Assume that X satisfies conditions (A1) and (A2).*

Assume that

$$\int_0^{\Delta_0 p}\left(N\left(u^{(ab+1)/(2a)}\right)\right)^{1/(ab+1)}du<\infty, \tag{16}$$

where $p=\theta^{2a/(ab+1)}$ is some number, $0\le\theta\le 1$, $\Delta_0=\varepsilon_0^{2a/(ab+1)}$, and

$$\varepsilon_0=\sup_{t,s\in T}\rho_X(t,s).$$

Then

$$\sup_{t\in T}|X(t)|\in D_{V,W}.$$

Moreover,

$$\mathsf{P}\left\{\sup_{t\in T}|X(t)|\ge x\right\}$$
$$\le\frac{1}{x^{ab}}\left(\inf_{t\in T}\|X(t)\|^{2a/(ab+1)}+\frac{1}{p(1-p)}\int_0^{\Delta_0 p}\left(N\left(u^{(ab+1)/(2a)}\right)\right)^{1/(ab+1)}du\right).$$

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Proof. For W and V as defined in the statement of the theorem, assumption (11) of Theorem 4.1 becomes of the following form:

$$\begin{aligned}&\sum_{k=1}^{\infty}\mu(N(\varepsilon_k))^{2a/(ab+1)}\varepsilon_{k-1}^{2a/(ab+1)}\\&\qquad=\varepsilon_0^{2a/(ab+1)}\sum_{k=1}^{\infty}\mu\left(N\left(\theta^k\right)\right)^{2a/(ab+1)}\left(\theta^{k-1}\right)^{2a/(ab+1)}<\infty,\end{aligned}$$

since $\varepsilon_k=\theta^k\varepsilon_0$. Putting $\Delta_0:=\varepsilon_0^{2a/(ab+1)}$ and $p:=\theta^{2a/(ab+1)}$, we get

$$\begin{aligned}&\varepsilon_0^{2a/(ab+1)}\sum_{k=1}^{\infty}\mu\left(N\left(\theta^k\right)\right)^{2a/(ab+1)}\left(\theta^{k-1}\right)^{2a/(ab+1)}\\&\qquad=\sum_{k=1}^{\infty}\left(\mu\left(N(\Delta_0 p)^{(ab+1)/(2a)}\right)\right)^{2a/(ab+1)}\Delta_0 p^{k-1}.\end{aligned}$$

Integrating both sides of the latter equality we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\mu\left(N\left(\left(\Delta_0 p^k\right)^{(ab+1)/(2a)}\right)\right)\Delta_0 p^{k-1}\\&\qquad\le\sum_{k=1}^{\infty}\int_{\Delta_0 p^{k+1}}^{\Delta_0 p^k}\mu\left(N\left(u^{(ab+1)/(2a)}\right)\right)du\frac{1}{p(1-p)}\\&\qquad=\frac{1}{p(1-p)}\int_0^{\Delta_0 p}\left(\mu\left(N\left(u^{(ab+1)/(2a)}\right)\right)\right)^{2a/(ab+1)}du.\end{aligned}$$

If the latter integral is finite, $\sup_{t\in T}|X(t)|\in D_{V,W}$.

Now

$$\mu(n)=\sup_{0<t<1/n}\left(\frac{W^{(-1)}(tn)}{W^{(-1)}(t)}\right)^{1/2}=n^{1/(2a)}$$

and Theorem 4.2 is proved. □

Theorem 4.3. *Let a process $X=\{X(t),t\in[0,T]\}$ be such that $X\in D_{V,W}$ and let $W(x)=|x|^a$ and $V(x)=|x|^b$ for $a>0$ and $b>0$. Assume that condition (A1) holds for the process X and moreover let X be separable on $[0,T]$. Let*

$$\sup_{|t-s|\le h}\|X(t)-X(s)\|\le Dh^c=\delta(h),$$

$D>0$, and let $c>(2a)^{-1}$. Then $\sup_{t\in[0,T]}|X(t)|\in D_{V,W}$. Moreover,

$$\begin{aligned}&\mathsf{P}\left\{\sup_{t\in[0,T]}|X(t)|>x\right\}\\&\quad\le\frac{1}{x^{ab}}\left(\inf_{t\in[0,T]}\|X(t)\|^{2a/(ab+1)}+\frac{1}{p(1-p)}\int_0^{\Delta_0 p}\left(\frac{TD^{1/c}}{2u^{(ab+1)/(2a)}}+1\right)^{1/(ab+1)}du\right)\end{aligned}$$

for all $x>0$, where p is defined in Theorem 4.2.

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Proof. By Theorem 16, $\sup_{t\in T}|X(t)| \in D_{V,W}$ if integral (16) is finite. We derive from the assumptions of the theorem that

$$N(\varepsilon) \leq \frac{T}{2\delta^{(-1)}(\varepsilon)} + 1.$$

Then the latter integral can be rewritten as follows:

$$\int_0^{\Delta_0 p} \left(\frac{T}{2\delta^{(-1)}\left(u^{(ab+1)/(2a)}\right)} + 1 \right)^{1/(ab+1)} du < \infty.$$

Using the precise value of $\delta(h)$ we obtain

$$\int_0^{\Delta_0 p} \left(\frac{TD^{1/c}}{2u^{(ab+1)/(2ac)}} + 1 \right)^{1/(ab+1)} du < \infty.$$

This integral is finite if

$$\int_0^{\Delta_0 p} \frac{1}{u^{1/(2ac)}}\, du$$

converges. This is the case if $c > (2a)^{-1}$. $\square$

5. Concluding remarks

We introduced the spaces of random variables $D_{V,W}$ in Section 2. The definition of the space $D_{V,W}$ is given in terms of prenorms. Some properties of these spaces are studied. We considered the convergence of infinite series of random variables belonging to these spaces. Sufficient conditions for the convergence of such series are given in Section 3. The bounds for the distribution of the supremum of stochastic processes belonging to the spaces $D_{V,W}$ are obtained in Section 4.

We plan to apply the results of Sections 3 and 4 for constructing the models of stochastic processes in the spaces $D_{V,W}$.

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