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LIPSCHITZ CONDITIONS FOR $\mathrm{Sub}_\varphi(\Omega)$-PROCESSES AND APPLICATIONS TO WEAKLY SELF-SIMILAR PROCESSES WITH STATIONARY INCREMENTS

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Abstract. We study the Lipschitz continuity of generalized sub-Gaussian processes and provide estimates for the distribution of the norms of such processes. The results are applied to the case of weakly self-similar generalized sub-Gaussian processes with stationary increments (the fractional Brownian motion is a particular case of these processes).

1. Introduction

Let (T,ρ) be some pseudometric space. We consider the Lipschitz continuity of stochastic processes $X=(X(t),t\in T)$ and provide some bounds for the distribution of norms of such processes. In particular, we find the modulus of continuity of a process X, that is, we find a function f such that

$$\limsup_{\varepsilon\to 0}\frac{\sup_{\rho(t,s)<\varepsilon}|X(t)-X(s)|}{f(\varepsilon)}<1.$$

We also provide some bounds for the tail probabilities

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{f(\rho(t,s))}>y\right\}.$$

The case where (T,ρ) is a subset of a d-dimensional Euclidean space is considered as a particular example.

The results obtained in the papers are applied to weakly self-similar processes with stationary increments belonging to the space $\mathrm{Sub}_\varphi(\Omega)$ of generalized sub-Gaussian processes.

The modulus of continuity f is found by Dudley [2] in the case of Gaussian processes. These results are generalized by Kozachenko [4] for some classes of processes belonging to the Orlicz spaces.

In the book [1], the modulus of continuity is evaluated for stochastic processes belonging to some classes Δ of Orlicz spaces, and bounds for the distribution of the norms of such processes are obtained in the Lipschitz spaces.

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2. Preliminaries

2.1. **Space** $\mathrm{Sub}_\varphi(\Omega)$**.** We briefly recall the basic notions related to the space of generalized sub-Gaussian random variables $\mathrm{Sub}_\varphi(\Omega)$ (see [1, 3]).

Definition 2.1. A continuous even convex function u is called an N *Orlicz function* if it increases for $x > 0$, $u(x)/x \to 0$ as $x \to 0$, and $u(x)/x \to \infty$ as $x \to \infty$.

More details concerning convex functions in some Orlicz spaces can be found in the book by Krasnosel'skiĭ and Rutickiĭ [5].

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a standard probability space.

Definition 2.2. Let φ be an Orlicz N-function such that

$$\liminf_{x\to 0} \frac{\varphi(x)}{x^2} = C > 0. \tag{Q}$$

We do not exclude the case of $C = +\infty$ in condition (Q). We say that a zero mean random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if there exists a positive constant a such that the inequality

$$\mathsf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda))$$

holds for all $\lambda \in \mathbb{R}$.

Example 2.1. The following are two examples of N-Orlicz functions satisfying condition (Q):

$$\varphi(x) = \frac{|x|^\alpha}{\alpha}, \qquad 1 < \alpha \le 2,$$
$$\varphi(x) = \begin{cases} \frac{|x|^2}{\alpha}, & |x| \le 1,\ \alpha > 2, \\ \frac{|x|^\alpha}{\alpha}, & |x| > 1. \end{cases}$$

Note that $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\xi) = \sup_{\lambda \ne 0} \frac{\varphi^{-1}(\ln \mathsf{E}\exp(\lambda\xi))}{|\lambda|}.$$

We also note that the inequalities

$$\begin{gathered} \mathsf{E}\exp(\lambda\xi) \le \exp\big(\varphi(\lambda\tau_\varphi(\xi))\big), \\ \big(\mathsf{E}\,\xi^2\big)^{\frac{1}{2}} \le C\tau_\varphi(\xi) \end{gathered} \tag{2.1}$$

hold for all $\lambda \in \mathbb{R}$, where $C > 0$ is a constant.

Definition 2.3. Let (T, ρ) be a pseudometric space. Let $N_{(T,\rho)}(u)$ denote the minimal number of closed ρ-balls covering the space T and such that their diameters do not exceed $2u$. Then the function

$$H(u) := \ln N_{(T,\rho)}(u)$$

is called the *metric entropy* of the space (T, ρ).

Definition 2.4. Let (T, ρ) be a pseudometric separable space. We say that a stochastic process $X = (X(t), t \in T)$ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ for all $t \in T$.

2.2. **An auxiliary result.** We recall the definition of the *Young–Fenchel transform* (denoted by φ^*) of an N Orlicz function φ. The Young–Fenchel transform is defined by

$$\varphi^*(x) := \sup_{y>0} (xy - \varphi(y)), \qquad x \geq 0.$$

The following result is rather technical, but we need it in the proofs of the main results of this paper.

Theorem 2.1. *Let $\{\xi_i\}_{i=1}^n \in \mathrm{Sub}_\varphi(\Omega)$, $n \geq 2$, $x > 2$, and let M and b be two numbers such that $b > 1$ and $M \geq \varphi^*(2)/\ln(2)$. Then*

$$\begin{aligned} \mathsf{P}\Big\{ \max_{j=1,\dots,n} |\xi_j| > xb \max_{j=1,\dots,n} \tau_\varphi(\xi_j) \cdot \varphi^{*(-1)}(M\ln(n)) \Big\} \\ \leq n^{1-M} \frac{b+1}{b-1} \exp\{-\varphi^*(x)\}. \end{aligned} \tag{2.2}$$

Proof. Let $\eta := \max_{j=1,\dots,n} |\xi_j|$, $a := b \max_{j=1,\dots,n} \tau_\varphi(\xi_j)$, and

$$u_n := \varphi^{*(-1)}(M\ln(n)).$$

Then

$$\begin{aligned} \mathsf{P}\{\eta > xau_n\} &= \mathsf{E}\mathbf{1}\{\omega \colon \eta > xau_n\} \\ &\leq \sum_{j=1}^n \mathsf{E}\mathbf{1}\{\eta = |\xi_j|\} \cdot \mathbf{1}\{\omega \colon |\xi_j| > xau_n\} \\ &\leq n \max_{j=1,\dots,n} \mathsf{E}\mathbf{1}\{\omega \colon |\xi_j| > xau_n\} \\ &\leq n^{1-M} n^M \max_{j=1,\dots,n} \mathsf{E}\mathbf{1}\{\omega \colon |\xi_j| > xau_n\} \cdot \frac{\exp\{\varphi^*(\frac{|\xi_j|}{au_n})\}}{\exp\{\varphi^*(x)\}}. \end{aligned} \tag{2.3}$$

Since $|\xi_j|/(au_n) > x > 2$, $n \geq 2$, and $M\ln(n) \geq \varphi^*(2)$ (that is, $u_n \geq 2$), we have

$$\begin{aligned} n^M \exp\Big\{\varphi^*\Big(\frac{|\xi_j|}{au_n}\Big)\Big\} &= \exp\Big\{M\ln(n) + \varphi^*\Big(\frac{|\xi_j|}{au_n}\Big)\Big\} \\ &= \exp\Big\{\varphi^*(\varphi^{*(-1)}(M\ln(n)) + \varphi^*\Big(\frac{|\xi_j|}{au_n}\Big)\Big\} \\ &\leq \exp\Big\{\varphi^*(\varphi^{*(-1)}(M\ln(n)) + \frac{|\xi_j|}{au_n}\Big\} \\ &= \exp\Big\{\varphi^*\Big(u_n + \frac{|\xi_j|}{au_n}\Big)\Big\} \leq \exp\Big\{\varphi^*\Big(\frac{|\xi_j|}{a}\Big)\Big\}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \mathsf{P}\{\eta > xau_n\} &= n^{1-M} \exp\{-\varphi^*(x)\} \max_{j=1,\dots,n} \mathsf{E}\exp\Big\{\varphi^*\Big(\frac{|\xi_i|}{a}\Big)\Big\} \\ &= n^{1-M} \exp\{-\varphi^*(x)\} \max_{j=1,\dots,n} \mathsf{E}\exp\Big\{\varphi^*\Big(\frac{|\xi_i|}{b\tau_\varphi|\xi_i|}\Big)\Big\}. \end{aligned} \tag{2.4}$$

Recall that if

$$\mathsf{P}\{|\xi| > x\} \leq C \exp\Big\{\varphi^*\Big(\frac{x}{D}\Big)\Big\},$$

for some $C > 0$ and $D > 0$, then

$$\mathsf{E}\exp\Big\{\varphi^*\Big(\frac{\xi}{A}\Big)\Big\} \leq 1 + \frac{CD}{A-D} \tag{2.5}$$

for all $A > D$ (see [1, Corollary 4.1]). It also follows from [1, Lemma 4.3] that

$$\mathsf{P}\{|\xi| > x\} \le 2\exp\left\{-\varphi^*\left(\frac{\xi}{\tau_\varphi(\xi)}\right)\right\}.$$

Thus (2.5) implies that

$$\mathsf{E}\exp\left\{\varphi^*\left(\frac{\xi_j}{b\tau_\varphi(\xi_j)}\right)\right\} \le \frac{b+1}{b-1}$$

for $b > 1$. Therefore

$$\mathsf{P}\{\eta > xau_n\} \le n^{1-M}\frac{b+1}{b-1}\exp\{-\varphi^*(x)\}.$$

The theorem is proved. □

3. Main results

Let (T,ρ) be a metric (pseudometric) separable compact space and let

$$X = \{X(t), t \in T\}$$

be a separable stochastic process belonging to the space $\mathrm{Sub}_\varphi(\Omega)$.

Suppose that there exists an increasing continuous function $\sigma = \{\sigma(h), h \ge 0\}$ such that $\sigma(0) = 0$ and

$$\sup_{\rho(t,s)\le h} \tau_\varphi(X(t) - X(s)) \le \sigma(h). \tag{3.1}$$

Denote by $N(u)$ the minimal number of closed ρ-balls that cover the space (T,ρ) and whose radii are equal to u.

Theorem 3.1. *Let $N(u) \to \infty$ as $u \to 0$, $M \ge \max(1, \varphi^*(2)/\ln(2))$, and let*

$$f_B(u) = \frac{1}{(11-2\sqrt{30})}\int_0^{\sigma(u)} \varphi^{*(-1)}\left(2M\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right)dv < \infty,$$

where $B > 1$ and $b > 1$ are some numbers and where v is a number such that $N(v) > 2$. Then

$$\begin{aligned}\mathsf{P}&\left\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{f_B(\rho(t,s))} > y\right\}\\ &\le \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)}\exp\left\{-\varphi^*\left(\frac{y}{b}\right)\right\}\end{aligned} \tag{3.2}$$

for all $y > 2b$.

Proof. Let $r \in (0,1)$. We consider the sequence $\{\nu_k, k = 0,1,2,\dots\}$ constructed as follows:

$$\nu_0 = \inf_{s\in T}\sup_{t\in T}\rho(t,s) \quad \text{and} \quad \nu_{k+1} = \min\{r\nu_k, \delta_k\},\ k \ge 0,$$

where

$$\delta_k = A\inf\left\{\nu\colon N\left(\sigma^{(-1)}(\nu)\right) < BN\left(\sigma^{(-1)}(\nu_k)\right)\right\}, \tag{3.3}$$

$\sigma^{(-1)}(\nu)$ is the inverse function to σ, $B > 1$ is a number, and where A is a number such that $A > 1$ and $Ar < 1$.

The sequence $\{\nu_k, k = 0,1,2,\dots\}$ constructed above is such that

$$\nu_{k+1} \le r\nu_k, \qquad k = 0,1,2,\dots, \tag{3.4}$$

that is,

$$\nu_k \le \frac{1}{1-r}(\nu_k - \nu_{k+1}). \tag{3.5}$$

We derive from (3.3) and (3.4) that

$$\begin{aligned} N\left(\sigma^{(-1)}(\nu_{k+2})\right) &\ge N\left(\sigma^{(-1)}(r\nu_{k+1})\right) \\ &\ge N\left(\sigma^{(-1)}(r\delta_k)\right) \ge BN\left(\sigma^{(-1)}(\nu_k)\right). \end{aligned} \tag{3.6}$$

This means that

$$N\left(\sigma^{(-1)}(\nu_k)\right) \ge BN\left(\sigma^{(-1)}(\nu_{k-2})\right) \ge B^2 N\left(\sigma^{(-1)}(\nu_{k-4})\right) \ge \cdots. \tag{3.7}$$

Let $\varepsilon_0 = \sigma^{(-1)}(\nu_0), \dots, \varepsilon_k = \sigma^{(-1)}(\nu_k)$. Denote by V_{ε_k}, $k = 0, 1, 2, \dots$, the set of the centers of all closed balls of radius ε_k belonging to the minimal covering of the space (T, ρ). The number of elements of the set V_{ε_k} is equal to $N(\varepsilon_k) = N\big(\sigma^{(-1)}(\nu_k)\big)$. Let

$$V_0 = \bigcup_{k=0}^{\infty} V_{\varepsilon_k}.$$

Inequality (3.1) together with the Chebyshev inequality implies that the process X is continuous in probability. Thus V_0 is the set of separability of the process X.

Further, let α_n be the mapping acting from the set V_0 to V_{ε_n} such that $\alpha_n(t) = t$ if $t \in V_{\varepsilon_n}$; otherwise $\alpha_n(t)$ is a point of the set V_{ε_n} for which $\rho(t, \alpha_n(t)) < \varepsilon_n$. The Chebyshev inequality, (3.1), and (3.4) yield

$$\begin{aligned} \mathsf{P}\left\{|X(t) - X(\alpha_n(t))| > r^{n/2}\right\} &\le \frac{\mathsf{E}\big(X(t) - X(\alpha_n(t))\big)^2}{r^n} \le \frac{C\tau_\varphi^2\big(X(t) - X(\alpha_n(t))\big)}{r^n} \\ &\le \frac{C\sigma^2\big(\rho(t, \alpha_n(t))\big)}{r^n} \le \frac{C\sigma^2(\varepsilon_n)}{r^n} \le \frac{C\nu_n^2}{r^n} \le \frac{C\nu_0^2 r^{2n}}{r^n} \\ &= C\nu_0^2 r^n, \end{aligned}$$

where $C > 0$ is a certain constant. Therefore

$$\sum_{n=1}^{\infty} \mathsf{P}\left\{|X(t) - X(\alpha_n(t))| > r^{n/2}\right\} < \infty.$$

The Borel–Cantelli lemma implies that $X(\alpha_n(t)) \to X(t)$ with probability one as $n \to \infty$. Since the set V_0 is countable, $X(\alpha_n(t)) \to X(t)$ as $n \to \infty$ with probability one for all $t \in V_0$.

Choose $0 < u \le \varepsilon_0$ and consider a number m such that $\varepsilon_{m+1} < u \le \varepsilon_m$. Since V_0 is a set of separability of the process X,

$$\sup_{\substack{\rho(t,s)<u \\ t,s\in T}} |X(t) - X(s)| = \sup_{\substack{\rho(t,s)<u \\ t,s\in V_0}} |X(t) - X(s)| \tag{3.8}$$

with probability one.

Let t and s belong to the set V_0 and be such that $\rho(t, s) < u$. Let $k > m + 1$. We introduce the following notation:

$$\begin{aligned} t_k &= \alpha_k(t), \qquad t_{k-1} = \alpha_{k-1}(t_k), \quad \dots, \quad t_m = \alpha_m(t_{m+1}), \\ s_k &= \alpha_k(s), \qquad s_{k-1} = \alpha_{k-1}(s_k), \quad \dots, \quad s_m = \alpha_m(t_{m+1}). \end{aligned}$$

Then

$$
\begin{aligned}
X(t)-X(s) &= (X(t)-X(t_k)) + \sum_{l=m+2}^{k} (X(t_l)-X(t_{l-1})) - (X(s)-X(s_k)) \\
&\quad - \sum_{l=m+2}^{k} (X(s_l)-X(s_{l-1})) + (X(t_{m+1})-X(s_{m+1}))
\end{aligned} \tag{3.9}
$$

for all t and s such that $\rho(t,s)<u$. Representation (3.9) implies that

$$
\begin{aligned}
X(t_{m+1})-X(s_{m+1}) &= (X(t)-X(s)) - (X(t)-X(t_k)) + (X(s)-X(s_k)) \\
&\quad - \sum_{l=m+2}^{k} (X(t_l)-X(t_{l-1})) + \sum_{l=m+2}^{k} (X(s_l)-X(s_{l-1}))
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_\varphi(X(t_{m+1})-X(s_{m+1})) &\le \tau_\varphi(X(t)-X(s)) + \tau_\varphi(X(t)-X(t_k)) + \tau_\varphi(X(t_s)-X(s_k)) \\
&\quad + \sum_{l=m+2}^{k} \tau_\varphi(X(t_l)-X(t_{l-1})) + \sum_{l=m+2}^{k} \tau_\varphi(X(s_l)-X(s_{l-1})) \\
&\le \sigma(\rho(t,s)) + \sigma(\rho(t,t_k)) + \sigma(\rho(s,s_k)) + \sum_{l=m+2}^{k} \sigma(\rho(t_l,t_{l-1})) \\
&\quad + \sum_{l=m+2}^{k} \sigma(\rho(s_l,s_{l-1})) \\
&\le \sigma(u) + 2\sigma(\varepsilon_k) + 2\sum_{l=m+2}^{k} \sigma(\varepsilon_{l-1}) \\
&\le \sigma(u) + 2\sum_{l=m+2}^{\infty} \sigma(\varepsilon_{l-1}) = \sigma(u) + 2\sum_{l=m+2}^{\infty} \nu_{l-1} \\
&\le \sigma(u) + 2\sum_{l=1}^{\infty} \nu_{m+l} \le \sigma(u) + 2\sum_{l=1}^{\infty} \nu_{m+1} r^{l-1} \\
&= \sigma(u) + \nu_{m+1}\frac{2}{1-r} \le \sigma(u)\left(1+\frac{2}{1-r}\right) = \sigma(u)\frac{3-r}{1-r}.
\end{aligned} \tag{3.10}
$$

Relations (3.9) and (3.10) yield

$$
\begin{aligned}
|X(t)-X(s)| &\le \sum_{l=m+2}^{k} |X(t_l)-X(t_{l-1})| + \sum_{l=m+2}^{k} |X(s_l)-X(s_{l-1})| \\
&\quad + |X(t)-X(t_k)| + |X(s)-X(s_k)| + |X(t_{m+1})-X(s_{m+1})| \\
&\le 2\sum_{l=m+2}^{k} \max_{w\in V_{\varepsilon_l}} |X(w)-X(\alpha_{l-1}(w))| \\
&\quad + \max_{\substack{w,v\in V_{\varepsilon_{m+1}}:\\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}} |X(w)-X(v)| \\
&\quad + |X(t)-X(t_k)| + |X(s)-X(s_k)|
\end{aligned} \tag{3.11}
$$

for all $t,s\in T$ such that $\rho(t,s)<u$.

Passing to the limit as $k \to \infty$, relation (3.11) implies that

$$|X(t) - X(s)| \le 2 \sum_{l=m+2}^{\infty} \max_{w \in V_{\varepsilon_l}} |X(w) - X(\alpha_{l-1}(w))| + \max_{\substack{w,v \in V_{\varepsilon_{m+1}}: \\ \tau_\varphi(X(w)-X(v)) \le \sigma(u)\frac{3-r}{1-r}}} |X(w) - X(v)|$$

with probability one. Now we deduce from (3.8) that

$$\sup_{\substack{\rho(t,s)\le u \\ t,s\in T}} |X(t) - X(s)| = \sup_{\substack{\rho(t,s)\le u \\ t,s\in V_0}} |X(t) - X(s)| \le 2 \sum_{k=m+2}^{\infty} \max_{w \in V_{\varepsilon_k}} |X(w) - X(\alpha_{l-1}(w))| + \max_{\substack{w,v \in V_{\varepsilon_{m+1}} \\ \tau_\varphi(X(w)-X(v)) \le \sigma(u)\frac{3-r}{1-r}}} |X(w) - X(v)|. \tag{3.12}$$

Let

$$c_l = b\sigma(\varepsilon_{l-1})\varphi^{*(-1)}\big(M \ln(N(\varepsilon_l))\big),$$

$$b_m(u) = b\varphi^{*(-1)}\left(M \ln\left(N^2(\varepsilon_{m+1})\right)\right)\sigma(u)\frac{3-r}{1-r},$$

$$\varepsilon_{m+1} < u \le \varepsilon_m.$$

Denote

$$\xi_l = \max_{t \in V_{\varepsilon_l}} |X(t) - X(\alpha_{l-1}(t))|$$

and put, for $\varepsilon_{m+1} < u \le \varepsilon_m$,

$$\eta_m(u) = \max_{\substack{w,z \in V_{\varepsilon_{m+1}} \\ \tau_\varphi(X(w)-X(z)) \le \sigma(u)\frac{3-r}{1-r}}} |X(w) - X(z)|.$$

Consider a number $v > 0$ such that $N(v) > 2$. We choose a number n for which

$$\varepsilon_{n+1} < V \le \varepsilon_n.$$

Let $\{G(u), u \ge 0\}$ be an increasing function such that

$$G(u) \ge 2 \sum_{l=m+2}^{\infty} c_l + b_m(u),$$

where m is an arbitrary number satisfying $\varepsilon_{m+1} < u \le \varepsilon_m$. We also put

$$w_m = \left(2 \sum_{l=m+2}^{\infty} c_l + b_m(\rho(t,s))\right)^{-1}.$$

Then

$$
\begin{aligned}
&\mathsf{P}\Big\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{G(\rho(t,s))}>x\Big\}\\
&\quad\le \mathsf{P}\Big\{\max\Big[\sup_{m\ge n+1}\ \sup_{\varepsilon_{m+1}<\rho(t,s)\le\varepsilon_m}\frac{|X(t)-X(s)|}{G(\rho(t,s))},\\
&\qquad\qquad \sup_{\varepsilon_{n+1}<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{G(\rho(t,s))}\Big]>x\Big\}\\
&\quad\le \mathsf{P}\Big\{\max\Big[\sup_{m\ge n+1}\ \sup_{\varepsilon_{m+1}<\rho(t,s)\le\varepsilon_m}\Big(2\sum_{l=m+2}^{\infty}\xi_l+\eta_m(\rho(t,s))\Big)w_m,\\
&\qquad\qquad \sup_{\varepsilon_{n+1}<\rho(t,s)\le v}\Big(2\sum_{l=n+2}^{\infty}\xi_l+\eta_n(\rho(t,s))\Big)w_n\Big]>x\Big\}\\
&\quad\le \sum_{l=n+2}^{\infty}\mathsf{P}\Big\{\frac{\xi_l}{c_l}>x\Big\}+\sum_{l=n+1}^{\infty}\mathsf{P}\Big\{\sup_{\varepsilon_{l+1}<u\le\varepsilon_l}\frac{\eta_l(u)}{b_l(u)}>x\Big\}\\
&\qquad+\mathsf{P}\Big\{\sup_{\varepsilon_{n+1}<u\le v}\frac{\eta_n(u)}{b_n(u)}>x\Big\}
\end{aligned}
\tag{3.13}
$$

for $x>2$ and $N(v)>2$.

Now we estimate the probabilities on the right-hand side of (3.13). We derive from Theorem 2.1 that

$$
\begin{aligned}
&\mathsf{P}\Big\{\sup_{\varepsilon_{l+1}<u\le\varepsilon_l}\frac{\eta_l(u)}{b_l(u)}>x\Big\}\\
&\quad\le \mathsf{P}\Big\{\sup_{\varepsilon_{l+1}<u\le\varepsilon_l}\ \max_{\substack{w,v\in V_{\varepsilon_{l+1}},\\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}}\frac{|X(w)-X(v)|}{b_l(u)}>x\Big\}\\
&\quad\le \mathsf{P}\Bigg\{\sup_{\varepsilon_{l+1}<u<\varepsilon_l}\ \max_{\substack{w,v\in V_{\varepsilon_{l+1}},\\ \tau_\varphi(X(w)-X(v))\ne0,\\ \tau_\varphi(X(w)-X(v))\le\sigma(u)\frac{3-r}{1-r}}} Y(l,u,r,w,v)>x\Bigg\}\\
&\quad\le \mathsf{P}\Bigg\{\max_{\substack{w,v\in V_{\varepsilon_{l+1}}\\ \tau_\varphi(X(w)-X(v))\ne0}}\frac{|X(w)-X(v)|}{\tau_\varphi(X(w)-X(v))}>xb\varphi^{*(-1)}\left(M\ln\left(N^2(\varepsilon_{l+1})\right)\right)\Bigg\}\\
&\quad\le \frac{b+1}{b-1}\left(N^2(\varepsilon_{l+1})\right)^{1-M}\exp\{-\varphi^*(x)\},
\end{aligned}
\tag{3.14}
$$

where

$$
Y(l,u,r,w,v)=\frac{|X(w)-X(v)|}{\tau_\varphi(X(w)-X(v))}\frac{\tau_\varphi(X(w)-X(v))}{\sigma(u)\frac{3-r}{1-r}}(b_l(u))^{-1}\sigma(u)\frac{3-r}{1-r}.
$$

Similar reasoning proves that

$$
\mathsf{P}\Big\{\sup_{\varepsilon_{n+1}<u\le v}\frac{\eta_n(u)}{b_n(u)}>x\Big\}\le\frac{b+1}{b-1}\left(N^2(\varepsilon_{n+1})\right)^{1-M}\exp\{-\varphi^*(x)\}. \tag{3.15}
$$

Another consequence of Theorem 2.1 is that

$$\mathsf{P}\left\{\frac{\xi_l}{c_l} > x\right\} \le \mathsf{P}\left\{\max_{\substack{t\in V_{\varepsilon_l}:\\ \tau_\varphi(X(t)-X(\alpha_{l-1}(t)))\neq 0}} \frac{\big(X(t)-X(\alpha_{l-1}(t))\big)}{\sigma(\varepsilon_{l-1})\varphi^{*(-1)}\big(M\ln(N(\varepsilon_l))\big)} > x\right\} \le \frac{b+1}{b-1}\,(N(\varepsilon_l))^{1-M}\exp\{-\varphi^*(x)\}. \tag{3.16}$$

Inequalities (3.14), (3.15), (3.16), and (3.6) imply that

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{G(\rho(t,s))} > x\right\}\\
&\le \left(\sum_{l=n+2}^{\infty}(N(\varepsilon_l))^{1-M} + \sum_{l=n+1}^{\infty}\left(N^2(\varepsilon_l)\right)^{1-M}\right)\frac{b+1}{b-1}\exp\{-\varphi^*(x)\}\\
&\le 2\sum_{l=n+1}^{\infty}(N(\varepsilon_l))^{1-M}\frac{b+1}{b-1}\exp\{-\varphi^*(x)\}\\
&\le \frac{4}{(N(\varepsilon_{n+1}))^{M-1}}\sum_{l=0}^{\infty}\left(\frac{1}{B^{M-1}}\right)^l\frac{b+1}{b-1}\exp\{-\varphi^*(x)\}\\
&= \frac{4B^{M-1}(b+1)}{(N(\varepsilon_{n+1}))^{M-1}(B^{M-1}-1)(b-1)}\exp\{-\varphi^*(x)\}\\
&\le \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)}\exp\{-\varphi^*(x)\}
\end{aligned} \tag{3.17}$$

if $x>2$ and if $v>0$ is such that $N(v)\ge 2$.

Now we estimate the sum $2\sum_{l=m+2}^{\infty}c_l + b_m(u)$. Put $Z(v) = b\varphi^{*(-1)}(Mv)$. Then

$$\sum_{l=m+2}^{\infty}c_l = \sum_{l=m+2}^{\infty}\nu_{l-1}Z\left(\ln\left(N\left(\sigma^{(-1)}(\nu_l)\right)\right)\right) = A_1 + A_2,$$

where

$$A_1 = \sum_{l\in D_1^{(m)}}\nu_{l-1}Z\left(\ln\left(N\left(\sigma^{(-1)}(\nu_l)\right)\right)\right), \qquad D_1^{(m)} = \{l\ge m+2, \nu_l = r\nu_{l-1}\},$$

$$A_2 = \sum_{l\in D_2^{(m)}}\nu_{l-1}Z\left(\ln\left(N\left(\sigma^{(-1)}(\nu_l)\right)\right)\right), \qquad D_2^{(m)} = \{l\ge m+2, \nu_l = \delta_{l-1}\}.$$

Inequality (3.5) implies that

$$\begin{aligned}
A_1 &= \frac{1}{r}\sum_{l\in D_1^{(m)}}\nu_l Z\left(\ln\left(N\left(\sigma^{(-1)}(\nu_l)\right)\right)\right)\\
&\le \frac{1}{r(1-r)}\sum_{l=m+2}^{\infty}(\nu_l-\nu_{l+1})Z\left(\ln\left(N\left(\sigma^{(-1)}(\nu_l)\right)\right)\right)\\
&\le \frac{1}{r(1-r)}\sum_{l=m+2}^{\infty}\int_{\nu_{l+1}}^{\nu_l}Z\left(\ln\left(N\left(\sigma^{(-1)}(u)\right)\right)\right)du\\
&= \frac{1}{r(1-r)}\int_0^{\nu_{m+2}}Z\left(\ln\left(N\left(\sigma^{(-1)}(u)\right)\right)\right)du.
\end{aligned}$$

Therefore

$$A_1 \leq \frac{1}{r(1-r)} \int_0^{\nu_{m+2}} Z\left(\ln\left(N\left(\sigma^{(-1)}(u)\right)\right)\right) du. \tag{3.18}$$

Since $N\big(\sigma^{(-1)}(\delta_l)\big) < BN\big(\sigma^{(-1)}(\nu_l)\big)$, we have

$$\begin{aligned} A_2 &= \sum_{l \in D_2^{(m)}} \nu_{l-1} Z\left(\ln\left(N\left(\sigma^{(-1)}(\delta_{l-1})\right)\right)\right) \\ &\leq \sum_{l \in D_2^{(m)}} \nu_{l-1} Z\left(\ln\left(BN\left(\sigma^{(-1)}(\nu_{l-1})\right)\right)\right) \\ &\leq \frac{1}{1-r} \sum_{l=m+2}^{\infty} (\nu_{l-1} - \nu_l) Z\left(\ln\left(BN\left(\sigma^{(-1)}(\nu_{l-1})\right)\right)\right) \\ &\leq \frac{1}{1-r} \int_0^{\nu_{m+1}} Z\left(\ln\left(BN\left(\sigma^{(-1)}(u)\right)\right)\right) du. \end{aligned} \tag{3.19}$$

Now we obtain from (3.18) and (3.19) that

$$2 \sum_{l=m+2}^{\infty} c_l \leq \frac{2(1+r)}{r(1-r)} \int_0^{\sigma(u)} Z\left(\ln\left(BN\left(\sigma^{(-1)}(u)\right)\right)\right) du, \tag{3.20}$$

since $\nu_{m+2} < \nu_{m+1} < \sigma(u)$. Further, if $\varepsilon_{m+1} < u \leq \varepsilon_m$ (or, $\nu_{m+1} < \sigma(u) \leq \nu_m$), then we have

$$b_m(u) \leq Z\left(2\ln\left(N\left(\sigma^{(-1)}(\nu_{m+1})\right)\right)\right) \sigma(u) \frac{3-r}{1-r}.$$

Recall that $\nu_{m+1} = \min(r\nu_m, \delta_m)$ and consider separately the two possible cases, namely $\nu_{m+1} = \delta_m$ and $\nu_{m+1} = r\nu_m$. First let $\nu_{m+1} = \delta_m$. By equality (3.3),

$$\begin{aligned} \sigma(u) Z\left(2\ln\left(N\left(\sigma^{(-1)}(\nu_{m+1})\right)\right)\right) &= \sigma(u) Z\left(2\ln\left(N\left(\sigma^{(-1)}(\delta_m)\right)\right)\right) \\ &\leq \sigma(u) Z\left(2\ln\left(BN\left(\sigma^{(-1)}(\nu_m)\right)\right)\right) \\ &\leq \sigma(u) Z\left(2\ln\left(BN\left(\sigma^{(-1)}(u)\right)\right)\right) \\ &\leq \int_0^{\sigma(u)} Z\left(2\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right) dv. \end{aligned}$$

Second, if $\nu_{m+1} = r\nu_m$, then

$$\begin{aligned} \sigma(u) Z\left(2\ln\left(N\left(\sigma^{(-1)}(\nu_{m+1})\right)\right)\right) &= \sigma(u) Z\left(2\ln\left(N\left(\sigma^{(-1)}(r\nu_m)\right)\right)\right) \\ &\leq \sigma(u) Z\left(2\ln\left(N\left(\sigma^{(-1)}(r\sigma(u))\right)\right)\right) \\ &\leq \int_0^{\sigma(u)} Z\left(2\ln\left(N\left(\sigma^{(-1)}(rv)\right)\right)\right) dv \\ &= \frac{1}{r} \int_0^{r\sigma(u)} Z\left(2\ln\left(N\left(\sigma^{(-1)}(t)\right)\right)\right) dt \\ &\leq \frac{1}{r} \int_0^{\sigma(u)} Z\left(2\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right) dv. \end{aligned}$$

Therefore

$$b_m(u) \leq \frac{3-r}{r(1-r)} \int_0^{\sigma(u)} Z\left(2\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right) dv.$$

Combining the above results we get the following bound:

$$2\sum_{l=m+2}^{\infty} c_l + b_m(u) \le \frac{5+r}{r(1-r)} b \int_0^{\sigma(u)} \varphi^{*(-1)}\left(M2\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right) dv. \tag{3.21}$$

This means that (3.17) implies that

$$\begin{aligned}\mathsf{P}&\left\{\sup_{0<\rho(t,s)\le v} \frac{|X(t)-X(s)|}{G_{r,b}(\rho(t,s))} > x\right\} \\ &\le \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)} \exp\{-\varphi^*(x)\}\end{aligned} \tag{3.22}$$

for $x > 2$, where

$$G_{r,b}(u) = b\frac{5+r}{r(1-r)} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(M2\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right) dv.$$

Since

$$\inf_{0<r<1} \frac{5+r}{r(1-r)} = \frac{1}{11-2\sqrt{30}},$$

we obtain

$$\begin{aligned}\mathsf{P}&\left\{\sup_{0<\rho(t,s)\le v} \frac{|X(t)-X(s)|}{bf_B(\rho(t,s))} > x\right\} \\ &\le \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1}-1)(b-1)} \exp\{-\varphi^*(x)\}\end{aligned} \tag{3.23}$$

for all $x > 2$. Inequality (3.2) follows from the latter inequality with $y = xb > 2b$. □

Theorem 3.2. *Let all the assumptions of Theorem* 3.1 *hold. Then, with probability one,*

$$\limsup_{\varepsilon\to 0} \frac{1}{2bf_B(\varepsilon)} \sup_{\rho(t,s)<\varepsilon} |X(t)-X(s)| < 1, \tag{3.24}$$

where

$$f_B(u) = \frac{1}{11-2\sqrt{30}} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(2M\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right) dv.$$

Proof. It follows from (3.12) that

$$\sup_{\rho(t,s)\le u} |X(t)-X(s)| \le 2\sum_{k=m+2}^{\infty} \xi_k + \eta_m(u) \tag{3.25}$$

with probability one.

Inequality (3.15) implies that $\eta_k(u) < 2b_k(u)$ with probability one if k is sufficiently large. This together with bound (3.16) yields that $\xi_k < 2c_k$ with probability one if k is sufficiently large. Therefore

$$\sup_{\rho(t,s)\le u} |X(t)-X(s)| \le 2\left(2\sum_{k=m+2}^{\infty} c_k + b_m(u)\right) \tag{3.26}$$

for sufficiently large m (in other words, for sufficiently small u). Now (3.21) and (3.23) imply that

$$\sup_{0<\rho(t,s)\le u} |X(t)-X(s)| \le 2bf_B(u)$$

with probability one if u is sufficiently small. □

The following result follows from Theorem 3.2.

Corollary 3.1. *If u is sufficiently small, then*

$$\sup_{\rho(t,s)\le u} |X(t)-X(s)| \le 2bf_B(u)$$

with probability one.

4. An application to $\mathrm{Sub}_\varphi(\Omega)$-processes defined on finite-dimensional spaces

Let T be a cube in a finite dimensional space, that is,

$$T = \underbrace{[T_1,T_2]\times\cdots\times[T_1,T_2]}_{d \text{ times}}, \qquad T_1<T_2,$$

and let $\rho(t,s)=\max_{1\le i\le d}|t_i-s_i|$ for $t=(t_i, i=1,\dots,d)$ and $s=(s_i, i=1,\dots,d)$.

Theorem 4.1. *Let $X=\{X(t), t\in T\}$ be a separable stochastic process belonging to the space $\mathrm{Sub}_\varphi(\Omega)$. Assume that there exists an increasing continuous function*

$$\sigma = \{\sigma(h), h\ge 0\}$$

such that $\sigma(0)=0$ and

$$\sup_{\rho(t,s)\le h} \tau_\varphi(X(t)-X(s)) \le \sigma(h). \tag{4.1}$$

Let $M\ge \max(1,\varphi^*(2)/\ln(2))$ and let $B>1$ and $b>1$ be some numbers. Then

$$\begin{aligned}&\mathsf{P}\left\{\sup_{0<\rho(t,s)\le v}\frac{|X(t)-X(s)|}{f_B^d(\rho(t,s))}>y\right\}\\ &\qquad\le \frac{4B^{M-1}(b+1)}{(B^{M-1}-1)(b-1)}\left(\frac{2v}{T_2-T_1}\right)^{d(M-1)}\exp\left\{-\varphi^*\left(\frac{y}{b}\right)\right\}\end{aligned} \tag{4.2}$$

for all $y>2b$ and $v\le T_2-T_1/(2\cdot 2^{1/d})$, where

$$f_B^d(u)=\frac{1}{11-2\sqrt{30}}\int_0^{\sigma(u)}\varphi^{*(-1)}\left(2Md\ln\left(B^{1/d}\left(\frac{T_2-T_1}{2\sigma^{(-1)}(s)}+1\right)\right)\right)ds.$$

Moreover,

$$\limsup_{\varepsilon\to 0}\frac{1}{2bf_B^d(\varepsilon)}\sup_{\rho(t,s)\le\varepsilon}|X(t)-X(s)|<1 \tag{4.3}$$

with probability one.

Proof. Theorem 4.1 follows from Theorems 3.1 and 3.2, since

$$\left(\frac{T_2-T_1}{2z}\right)^d \le N(z) \le \left(\frac{T_2-T_1}{2z}+1\right)^d \tag{4.4}$$

for all $z>0$. □

Remark 4.1. The normalization $f_B^d(u)$ in inequality (4.2) is such that

$$f_B^d(u)\le\frac{1}{11-2\sqrt{30}}\int_0^{\sigma(u)}\varphi^{*(-1)}\left(2Md\ln\left(B^{1/d}\left(\frac{T_2-T_1}{\sigma^{(-1)}(s)}\right)\right)\right)ds. \tag{4.5}$$

Indeed,

$$\sigma^{(-1)}(s)\le\sigma^{(-1)}(\sigma(v))=v\le\frac{T_2-T_1}{2^{1+1/d}}$$

in (4.2), whence

$$\frac{T_2-T_1}{2\sigma^{(-1)}(s)}\ge 2^{1/d}>1.$$

Example 4.1. Let $\varphi(x) = |x|^p/p$, $p > 1$, for sufficiently large $|x|$. In this case,

$$\varphi^*(x) = \frac{|x|^q}{q} \quad \text{and} \quad \varphi^{*(-1)}(x) = (qx)^{1/q},$$

where q is defined by $p^{-1} + q^{-1} = 1$. Assume that $T_2 - T_1 > 1$ and $\sigma(h) = c/(\ln h^{-1})^\alpha$ for some $c > 0$ and all $h \in (0,1)$, where $\alpha > q^{-1}$. Then $\sigma^{(-1)}(h) = \exp\left\{-(c/h)^{1/\alpha}\right\}$ and

$$\begin{aligned}
f_B^d(u) &\le \frac{1}{11 - 2\sqrt{30}} \int_0^{\sigma(u)} q^{1/q} \left(2Md \ln\left(B^{1/d}(T_2 - T_1) \exp\left\{\left(\frac{c}{t}\right)^{1/\alpha}\right\}\right)\right)^{1/q} dt \\
&\le \frac{(2Mdq)^{1/q}}{11 - 2\sqrt{30}} \left(\int_0^{\sigma(u)} \left(\ln\left(B^{1/d}(T_2 - T_1)\right)\right)^{1/q} dt + \int_0^{\sigma(u)} \left(\frac{c}{t}\right)^{\frac{1}{\alpha q}} dt\right) \\
&= \frac{(2Mdq)^{1/q}}{11 - 2\sqrt{30}} \left(\sigma(u) \left(\ln\left(B^{1/d}(T_2 - T_1)\right)\right)^{1/q} + \frac{c^{\frac{1}{\alpha q}}}{1 - \frac{1}{\alpha q}} (\sigma(u))^{1 - \frac{1}{\alpha q}}\right) \\
&\le A \cdot (\sigma(u))^{1 - \frac{1}{\alpha q}} = \frac{Ac}{\left(\ln \frac{1}{u}\right)^{\alpha - \frac{1}{q}}}
\end{aligned} \tag{4.6}$$

for all sufficiently small u, where

$$A = \frac{(2Mdq)^{1/q}}{11 - 2\sqrt{30}} \left(\left(\ln B^{1/d}(T_2 - T_1)\right)^{1/q} + \frac{c^{\frac{1}{\alpha q}}}{1 - \frac{1}{\alpha q}}\right).$$

Example 4.2. Consider the same function $\varphi(x)$ as in Example 4.1 and let $\sigma(h) = Dh^\alpha$, $h > 0$, $D > 0$, $0 < \alpha \le 1$, and $T_2 - T_1 > 1$. In this case, $\sigma^{(-1)}(u) = (u/D)^{1/\alpha}$, whence

$$\begin{aligned}
f_B^d(u) &\le \frac{1}{11 - 2\sqrt{30}} \int_0^{Du^\alpha} q^{1/q} \left(2Md \ln\left(B^{1/d}(T_2 - T_1)\left(\frac{D}{t}\right)^{1/\alpha}\right)\right)^{1/q} dt \\
&\le \frac{(2Mdq)^{1/q}}{11 - 2\sqrt{30}} \int_0^{Du^\alpha} \left[\left(\ln B^{1/d}(T_2 - T_1)\right)^{1/q} + \left(\frac{1}{\alpha} \ln \frac{D}{t}\right)^{1/q}\right] dt \\
&= \frac{(2Mdq)^{1/q}}{11 - 2\sqrt{30}} \left(Du^\alpha \left(\ln B^{1/d}(T_2 - T_1)\right)^{1/q} + \left(\frac{1}{\alpha}\right)^{1/q} \int_0^{Du^\alpha} \left(\ln \frac{D}{t}\right)^{1/q} dt\right)
\end{aligned}$$

and

$$\int_0^{Du^\alpha} \left(\ln \frac{D}{t}\right)^{1/q} dt = D \int_0^{u^\alpha} \left(\ln \frac{1}{t}\right)^{1/q} dt$$

for sufficiently small u. Since

$$\int_0^{u^\alpha} \left(\ln \frac{1}{t}\right)^{1/q} dt \le u^\alpha \left(\ln \frac{1}{u^\alpha}\right)^{1/q} \left(1 + \frac{1}{q \ln \frac{1}{u^\alpha}}\right) \le u^\alpha \left(\ln \frac{1}{u}\right)^{1/q} \alpha^{1/q} \left(1 + \frac{1}{q\alpha \ln \frac{1}{\varkappa}}\right)$$

for $u < \varkappa < \frac{1}{e}$, we have

$$f_B^d(u) \le C_1 u^\alpha + C_2 u^\alpha \left(\ln \frac{1}{u}\right)^{1/q} \le C_3 u^\alpha \left(\ln \frac{1}{u}\right)^{1/q}$$

for sufficiently small u, where C_1, C_2, and C_3 are some constants.

Now let $T = [T_1, T_2]$ for $-\infty < T_1 < T_2 < \infty$. Then

$$\frac{T_2 - T_1}{2u} \le N(u) \le \frac{T_2 - T_1}{2u} + 1$$

and the following result holds.

Corollary 4.1. *Let $X = \{X(t), t \in [T_1, T_2]\}$ be a separable process belonging to the space $\mathrm{Sub}_\varphi(\Omega)$. Assume that there exists a continuous increasing function $\sigma = \{\sigma(h), h \geq 0\}$ such that $\sigma(0) = 0$ and*

$$\sup_{t,s \in [T_1,T_2]:\, \rho(t,s) \leq h} \tau_\varphi(X(t) - X(s)) \leq \sigma(h). \tag{4.7}$$

Let $M \geq \max(1, \varphi^*(2)/\ln 2)$, $B > 1$, and $b > 1$. Assume that u is a number such that $(T_2 - T_1)/(2u) > 2$. Then

$$\mathsf{P}\left\{ \sup_{0<|t-s|\leq u} \frac{|X(t) - X(s)|}{\tilde{f}_B(|t-s|)} > y \right\} \leq \frac{4(b+1)(2u)^{M-1} B^{M-1}}{(b-1)(T_2 - T_1)^{M-1}(B^{M-1} - 1)} \exp\left\{ -\varphi^*\left(\frac{y}{b}\right) \right\}$$

for all $y > 2b$, where

$$\begin{aligned} \tilde{f}_B(u) &= \frac{1}{(11 - 2\sqrt{30})} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(2M \ln\left(B\left(\frac{T_2 - T_1}{2\sigma^{(-1)}(v)} + 1\right)\right)\right) dv \\ &\leq \frac{1}{11 - 2\sqrt{30}} \int_0^{\sigma(u)} \varphi^{*(-1)}\left(2M \ln\left(\frac{B(T_2 - T_1)}{\sigma^{(-1)}(v)}\right)\right) dv. \end{aligned}$$

5. Lipschitz spaces

Definition 5.1. A function $q = \{q(t), t \in \mathbb{R}\}$ is called a modulus of continuity if $q(t) \geq 0$, $q(0) = 0$, and $q(t) < q(t+s) \leq q(t) + q(s)$ for $t > 0$, $s > 0$.

Example 5.1. The function $q(t) = c|t|^\alpha$ with $c > 0$ and $0 < \alpha \leq 1$ is a modulus of continuity.

Definition 5.2. Let (T, ρ) be a metric space and let q be a modulus of continuity. The family of functions $\{x(t), t \in T\}$ such that

$$\sup_{\substack{t,s \in T \\ t \neq s}} \frac{|x(t) - x(s)|}{q(\rho(t,s))} < \infty \tag{5.1}$$

is called the Lipschitz space $\Lambda_q(T, \rho)$.

A family of functions $\{x(t), t \in T\}$ such that

$$\sup_{\rho(t,s) \leq h} |x(t) - x(s)| = o(q(h)), \qquad h \to 0,$$

is called the Lipschitz space $\Lambda_q^o(T, \rho)$.

Theorem 5.1. *Let $X = \{X(t), t \in T\}$ be a stochastic process satisfying all the assumptions of Theorem 3.1. If $f_B(u) \leq q(u)$ (or $f_B(u) = o(q(u))$), then the process X belongs to the space $\Lambda_q(T, \rho)$ (or to $\Lambda_q^o(T, \rho)$) with probability one and*

$$\begin{aligned} &\mathsf{P}\left\{ \sup_{0<\rho(t,s)\leq v} \frac{|X(t) - X(s)|}{q(\rho(t,s))} > y \right\} \\ &\qquad \leq \frac{4B^{M-1}(b+1)}{(N(v))^{M-1}(B^{M-1} - 1)(b-1)} \exp\left\{ -\varphi^*\left(\frac{y}{b}\right) \right\}. \end{aligned} \tag{5.2}$$

Theorem 5.1 follows from Theorem 3.1.

Corollary 5.1. *Let $X = \{X(t), t \in T\}$ be a stochastic process satisfying all the assumptions of Theorem 3.1 and let q be a modulus of continuity. If*

$$f_B^q(u) = \int_0^{\sigma(u)} \frac{\varphi^{*(-1)}\left(2M \ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right)}{q(v)} dv < \infty,$$

then the stochastic process X belongs to the space $\Lambda_q^o(T, \rho)$ with probability one.

Proof. Indeed,

$$f_B(u) \le c\int_0^{\sigma(u)} \frac{q(u)\varphi^{*(-1)}\left(2M\ln\left(BN\left(\sigma^{(-1)}(v)\right)\right)\right)}{q(v)}\,dv$$
$$\le q(u)cf_B^q(u) = o(q(u)),$$

and Corollary 5.1 follows from Theorem 5.1. □

6. An application to self-similar processes with stationary increments belonging to the space $\mathrm{Sub}_\varphi(\Omega)$

Let $H \in (0,1)$. Consider a centered square integrable process $Z_H = (Z_H(t)\colon t\in[0,1])$ whose covariance function is given by

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Assume that the process belongs to the space $\mathrm{Sub}_\varphi(\Omega)$. We say in this case that Z_H is a weakly self-similar process with stationary increments belonging to the space $\mathrm{Sub}_\varphi(\Omega)$ (an SSSI-$\mathrm{Sub}_\varphi(\Omega)$-process for short).

Remark 6.1. If a second order process Z_H with stationary increments is self-similar, that is, if the finite dimensional distributions of $Z_H(t)$ and $a^{-H}Z(at)$ are the same, then the covariance function of the process Z_H is necessarily equal to R_H.

Corollary 6.1. *Let Z_H be a weakly self-similar process with stationary increments belonging to the space $\mathrm{Sub}_\varphi(\Omega)$. If $M \ge \max(1, \varphi^*(2)/\ln 2)$, $B > 1$, $b > 1$, and $u \in (0, \frac14)$, then*

$$\mathsf{P}\left\{\sup_{0<|t-s|\le u} \frac{|Z_H(t)-Z_H(s)|}{b\tilde f_B(|t-s|)} > y\right\} \le \frac{4(b+1)B^{M-1}(2u)^{M-1}}{(b-1)(B^{M-1}-1)} \exp\left\{-\varphi^*\left(\frac{y}{b}\right)\right\}$$

for an arbitrary $y > 2b$, where

$$\tilde f_B(u) = \frac{1}{(11-2\sqrt{30})}\int_0^{u^H} \varphi^{*(-1)}\left(2M\ln\left(B\left(\frac{1}{2v^{1/H}}+1\right)\right)\right)dv$$
$$\le \frac{1}{11-2\sqrt{30}}\int_0^{u^H} \varphi^{*(-1)}\left(2M\ln\left(\frac{B}{v^{1/H}}\right)\right)dv.$$

Corollary 6.1 follows from Corollary 4.1 with $\sigma(u) = u^H$, $u \ge 0$.

Example 6.1. Let $1 < p \le 2$ and let $\varphi(x) = \frac{|x|^p}{p}$ if $|x|$ is sufficiently large. In this case, $\varphi^*(x) = \frac{|x|^r}{r}$ and $\varphi^{*(-1)}(x) = (rx)^{1/r}$, where $\frac1p + \frac1r = 1$. For the process Z_H, we have $\sigma(u) = u^H$, $u > 0$, and the inverse function is equal to $\sigma^{(-1)}(u) = (u)^{\frac1H}$. According to Corollary 6.1, $u \in (0, \frac14)$. Hence

$$\tilde f_B(u) \le \frac{1}{11-2\sqrt{30}}\int_0^{u^H} r^{1/r}\left(2M\ln\left(B\left(\frac1t\right)^{1/H}\right)\right)^{1/r} dt$$
$$\le \frac{(2Mr)^{1/r}}{11-2\sqrt{30}}\int_0^{u^H}\left[(\ln B)^{1/r} + \left(\frac1H\ln\frac1t\right)^{1/r}\right]dt$$
$$= \frac{(2Mr)^{1/r}}{11-2\sqrt{30}}\left(u^H(\ln B)^{1/r} + \left(\frac1H\right)^{1/r}\int_0^{u^H}\left(\ln\frac1t\right)^{1/r}dt\right).$$

Since

$$\int_0^{u^H} \left(\ln \frac{1}{t}\right)^{1/r} dt \le u^H \left(\ln \frac{1}{u}\right)^{1/r} H^{1/r} \left(1 + \frac{1}{rH \ln \frac{1}{\varkappa}}\right),$$

for $u < \varkappa < e^{-1}$, we obtain

$$\tilde{f}_B(u) \quad \le \quad \left[\frac{(2Mr)^{1/r}}{11 - 2\sqrt{30}} (\ln B)^{1/r}\right] u^H + \left[\frac{(2Mr)^{1/r}}{11 - 2\sqrt{30}} \left(1 + \frac{1}{rH \ln \frac{1}{\varkappa}}\right)\right] u^H \left(\ln \frac{1}{u}\right)^{1/r}$$

if u is sufficiently small, whence

$$\tilde{f}_B(u) \le C_B\, u^H \left(\ln \frac{1}{u}\right)^{1/r},$$

where

$$C_B = \frac{(2Mr)^{1/r}}{11 - 2\sqrt{30}} \left((\ln B)^{1/r} + 1 + \frac{1}{rH \ln \frac{1}{\varkappa}}\right). \tag{6.1}$$

Collecting the above bounds we prove the following result.

Theorem 6.1. *Let $0 < p \le 1$. Assume that Z_H is a weakly self-similar process with stationary increments belonging to the space $\mathrm{Sub}_\varphi(\Omega)$ with $\varphi(x) = |x|^p/p$. Then the stochastic process Z_H belongs to the space $\Lambda_q(T, \rho)$ with probability one, where $T = [0, 1]$, $\rho(t, s) = |t - s|$, $q(x) = C_B x^H (\ln x^{-1})^{1/r}$, and where the constant C_B is defined by (6.1).*

Moreover, if $u \in (0, \frac{1}{4})$, then the norm in this space satisfies the inequality

$$\begin{aligned} &\mathsf{P}\left\{\sup_{0<|t-s|\le u} \frac{|Z_H(t) - Z_H(s)|}{C_B |t-s|^H \left(\ln \frac{1}{|t-s|}\right)^{\frac{1}{r}}} > y\right\} \\ &\qquad \le \frac{4(b+1)B^{M-1}(2u)^{M-1}}{(b-1)(B^{M-1}-1)} \exp\left\{-\frac{y^r}{rb^r}\right\} \end{aligned} \tag{6.2}$$

for all $y > 2b$.

Remark 6.2. If Z_H is a Gaussian process (thus Z_H is a fractional Brownian motion), then all the assumptions of Theorem 6.1 hold with $p = 2$, $r = 2$, and

$$q(x) = \tilde{C}_B x^H \left(\ln \frac{1}{x}\right)^{\frac{1}{2}}, \qquad \tilde{C}_B = \frac{2\sqrt{M}}{11 - 2\sqrt{30}} \left((\ln B)^{1/2} + 1 + \frac{1}{2H \ln \frac{1}{\varkappa}}\right).$$

If $u \in (0, \frac{1}{4})$ and $y > 2b$, then

$$\begin{aligned} &\mathsf{P}\left\{\sup_{0<|t-s|<u} \frac{|Z_H(t) - Z_H(s)|}{\tilde{C}_B |t-s|^H \left(\ln \frac{1}{|t-s|}\right)^{1/2}} > y\right\} \\ &\qquad \le \frac{4(b+1)B^{M-1}(2u)^{M-1}}{(b-1)(B^{M-1}-1)} \exp\left\{-\frac{y^2}{2b^2}\right\}. \end{aligned} \tag{6.3}$$

Remark 6.3. To make the bound more precise, one can choose the constants b, B, and M such that the right-hand side of (6.2) is minimal.

Example 6.2. Let

$$\varphi(x) = \begin{cases} \frac{|x|^2}{p}, & |x| < 1, \\ \frac{|x|^p}{p}, & |x| \ge 1. \end{cases}$$

In this case, $\varphi^*(x) = |x|^r/r$ for $|x| \ge 1$ and $\varphi^{*(-1)}(x) = (rx)^{1/r}$ for $|x| \ge 1/r$, where $p^{-1} + r^{-1} = 1$.

If

$$2M \ln \left(B \left(\frac{1}{u} \right)^{1/H} \right) \geq \frac{1}{r}, \quad \text{or} \quad 0 < u \leq B^H \exp\left\{ -\frac{H}{2Mr} \right\},$$

or if

$$u \in \left(0, \min \left(B^H \exp\left\{ -\frac{H}{2Mr} \right\}, \frac{1}{4} \right) \right),$$

then we obtain the same bound (and by using the same method) as in Example 6.1. Here

$$\tilde{f}_B(u) \leq C_B u^H \left(\ln \frac{1}{u} \right)^{1/r}.$$

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