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# FUNCTIONAL LIMIT THEOREMS FOR STOCHASTIC INTEGRALS WITH APPLICATIONS TO RISK PROCESSES AND TO VALUE PROCESSES OF SELF-FINANCING STRATEGIES IN A MULTIDIMENSIONAL MARKET. II

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Abstract. We study sufficient conditions for the convergence of value processes of self-financial strategies in the case of a $d$-dimensional financial market with continuous time. The conditions for the weak convergence of value processes are discussed in detail for the Black–Scholes market model. We also consider the "inverse" problem for the weak convergence of risk-minimizing strategies.

## 1. Introduction

In the current paper, we consider the convergence of value processes of self-financing strategies for a $d$-dimensional market with continuous time as the strategies converge. We find conditions for the weak convergence of the corresponding probability measures in the Skorokhod space $D[0,T]$. The proofs below use the results of [1] concerning the weak convergence of stochastic integrals with respect to semimartingales. Conditions for the convergence of value processes are discussed in detail for the Black–Scholes model.

In Section 5, we consider the inverse problem. Namely, we study the convergence of strategies minimizing the mean square and local square risk as the sequence of the corresponding contingent claims converge. A similar problem is considered in [2] and [3]. The paper [2] deals with the one-dimensional case. For this case, conditions are found in [2] for the convergence of strategies minimizing the local square risk in the space $L^2$ as the sequence of contingent claims, as well as discounted price processes, converges. The paper [3] studies a $d$-dimensional finance market with continuous time. However the problem is solved in [3] only for a certain class of discounted price processes, namely for those processes that satisfy the so-called *structure condition*: $B^i \ll \langle M^i \rangle$. Moreover, the problem on the convergence of a single component of a strategy $\xi_n$ is still open. We use some of the results of the paper [1] and determine the conditions for such a convergence.

## 2. A model of a financial market with continuous time

We briefly describe a model of a financial market with continuous time considered in [6]. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space with a given filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying

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the standard assumptions on the right continuity and completeness. Moreover we assume that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Let

$$S(t) := \left(S^0(t), S^1(t), \dots, S^d(t)\right), \qquad t \in [0,T],$$

be an $\mathcal{F}_t$-adapted $d$-dimensional price process for risky assets whose components are positive almost surely. As a rule we will assume that $S^0(t)$ is a riskfree asset. Then the discounted price process for the risky asset is defined by $X(t) := S(t)/S^0(t)$, $t \in [0,T]$.

In this model, $S^i(t)$ for all $1 \le i \le d$ means the price for the asset $i$ at the moment $t$, while $S^0(t)$ is the price of a riskfree asset (say, the price process for a bank account). All the processes are assumed to be right continuous and to have limits on the left.

In what follows we assume that the process $X(t)$ is a square integrable semimartingale admitting the following decomposition:

$$X^i(t) = X^i(0) + B^i(t) + M^i(t), \tag{1}$$

where

1) $M^i$ is a square integrable martingale such that $M^i(0) = 0$;
2) $B^i$ is a predictable process of bounded variation such that $B^i(0) = 0$.

*Notation* 1. In what follows we use the following notation:

$$\xi(t) \cdot X(t) := \sum_{i=1}^d \xi^i(t) X^i(t) \quad \text{and} \quad \int_0^t (\xi(s), dX(s)) := \sum_{i=1}^d \int_0^t \xi^i(s)\, dX^i(s).$$

**Definition 1.** Any $(d+1)$-dimensional process $(\xi^0(t), \xi^1(t), \dots, \xi^d(t))$ is called an investor strategy or an investor portfolio if

1) $(\xi^1(t), \dots, \xi^d(t))$ is a predictable process;
2) $\xi^0(t)$ is an $\mathcal{F}_t$-measurable process;
3) for each $1 \le i \le d$,

$$\mathsf{E}\left[\int_0^T \left(\xi^i(u)\right)^2 \, d\langle M^i\rangle(u) + \left(\int_0^T \left|\xi^i(u)\right| \, d\left|B^i\right|(u)\right)^2\right] < \infty;$$

4) the process $V(t) := \xi^0(t) + \xi(t) \cdot X(t)$ is square integrable, that is, $V(t) \in L^2(\mathsf{P})$ for all $0 \le t \le T$.

Note that $\xi^i(t)$ is the total amount of shares of the asset $i$ in the investor portfolio at the moment $t$ if $1 \le i \le d$, while $\xi^0(t)$ is the amount of units of the riskfree asset. Then $V(t)$ is the value of the portfolio. Condition 1) means that the investor should determine the amount of shares for every asset in the portfolio prior to the next change of prices in the financial market.

**Definition 2.** The process $V(t)$ defined above is called the value process for the strategy $(\xi^0, \xi)$. The price process for the strategy $(\xi^0, \xi)$ is defined by

$$C(t) := V(t) - \sum_{i=1}^d \int_0^t \xi^i(s)\, dX^i(s)$$

for all $0 \le t \le T$.

**Definition 3.** A strategy $(\xi^0, \xi)$ is called self-financing if $C(t)$ is a constant.

**Definition 4.** A strategy $(\xi^0, \xi)$ is called self-financing in the mean sense if

$$\mathsf{E}[C(t) - C(t) \mid \mathcal{F}_t] = 0.$$

Below we provide some important examples of self-financing and self-financing in the mean sense strategies.

**Example 1.** The strategy

$$\left(\xi^0(t), \xi^1(t), \dots, \xi^d(t)\right) = (c_0, c_1, \dots, c_d), \qquad 0 \le t \le T,$$

where the $c_i$, $0 \le i \le d$, are some constants, is self-financing (it is known as the "buy and hold" strategy). Indeed,

$$\begin{aligned}
C(t) &= V(t) - \sum_{i=1}^{d} \int_0^t \xi^i(s)\, dX^i(s) \\
&= \xi^0(t) + \sum_{i=1}^{d} \xi^i(t) \cdot X^i(t) - \sum_{i=1}^{d} \int_0^t \xi^i(s)\, dX^i(s) \\
&= c_0 + \sum_{i=1}^{d} c_i X^i(t) - \sum_{i=1}^{d} c_i \left(X^i(t) - X^i(0)\right) \\
&= c_0 + \sum_{i=1}^{d} c_i X^i(0) = \text{const}.
\end{aligned}$$

**Example 2.** A strategy

$$\left(\xi^0(t), \xi^1(t), \dots, \xi^d(t)\right) = \left(\xi^0(t), c_1, \dots, c_d\right), \qquad 0 \le t \le T,$$

where the $c_i$, $0 \le i \le d$, are some constants and where $\xi^0(t)$ and $X(t)$ are martingales, is self-financing in the mean sense. Indeed,

$$\begin{aligned}
&\mathsf{E}[C(T) - C(t) \mid \mathcal{F}_t] \\
&\quad = \mathsf{E}\left[V(T) - \sum_{i=1}^{d} \int_0^T \xi^i(s)\, dX^i(s) - \left(V(t) - \sum_{i=1}^{d} \int_0^t \xi^i(s)\, dX^i(s)\right) \,\middle|\, \mathcal{F}_t\right] \\
&\quad = \mathsf{E}\Bigg[\xi^0(T) + \sum_{i=1}^{d} \xi^i(T) X^i(T) - \sum_{i=1}^{d} \int_0^T \xi^i(s)\, dX^i(s) \\
&\qquad\qquad - \left(\xi^0(t) + \sum_{i=1}^{d} \xi^i(t) \cdot X^i(t) - \sum_{i=1}^{d} \int_0^t \xi^i(s)\, dX^i(s)\right) \,\Bigg|\, \mathcal{F}_t\Bigg] \\
&\quad = \mathsf{E}\Bigg[\xi^0(T) + \sum_{i=1}^{d} c_i X^i(T) - \sum_{i=1}^{d} c_i \left(X^i(T) - X^i(0)\right) \\
&\qquad\qquad - \left(\xi^0(t) + \sum_{i=1}^{d} c_i X^i(t) - \sum_{i=1}^{d} c_i \left(X^i(t) - X^i(0)\right)\right) \,\Bigg|\, \mathcal{F}_t\Bigg] \\
&\quad = \mathsf{E}\left[\xi^0(T) - \xi^0(t) + \sum_{i=1}^{d} c_i X^i(0) - \sum_{i=1}^{d} c_i X^i(0) \,\middle|\, \mathcal{F}_t\right] \\
&\quad = \mathsf{E}\left[\xi^0(T) - \xi^0(t) \mid \mathcal{F}_t\right] = 0.
\end{aligned}$$

**Example 3.** Consider an arbitrage free market. Let the objective measure be risk neutral. This means that the processes $X^i(t)$ are martingales with respect to the measure $\mathsf{P}$. Then the strategy

$$\left(\xi^0(t), \xi^1(t), \dots, \xi^d(t)\right)$$

is self-financing in the mean sense if and only if $V(t)$ is a martingale with respect to the measure $\mathsf{P}$. Indeed,

$$\begin{aligned}
&\mathsf{E}[C(T) - C(t) \mid \mathcal{F}_t] \\
&\quad = \mathsf{E}\left[V(T) - \sum_{i=1}^d \int_0^T \xi^i(s)\, dX^i(s) - \left(V(t) - \sum_{i=1}^d \int_0^t \xi^i(s)\, dX^i(s)\right) \Bigg| \mathcal{F}_t\right] \\
&\quad = \mathsf{E}\left[V(T) - V(t) + \sum_{i=1}^d \int_t^T \xi^i(s)\, dX^i(s) \Bigg| \mathcal{F}_t\right] \\
&\quad = \mathsf{E}\left[V(T) - V(t) \mid \mathcal{F}_t\right] + \sum_{i=1}^d \mathsf{E}\left[\int_t^T \xi^i(s)\, dX^i(s) \Bigg| \mathcal{F}_t\right] \\
&\quad = \mathsf{E}\left[V(T) - V(t) \mid \mathcal{F}_t\right] = 0.
\end{aligned}$$

## 3. Weak convergence of value processes for self-financing strategies

In this section, we consider the problem of convergence of value processes for self-financing strategies as the strategies and the corresponding price processes converge. First we set the assumptions for this problem.

Let $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t\in\mathbf{R}_+}, P^n)$ be a sequence of stochastic bases. In what follows the symbol $E^n$ denotes the expectation with respect to the measure $P^n$. Let

$$\{X_n(t), \mathcal{F}^n_t, t \in \mathbf{R}_+\} = \left\{\left(X^1_n(t), X^2_n(t), \dots, X^d_n(t)\right), \mathcal{F}^n_t, t \in \mathbf{R}_+, n \in \mathbf{Z}_+\right\}$$

be a sequence of $d$-dimensional semimartingales whose components admit the following decomposition:

$$X^i_n(t) = X^i_n(0) + M^i_n(t) + B^i_n(t), \qquad 1 \le i \le d, \tag{2}$$

where $\{M^i_n(t), \mathcal{F}^n_t, t \in \mathbf{R}_+, n \in \mathbf{Z}_+\}$ is a sequence of square integrable martingales, $M^i_n(0) = 0$, and where $\{B^i_n(t), t \in \mathbf{R}_+, n \in \mathbf{Z}_+\}$ is a sequence of processes of bounded variation, $B^i_n(0) = 0$. Let $\mu^i_n(t) := \langle M^i_n\rangle(t)$ be the square characteristics of the corresponding martingales.

We further assume that the sequence of the investor strategies

$$\left\{\xi^i_n(t), \mathcal{F}^n_t, t \in \mathbf{R}_+, n \in \mathbf{Z}_+, 1 \le i \le d\right\}$$

is such that

$$E^n \int_0^T \left(\xi^i_n\right)^2(t)\, d\mu^i_n(t) < \infty, \qquad \int_0^T \xi^i_n(t)\, d|B^i_n|(t) < \infty,$$

$P^n$-almost surely and for all $n \in \mathbf{Z}_+$ and $1 \le i \le d$.

Below we use the notation introduced in the first part of this paper [1].

Fix some countable and everywhere dense in the interval $[0, T]$ set $I \subset \mathbf{R}_+$. Put $T_I := I \cap [0, T]$. We also denote by $L_{T_I}$ the class of all sequences

$$\alpha_k = \{0 = t_{0k} < t_{1k} < \dots < t_{k_T k} < T\}$$

of finite partitions of the interval $[0, T]$ satisfying the following two conditions:

1) $\alpha_k \subseteq \alpha_{k+1} \subseteq T_I$;
2) for all $t \in T_I$, there exists $k(t)$ such that $t \in \alpha_k$ for $k > k(t)$.

Put $\triangle_{jk} x := x(t_{j\,k}) - x(t_{j-1\,k})$, $\Delta x(t) = x(t) - x(t-)$, and

$$\omega_{j\,k} x = \sup_{t_{j-1\,k} \le s < t \le t_{j\,k}} |x(t) - x(s)|, \qquad k_t = \sup\{j \colon t_{jk} \le t\}.$$

The symbol "$\Rightarrow$" stands for the weak convergence of finite-dimensional distributions, while the symbol "$\xrightarrow{D[0,T]}$" denotes the weak convergence of probability measures on the interval $[0,T]$ in the Skorokhod topology.

We also introduce the following notation:

$$\triangle_D(x(\cdot),\delta,T) := \sup_{0\le t<t'<t''<t+\delta\le T} \left(|x(t'')-x(t')| \wedge |x(t')-x(t)|\right)$$

and

$$\mathcal{T}(\mathcal{F}^n) = \{0\le\tau\le T\colon \tau \text{ is an } \mathcal{F}.^n\text{-stopping time}\}.$$

**Theorem 1.** *Let the strategies* $\{\xi_n^i(t),\mathcal{F}_t^n, t\in\mathbf{R}_+, n\in\mathbf{Z}_+, 1\le i\le d\}$ *be self-financing. Assume that either the set of assumptions* ($A_1$) *or* ($A_2$) *holds where*

($A_1$)

1) $(\xi_n(t),X_n(t)),\ t\in T_I \Rightarrow (\xi_0(t),X_0(t)),\ t\in T_I;$

*for all* $1\le i\le d$,

2) $\lim\limits_{C\to\infty}\limsup\limits_{n\to\infty} P^n\left(\sup\limits_{0\le t\le T}\left|\xi_n^i(t)\right|\ge C\right)=0;$

3) $\lim\limits_{C\to\infty}\limsup\limits_{n\to\infty} P^n\left(\sup\limits_{0\le t\le T}\left|X_n^i(t)\right|\ge C\right)=0;$

4) $\lim\limits_{\delta\to 0}\limsup\limits_{n\to\infty}\sup\limits_{\sigma\in\mathcal{T}(\mathcal{F}^n)} P^n\left(\sup\limits_{t\le\delta}\left|\xi_n^i(t+\sigma)-\xi_n^i(\sigma)\right|\ge\eta\right)=0;$

5) $\lim\limits_{\delta\to 0}\limsup\limits_{n\to\infty}\sup\limits_{\sigma\in\mathcal{T}(\mathcal{F}^n)} P^n\left(\sup\limits_{t\le\delta}\left|X_n^i(t+\sigma)-X_n^i(\sigma)\right|\ge\eta\right)=0.$

($A_2$)

1) $(\xi_n(t),M_n(t),B_n(t),\mu_n(t)),\ t\in T_i \Rightarrow (\xi_0(t),M_0(t),B_0(t),\mu_0(t)),\ t\in T_i;$

*for all* $1\le i\le d$,

2) $\sup\limits_{n\ge 0} E^n\displaystyle\int_0^T \left(\xi_n^i\right)^2(s)\,d\mu_n^i(s)<\infty;$

3) $\lim\limits_{C\to\infty}\limsup\limits_{n\to\infty} P^n\left\{\sup\limits_{0\le t\le T}\left|\xi_n^i(t)\right|\ge C\right\}=0;$

4) $\lim\limits_{k\to\infty}\limsup\limits_{n\to\infty} E^n\displaystyle\sum_{j=1}^{k_T}\omega_{j\,k}\xi_n^i\omega_{j\,k}B_n^i=0;$

5) $\lim\limits_{\delta\to 0}\limsup\limits_{n\to\infty} P^n\left\{\triangle_D\left(\left|B_n^i\right|(\cdot),\delta,T\right)>\alpha\right\}=0;$

6) $\lim\limits_{k\to\infty}\limsup\limits_{n\to\infty} E^n\displaystyle\sum_{j=1}^{k_T}\omega_{j\,k}\xi_n^i\omega_{j\,k}\mu_n^i=0;$

7) $\lim\limits_{\delta\to 0}\limsup\limits_{n\to\infty}\sup\limits_{\sigma\in\mathcal{T}(\mathcal{F}^n)} E^n\left(\mu_n^i(\sigma+\delta)-\mu_n^i(\sigma)\right)=0;$

8) $\limsup\limits_{n\to\infty} P^n\left\{\sup\limits_{t\in[0,T]}\left|\Delta B_n^i(t)\right|>\alpha\right\}=0$ *for all* $\alpha>0$.

*Then* $V_n(\cdot)\xrightarrow{D[0,T]} V_0(\cdot),\ n\to\infty.$

*Proof.* 1. First we assume that the set of conditions $(A_1)$ holds.

Condition 1) implies the weak convergence of finite-dimensional distributions:

$$V_n(t) \Rightarrow V_0(t), \qquad n \to \infty,\ t \in T_I. \tag{3}$$

Now we use conditions 2)–4) to prove that the sequence of processes $V_n(t)$ is dense:

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\sigma\in\mathcal{T}(\mathcal{F}^n)} P^n\left(\sup_{t\le\delta}|V_n(t+\sigma)-V_n(\sigma)|\ge\eta\right)\\
&\quad=\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\sigma\in\mathcal{T}(\mathcal{F}^n)} P^n\left(\sup_{t\le\delta}\left|\xi_n^0(t+\sigma)-\xi_n^0(\sigma)+\sum_{i=1}^d\xi_n^i(t+\sigma)X_n^i(t+\sigma)\right.\right.\\
&\qquad\qquad\left.\left.-\sum_{i=1}^d\xi_n^i(\sigma)X_n^i(\sigma)\right|\ge\eta\right)\\
&\quad\le\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\sigma\in\mathcal{T}(\mathcal{F}^n)} P^n\left(\sup_{t\le\delta}\left|\xi_n^0(t+\sigma)-\xi_n^0(\sigma)\right|\ge\frac{\eta}{2d+1}\right)\\
&\qquad+\sum_{i=1}^d\lim_{C\to\infty}\left[\limsup_{n\to\infty}P^n\left(\sup_{0\le t\le T}\left|\xi_n^i(t)\right|\ge C\right)\right.\\
&\qquad\qquad+C\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\sigma\in\mathcal{T}(\mathcal{F}^n)} P^n\left(\sup_{t\le\delta}\left|X_n^i(t+\sigma)-X_n^i(\sigma)\right|\ge\frac{\eta}{2d+1}\right)\\
&\qquad\qquad+\limsup_{n\to\infty}P^n\left(\sup_{0\le t\le T}\left|X_n^i(t)\right|\ge C\right)\\
&\qquad\qquad\left.+C\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\sigma\in\mathcal{T}(\mathcal{F}^n)} P^n\left(\sup_{t\le\delta}\left|\xi_n^i(t+\sigma)-\xi_n^i(\sigma)\right|\ge\frac{\eta}{2d+1}\right)\right]\\
&\quad=0.
\end{aligned} \tag{4}$$

Then Theorem 1 (Chapter 2, Section 6, §3 of [4]) implies that the sequence $V_n(\cdot)$ is relatively compact in the Skorokhod topology. Now we derive the convergence in the Skorokhod topology of the sequence of stochastic integrals from the weak convergence of their finite-dimensional distributions.

2. Now we consider the case where all the conditions of the set $(A_2)$ hold. Since the strategies $\{\xi_n^i(t),\mathcal{F}_t^n,t\in\mathbf{R}_+,n\in\mathbf{Z}_+,1\le i\le d\}$ are self-financing,

$$V(t)=C+\int_0^t(\xi_n(s),dX_n(s)),$$

where $C$ is a constant. Applying Theorem 5 of [1] we complete the proof. □

## 4. The weak convergence of value processes in the Black–Scholes market

Consider a particular case where all the processes are defined on the same stochastic basis and the price process is defined as follows:

$$\begin{aligned}
S_n^i(t)&=S_0^i\exp\left\{\int_0^t\mu_n^i(s)\,ds+\int_0^t\sigma_n^i(s)\,dW^i(s)\right\},\qquad 1\le i\le d,\ n\in\mathbf{Z}_+,\\
S_n^0(t)&=\exp\left\{\int_0^t r_n(s)\,ds\right\},\qquad n\in\mathbf{Z}_+,\ t\in[0,T],
\end{aligned} \tag{5}$$

where $W=(W^1,W^2,\dots,W^d)$ is a vector-valued Wiener process, $S_0^i$ are constants, and where $\mu_n^i(s)$, $\sigma_n^i(s)$, and $r_n(s)$ are stochastic processes for which the integrals in equalities (5) are well defined. We also assume that

$$\mathsf{E}\int_0^T \left(\sigma_n^i(s)\right)^2 ds < \infty, \qquad n \in \mathbf{Z}_+.$$

Then the discounted price process is defined as follows:

$$\begin{gathered} X_n^i(t) = S_0^i \exp\left\{\int_0^t \hat{\mu}_n^i(s)\,ds + \int_0^t \sigma_n^i(s)\,dW^i(s)\right\}, \qquad 1 \le i \le d,\ n \in \mathbf{Z}_+, \\ \hat{\mu}_n^i(s) = \mu_n^i(s) - r_n(s), \qquad n \in \mathbf{Z}_+. \end{gathered} \tag{6}$$

**Theorem 2.** *Let the discounted price processes be defined by equalities* (6). *Assume that*

1) $\xi_n(t),\ t \in T_I,\ \Rightarrow\ \xi_0(t),\ t \in T_I;$

*for all* $1 \le i \le d$,

2) $\displaystyle\int_0^T \left|\hat{\mu}_n^i(s) - \hat{\mu}_0^i(s)\right| ds \xrightarrow{\mathsf{P}} 0,\ \mathsf{E}\int_0^T \left(\sigma_n^i(s) - \sigma_0^i(s)\right)^2 ds \to 0,\ n \to \infty,$

3) $\displaystyle\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)} \mathsf{P}\left(\int_\tau^{\delta+\tau} \left|\hat{\mu}_n^i(s)\right| ds \ge \eta\right) = 0,\ \eta > 0;$

4) $\displaystyle\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)} \mathsf{P}\left(\int_\tau^{\delta+\tau} (\sigma_n^i(s))^2 ds \ge \eta\right) = 0,\ \eta > 0;$

5) $\displaystyle\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}|\xi_n^i(t)| \ge C\right\} = 0;$

6) $\displaystyle\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)} \mathsf{P}\left(\sup_{t\le\delta}\left|\xi_n^i(t+\tau) - \xi_n^i(\tau)\right| \ge \eta\right) = 0.$

*Then* $V_n(\cdot) \xrightarrow{D[0,T]} V_0(\cdot)$, $n \to \infty$.

*Proof.* Condition 2) implies that, for all $1 \le i \le d$,

$$\int_0^t \sigma_n^i(s)\,dW^i(s) \xrightarrow{\mathsf{P}} \int_0^t \sigma_0^i(s)\,dW^i(s), \qquad n \to \infty,\ t \in T_I. \tag{7}$$

Then conditions 1), 2), equalities (7), and Theorem 4.4 of [5] yield

$$\begin{aligned} &\left(\xi_n(t), \left\{\int_0^t \hat{\mu}_n^i(s)\,ds, \int_0^t \sigma_n^i(s)\,dW^i(s)\right\}_{1\le i\le d}\right) \\ &\qquad \Rightarrow \left(\xi_0(t), \left\{\int_0^t \hat{\mu}_0^i(s)\,ds, \int_0^t \sigma_0^i(s)\,dW^i(s)\right\}_{1\le i\le d}\right), \qquad t \in T_I. \end{aligned}$$

Clearly, all the conditions of the set $(A_1)$ follow from conditions 1)–6), whence we obtain the weak convergence

$$V_n(t) \Rightarrow V(t) \quad \text{as } n \to \infty$$

for $t \in T_I$.

Now we prove that the sequence of processes $V_n(t)$ is dense:

$$\begin{aligned}
&\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}|V_n(t+\tau)-V_n(\tau)|\ge\eta\right)\\
&\quad=\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\Bigg(\sup_{t\le\delta}\Bigg|\xi_n^0(t+\tau)-\xi_n^0(\tau)\\
&\qquad\qquad+\sum_{i=1}^d\xi_n^i(t+\tau)X_n^i(t+\tau)-\sum_{j=1}^d\xi_n^i(\tau)X_n^i(\tau)\Bigg|\ge\eta\Bigg)\\
&\quad\le\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|\xi_n^0(t+\tau)-\xi_n^0(\tau)\right|\ge\frac{\eta}{2d+1}\right)\\
&\qquad+\sum_{i=1}^d\lim_{C\to\infty}\Bigg[\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}\left|\xi_n^i(t)\right|\ge C\right)\\
&\qquad\qquad+C\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|X_n^i(t+\tau)-X_n^i(\tau)\right|\ge\frac{\eta}{2d+1}\right)\\
&\qquad\qquad+\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}\left|X_n^i(t)\right|\ge C\right)\\
&\qquad\qquad+C\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|\xi_n^i(t+\tau)-\xi_n^i(\tau)\right|\ge\frac{\eta}{2d+1}\right)\Bigg]\\
&\quad=\sum_{i=1}^d\lim_{C\to\infty}\Bigg[C\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|X_n^i(t+\tau)-X_n^i(\tau)\right|\ge\frac{\eta}{2d+1}\right)\\
&\qquad\qquad+\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}\left|X_n^i(t)\right|\ge C\right)\Bigg].
\end{aligned}$$

According to condition 2), we have for all $1\le i\le d$ that

$$\begin{aligned}
&\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}\left|X_n^i(t)\right|\ge C\right)\\
&\quad=\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}S_0^i\exp\left\{\int_0^t\hat\mu_n^i(s)\,ds+\int_0^t\sigma_n^i(s)\,dW^i(s)\right\}\ge C\right)\\
&\quad=\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}\left|\int_0^t\hat\mu_n^i(s)\,ds+\int_0^t\sigma_n^i(s)\,dW^i(s)\right|\ge\ln\left(\frac{C}{S_0^i}\right)\right)\\
&\quad\le\lim_{C\to\infty}\limsup_{n\to\infty}\Bigg[\mathsf{P}\left(\sup_{0\le t\le T}\left|\int_0^t\hat\mu_n^i(s)\,ds\right|\ge\frac12\ln\left(\frac{C}{S_0^i}\right)\right)\\
&\qquad\qquad+\mathsf{P}\left(\sup_{0\le t\le T}\left|\int_0^t\sigma_n^i(s)\,dW^i(s)\right|\ge\frac12\ln\left(\frac{C}{S_0^i}\right)\right)\Bigg]\\
&\quad\le\lim_{\hat C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left(\int_0^T\left|\hat\mu_n^i(s)\right|\,ds\ge\hat C\right)\\
&\qquad+\lim_{\acute C\to\infty}\limsup_{n\to\infty}\frac{1}{\acute C}\int_0^t\mathsf{E}\left(\sigma_n^i(s)\right)^2\,ds\\
&\quad=0
\end{aligned}\tag{8}$$

and

$$
\begin{aligned}
&\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|X_n^i(t+\tau)-X_n^i(\tau)\right|\ge\eta\right)\\
&\quad=\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\Bigg(\sup_{t\le\delta}\Bigg|S_0^i\exp\left\{\int_0^\tau\hat{\mu}_n^i(s)\,ds+\int_0^\tau\sigma_n^i(s)\,dW^i(s)\right\}\\
&\qquad\qquad-S_0^i\exp\Bigg\{\int_0^{t+\tau}\hat{\mu}_n^i(s)\,ds\\
&\qquad\qquad\qquad+\int_0^{t+\tau}\sigma_n^i(s)\,dW^i(s)\Bigg\}\Bigg|\ge\eta\Bigg)\\
&\quad\le\lim_{C\to\infty}\Bigg[\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}S_0^i\exp\left\{\int_0^t\hat{\mu}_n^i(s)\,ds+\int_0^t\sigma_n^i(s)\,dW^i(s)\right\}\ge C\right)\\
&\qquad+\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\Bigg(\sup_{t\le\delta}\Bigg|\exp\Bigg\{\int_\tau^{t+\tau}\hat{\mu}_n^i(s)\,ds\\
&\qquad\qquad+\int_\tau^{t+\tau}\sigma_n^i(s)\,dW^i(s)\Bigg\}-1\Bigg|\ge\frac{\eta}{C}\Bigg)\Bigg]\\
&\quad=\lim_{C\to\infty}\Bigg[\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\Bigg(\sup_{t\le\delta}\left|\int_\tau^{t+\tau}\hat{\mu}_n^i(s)\,ds+\int_\tau^{t+\tau}\sigma_n^i(s)\,dW^i(s)\right|\\
&\qquad\qquad\times\exp\Bigg\{\Bigg|\int_\tau^{t+\tau}\hat{\mu}_n^i(s)\,ds\\
&\qquad\qquad\qquad+\int_\tau^{t+\tau}\sigma_n^i(s)\,dW^i(s)\Bigg|\Bigg\}\ge\frac{\eta}{C}\Bigg)\Bigg].
\end{aligned}
\tag{9}
$$

Here we used bounds (8) and equality

$$
\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left(\sup_{0\le t\le T}\left|S_0^i\exp\left\{\int_0^t\hat{\mu}_n^i(s)\,ds+\int_0^t\sigma_n^i(s)\,dW^i(s)\right\}\right|\ge C\right)=0
$$

in (9).

Now, recalling (8), we see that it remains to prove that

$$
P:=\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|\int_\tau^{t+\tau}\hat{\mu}_n^i(s)\,ds+\int_\tau^{t+\tau}\sigma_n^i(s)\,dW^i(s)\right|\ge\eta\right)=0
$$

for all $\eta>0$. For this, we use the Lenglart inequality (Theorem 3 in [4], p. 57) and conditions 3) and 4):

$$
\begin{aligned}
P&\le\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|\int_\tau^{t+\tau}\hat{\mu}_n^i(s)\,ds\right|\ge\frac{\eta}{2}\right)\\
&\quad+\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in\mathcal{T}(\mathcal{F}^n)}\mathsf{P}\left(\sup_{t\le\delta}\left|\int_\tau^{t+\tau}\sigma_n^i(s)\,dW^i(s)\right|\ge\frac{\eta}{2}\right)\\
&=0.
\end{aligned}
$$

Therefore Theorem 1 in [4] (see Chapter 2, Section 6, §3 in [4]) yields that the sequence $V_n(t)$ is relatively compact in the Skorokhod topology. Finally, we derive the convergence of the sequence of stochastic integrals in the Skorokhod topology from the convergence of their finite-dimensional distributions. □

## 5. Risk minimizing strategies and their stability

In this section, we consider the hedging problem for a contingent claim $H$ at a moment $T$. The simplest example of such a contingent claim is the discounted European option of the form $H=(X_T-K)^+$ for an underlying asset $\{X_t, t\in[0,T]\}$. The hedging problem in our case is to find a strategy that generates the amount $H$ at the moment $T$ and, simultaneously, minimizes a certain measure of risk. We also consider the stability of such strategies under the convergence of a sequence of contingent claims $H_n$ (the mode of their convergence will be specified later).

Consider a model of a financial market constructed above and let the corresponding value process be given by (1). Let $H$ be a contingent claim with expiration time $T$, that is, $H$ is a random variable such that $H\in L^2(\Omega,\mathcal{F}_T,\mathsf{P})$.

**Definition 5.** A strategy $(\xi^0,\xi)$ is called $H$-admissible if $V(T)=H$ almost surely with respect to the measure $\mathsf{P}$.

Note that an $H$-admissible strategy always exists. For example, if $\xi^0_T=H$ and $\xi\equiv 0$ for all other cases, then $(\xi^0,\xi)$ is an $H$-admissible strategy.

**Definition 6.** The process

$$R_\xi(t):=\mathsf{E}\left[(C(T)-C(t))^2\mid\mathcal{F}_t\right]$$

is called the mean square risk process of a strategy.

Let a contingent claim admit the following decomposition:

$$H=H_0+\int_0^T(\xi_H(t),dX(t))\quad \mathsf{P}\text{-almost surely}$$

(see Notation 1), where $\xi_H$ satisfies the assumptions imposed in Definition 1. Then one can follow the following strategy:

$$\xi^i=\xi^i_H,\qquad 1\le i\le d;$$
$$\xi^0=V-\xi\cdot X,\qquad V(t):=H_0+\int_0^t(\xi_H(s),dX(s)),\quad 0\le t\le T.$$

This strategy is $H$-admissible, self-financing (since $C_t=C_T=H_0$ for $0\le t\le T$), and minimizes the mean square risk (namely, $R_\xi(t)=0$ for $0\le t\le T$).

Consider a sequence of contingent claims $\{H_n, n\in\mathbf{Z}_+\}$ admitting the following decomposition:

$$H_n=H^0_n+\int_0^T(\xi_{H,n}(t),dX(t)) \tag{10}$$

$\mathsf{P}$-almost surely.

**Definition 7.** We say that the price process associated with assets $\{S^i, 0\le i\le d\}$ has conditionally linearly independent components if whenever

$$\sum_{i=0}^d\zeta^i(t)S^i(t)=0$$

for every $\mathcal{F}_t$-predictable process $\zeta\in\mathbb{R}^d$, for every $\mathcal{F}_t$-measurable process $\zeta^0$, and for all $t\in[0,T]$, then $(\zeta^0(t),\zeta^1(t),\dots,\zeta^d(t))=0$ for all $t\in[0,T]$ almost surely with respect to the measure $\mathsf{P}$.

**Lemma 1.** *If the price process has conditionally linearly independent components, then*

$$\sum_{i=1}^{d}\int_0^t \xi^i(s)\,dX^i(s)=0 \Longleftrightarrow \big(\xi^1(t),\dots,\xi^d(t)\big)=0$$

*for every predictable vector process* ($\xi^i, 1\le i\le d$) *and for the corresponding discounted price process associated with risky assets.*

*Proof.* The sufficiency is obvious.

*Necessity.* Let

$$\sum_{i=1}^{d}\int_0^t \xi^i(s)\,dX^i(s)=0.$$

One can define the process $\xi^0$ in such a way that the strategy $(\xi^0,\xi)$ is self-financing, that is, $C(t)=C$ almost surely with respect to the measure $\mathsf{P}$, where $C$ is a constant. Indeed, if

$$\xi^0(t)=C+\sum_{i=1}^{d}\int_0^t \xi^i(s)\,dX^i(s)-\sum_{i=1}^{d}\xi^i(s)X^i(s)=C-\sum_{i=1}^{d}\xi^i(s)X^i(s),$$

then

$$\big(\xi^0(t)-C\big)+\sum_{i=1}^{d}\xi^i(t)X^i(t)=0. \tag{11}$$

Multiplying equality (11) by $S^0(t)$ we obtain

$$\big(\xi^0(t)-C\big)\,S^0(t)+\sum_{i=1}^{d}\xi^i(t)S^i(t)=0.$$

Now the result follows, since the prices associated with the assets are conditionally linearly independent. □

**Theorem 3.** *Let the discounted price process associated with risky assets be of the form* (1). *We further assume that*

1) *the contingent claims* $\{H_n, n\in\mathbf{Z}_+\}$ *admit decomposition* (10);
2) *for all* $0\le t\le T$,
$$H_n^0+\int_0^t(\xi_{H,n}(s),dX(s))\xrightarrow{\mathsf{P}}H_0^0+\int_0^t(\xi_{H,0}(s),dX(s))$$
*as* $n\to\infty$;
3) *the price process associated with the assets* $\{S^i, 1\le i\le d\}$ *has conditionally linearly independent components;*
4) *the sequence of measures corresponding to the processes* $\{\xi_{H,n}(\cdot), n\in\mathbf{Z}_+\}$ *is weakly precompact in* $D[0,T]$.

*Then, for every* $1\le i\le d$,

$$\xi^i_{H,n}(\cdot)\xrightarrow{D[0,T]}\xi^i_{H,0}(\cdot)$$

*as* $n\to\infty$.

*Proof.* Since the sequence of contingent claims $\{H_n,\, n\in\mathbf{Z}_+\}$ admits decomposition (10), the strategies

$$\big\{\big(V_n(t)-\xi_{H,n}(t)\cdot X,\xi^1_{H,n}(t),\xi^2_{H,n}(t),\dots,\xi^d_{H,n}(t)\big)\,,n\in\mathbf{Z}_+\big\}$$

are self-financing. Applying the above Lemma 1 and Theorems 6 of [1] we complete the proof. □

**Definition 8.** Two square integrable martingales $Y(t)$ and $Z(t)$ are called orthogonal if their product $Y(t)X(t)$ is a martingale, too.

Now we consider the martingale case; namely, we assume that the process $X$ is a square integrable martingale with respect to the measure $\mathsf{P}$. Then any square integrable contingent claim $H$ admits the following Kunita–Watanabe decomposition:

$$H = H_0 + \int_0^T (\xi_H(t), dX(t)) + L^H(T) \tag{12}$$

$\mathsf{P}$-almost surely, where $\xi_H$ satisfies conditions imposed in Definition 1 and the process $L^H$ is a square integrable martingale being orthogonal to $X$ and such that $L^H(0) = 0$. This decomposition is unique in the sense that

$$H_0 = \hat{H}_0, \qquad \int_0^T (\xi_H(t), dX(t)) = \int_0^T \left(\hat{\xi}_H(t), dX(t)\right), \qquad L^H(T) = \hat{L}^H(T)$$

almost surely with respect to the measure $\mathsf{P}$ if

$$H = H_0 + \int_0^T (\xi_H(t), dX(t)) + L^H(T) = \hat{H}_0 + \int_0^T \left(\hat{\xi}_H(t), dX(t)\right) + \hat{L}^H(T).$$

It is worth mentioning that decomposition (12) determines the strategies that minimize $R_\xi(t)$ in this case, namely

$$\xi^i = \xi^i_H, \qquad 1 \le i \le d; \qquad \xi^0 = V - \xi \cdot X,$$
$$V(t) := H_0 + \int_0^t (\xi_H(s), dX(s)) + L^H(t), \qquad 0 \le t \le T$$

(see [7]). It is obvious that these strategies are self-financing in the mean sense.

Consider a sequence of square integrable contingent claims $\{\bar{H}_n, n \in \mathbf{Z}_+\}$ whose elements admit the following decomposition:

$$\bar{H}_n = \bar{H}_n^0 + \int_0^T \left(\xi_{\bar{H},n}(t), dX(t)\right) + L_n^{\bar{H}}(T) \quad \mathsf{P}\text{-almost surely.} \tag{13}$$

Then the integrands in the components of the latter decomposition converge (in other words, the risk-minimizing strategies converge) as stated in the following assertion.

**Theorem 4.** *Let the discounted price process $X(t)$ be a martingale with respect to the measure $\mathsf{P}$ and let $\mathsf{E}[\bar{H}_n - \bar{H}_0]^2 \to 0$ as $n \to \infty$.*

*1. If the discounted price processes $\{X^i, 1 \le i \le d\}$ associated with the risky assets are mutually orthogonal martingales, then*

$$\mathsf{E}\left[\int_0^T \left(\xi^i_{\bar{H},n}(t) - \xi^i_{\bar{H},0}(t)\right)^2 d\langle X^i\rangle(t)\right] \to 0$$

*for every $1 \le i \le d$ as $n \to \infty$.*

*2. If*

1) *the price process associated with the risky assets $\{S^i, 1 \le i \le d\}$ has conditionally linearly independent components;*
2) *the sequence of measures corresponding to the processes $\{\xi_{\bar{H},n}(\cdot), n \in \mathbf{Z}_+\}$ is weakly precompact in the space $D[0,T]$,*

*then, for every $1 \le i \le d$,*

$$\xi^i_{\bar{H},n}(\cdot) \overset{D[0,T]}{\longrightarrow} \xi^i_{\bar{H},0}(\cdot)$$

*as $n \to \infty$.*

*Proof.* Since $\mathsf{E}[\bar{H}_n - \bar{H}_0]^2 \to 0$ as $n \to \infty$ and since the components of decomposition (13) are orthogonal,

$$\mathsf{E}\left[\bar{H}_n^0 - \bar{H}_0^0\right]^2 \to 0, \qquad \mathsf{E}\left[L_n^{\bar{H}}(T) - L_0^{\bar{H}}(T)\right]^2 \to 0,$$
$$\mathsf{E}\left[\int_0^T \big((\xi_{\bar{H},n}(t) - \xi_{\bar{H},0}(t)), dX(t)\big)\right]^2 \to 0$$

as $n \to \infty$.

The proof of the first statement of the theorem follows, since the discounted price processes $\{X^i, 1 \le i \le d\}$ are mutually orthogonal. The proof of part 2) of the theorem follows from Theorem 6 of [1]. $\square$

Now we consider the general semimartingale case, that is, the case where the discounted price process is defined by equality (1). In this case, a strategy does not always exist that minimizes the risk $R_\xi(t)$. Thus, another criterion of the optimality is considered in [6] which yields the existence of a strategy that minimizes the local square risk. The existence of such a strategy is equivalent to the following decomposition of the contingent claim:

$$H = H_0 + \int_0^T (\xi_H(t), dX(t)) + L^H(T) \quad \mathsf{P}\text{-almost surely,} \tag{14}$$

where $\xi_H$ satisfies all the conditions of Definition 1 and where the process $L^H$ is a square integrable martingale that is orthogonal to $M$ and such that $L^H(0) = 0$.

Note that decomposition (14) determines the following strategies that minimize the local square risk:

$$\xi^i = \xi_H^i, \quad 1 \le i \le d; \quad \xi^0 = V - \xi \cdot X,$$
$$V(t) := H_0 + \int_0^t (\xi_H(s), dX(s)) + L^H(t), \quad 0 \le t \le T$$

(see [6] and [7]). It is obvious that such strategies are self-financing in the mean sense.

In contrast to the martingale case, decomposition (14) holds and is unique only if some additional conditions are satisfied.

**Definition 9.** A martingale measure $\hat{P} \approx \mathsf{P}$ is called minimal if $\hat{P} = \mathsf{P}$ on $\mathcal{F}_0$,

$$\mathsf{E}\left[\frac{d\hat{P}}{d\,\mathsf{P}}\right]^2 < \infty,$$

and if every square integrable $\mathsf{P}$-martingale that is orthogonal to $M$ with respect to the measure $\mathsf{P}$ is a martingale with respect to the measure $\hat{P}$, as well.

It is proved in [6] and [7] that the existence and uniqueness of decomposition (14) are equivalent to the existence of a minimal martingale measure.

Assume that a martingale measure $\hat{P}$ exists, that $\{H_n, n \in \mathbf{Z}_+\}$ is a sequence of contingent claims whose elements admit the following decomposition

$$H_n = H_n^0 + \int_0^T (\xi_{H,n}(t), dX(t)) + L_n^H(T) \tag{15}$$

$\mathsf{P}$-almost surely, and that the semimartingale $X$ is a continuous process. Then the following convergence of integrands of the components of this decomposition (that is, the convergence of strategies that minimize the local square risk) holds.

**Theorem 5.** *Assume that*

1) $E_{\hat{P}}[H_n - H_0]^2 \to 0$ *as* $n \to \infty$*;*

2) *the price process associated with the risky assets* $\{S^i, 1 \le i \le d\}$ *has conditionally linearly independent components;*
3) *the sequence of measures corresponding to the processes* $\{\xi_{H,n}(\cdot), n \in \mathbf{Z}_+\}$ *is weakly precompact with respect to the measure* $\hat{P}$ *in* $D[0,T]$.

*Then, for all* $1 \le i \le d$,

$$\xi^i_{H,n}(\cdot) \xrightarrow{D[0,T]} \xi^i_{H,0}(\cdot)$$

*as* $n \to \infty$.

*Proof.* Since $\hat{P}$ is a minimal martingale measure, the components of decomposition (15) are such that

1) the process $L^H_n(t)$ is a martingale with respect to the measure $\hat{P}$;
2) the process $L^H_n(t)$ is orthogonal to every component of the vector-process $X$ (see Theorem 3.5 in [7] concerning the property of orthogonality in the continuous case).

Thus equality (15) is, in fact, the Kunita–Watanabe decomposition with respect to the measure $\hat{P}$. Applying Theorem 4, we complete the proof. □

*Remark* 1. Since

$$\mathsf{E}\left[\frac{d\hat{P}}{d\,\mathsf{P}}\right]^2 < \infty,$$

one can use the assumption that $\mathsf{E}[H_n - H_0]^4 \to 0$ as $n \to \infty$ instead of condition 1) of the latter theorem. It is also clear that the conditions $\mathsf{E}[H_n - H_0]^2 \to 0$ as $n \to \infty$ and $d\hat{P}/d\,\mathsf{P} \le C$ for some $C > 0$ with respect to the measure $\mathsf{P}$ also imply condition 1) of the latter theorem.

## 6. Concluding remarks

In this paper, we studied the problem of convergence of value processes of self-financing strategies for a $d$-dimensional financial market with continuous time under the assumption that the strategies themselves converge. We found the sufficient conditions for the weak convergence of the corresponding probability measures in the Skorokhod space $D[0,T]$.

The conditions for the convergence of value processes are discussed in detail for the Black–Scholes market.

We also considered the "inverse" problem on the convergence of strategies that minimize the square risk and local square risk if the sequence of the corresponding contingent claims converges. We obtained conditions for the weak convergence of every component of the strategies $\xi_n$.

The proofs in this paper use the results obtained in [1].

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