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CONSISTENCY OF QUANTILE ESTIMATORS IN REGRESSION MODELS WITH LONG-RANGE DEPENDENT NOISE

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Abstract. Sufficient conditions for weak consistency of the Koenker–Bassett estimator are obtained for the parameter of a nonlinear regression model with continuous time and random noise possessing the property of the long-range dependence.

1. Introduction

Mathematical models for observations of the form "signal plus noise" have a wide area of applications in various fields of natural and social sciences such as turbulence theory, meteorology, hydrology, geophysics, statistical radiophysics, chemical kinetics, econometrics, finance, sociology, etc.

Models with continuous time and random noise with long-range memory are of special interest. Some recent applied studies prove that various data observed in the sciences mentioned above show the property of long-range dependence (see, for example, the books by Beran [1], Leonenko [2], Doukhan et al. [3]). Parameters of nonlinear regression models can be estimated by a method proposed by Bassett and Koenker [4]. The Koenker–Bassett estimator can be used to estimate an unknown parameter of the unknown β-quantile of observations. The value $\beta \in (0,1)$ is defined from the distribution of the random noise.

Earlier studies [5, 6] dealt with sufficient conditions for weak consistency and asymptotic normality of the Koenker–Bassett estimator in the case of nonlinear models with discrete time and with independent and identically distributed errors of observations.

The main aim of this paper is to obtain sufficient conditions for weak consistency of the Koenker–Bassett estimator in the case of nonlinear models with continuous time.

2. Setting of the problem

Consider the following nonlinear regression model:

$$X(t) = g(t,\theta) + \varepsilon(t), \qquad t \geq 0, \tag{1}$$

where $g(t,\theta)$ is a real function that is continuous with respect to the set of all its arguments $(t,\theta) \in \mathbb{R}_+ \times \Theta^c$, $\Theta \subset \mathbb{R}^q$ is an open bounded set of parameters containing θ, and Θ^c is the closure in $\mathbb{R}^q$ of the set Θ.

We also assume that $\varepsilon(t)$ satisfies the following conditions.

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A1. $\varepsilon(t)$, $t \in \mathbb{R}$, is a local functional of the Gaussian stationary process $\xi(t)$, that is, $\varepsilon(t) = G(\xi(t))$, where $G(x)$, $x \in \mathbb{R}$, is a Borel function; moreover,

$$\mathsf{E}\,\varepsilon(0) = 0, \qquad \mathsf{E}\,\varepsilon^2(0) < \infty.$$

A2. $\xi(t)$, $t \in \mathbb{R}$, is a measurable real stationary Gaussian process defined on the probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. Furthermore, $\xi(t)$ is a mean square continuous process with long-range dependence, and its covariance function is given by

$$\mathsf{E}\,\xi(t)\xi(0) = B(t) = L\left(|t|\right)/|t|^{\alpha}, \qquad \alpha \in (0,1),$$

where $L(t)$ is a slowly varying at infinity function (see, for example, [7]). Finally we assume that

$$\mathsf{E}\,\xi(t) = 0, \qquad \mathsf{E}\,\xi^2(t) = B(0) = 1.$$

We denote by $F(x)$ the distribution function of $\varepsilon(0)$ and assume that

A3. $F(0) = \beta$ for some $\beta \in (0,1)$.

Consider the function

$$\rho_\beta(x) = \begin{cases} \beta x, & x \ge 0, \\ (\beta - 1)x, & x < 0, \end{cases} \qquad \beta \in (0,1).$$

Definition 1. Any random vector $\hat\theta_T = \hat\theta_T\left(X(t), t \in [0,T]\right) \in \Theta^c$ such that

$$Q_T(\hat\theta_T) = \inf_{\tau \in \Theta^c} Q_T(\tau),$$

where

$$Q_T(\tau) = \int_0^T \rho_\beta\left(X(t) - g(t,\tau)\right) dt, \tag{2}$$

is called the Koenker–Bassett estimator (or generalized least modules estimator) of the unknown parameter $\theta \in \Theta$ constructed from observations $X(t)$, $t \in [0,T]$, represented in the form of (1) with respect to the loss function $\rho_\beta(x)$, $x \in \mathbb{R}$.

The estimator $\hat\theta_T$ exists if the above conditions are satisfied for the model (1) (see, for example, [8]).

Since

$$\mathsf{P}\left(X(t) < g(t,\theta)\right) = \mathsf{P}(\varepsilon(t) < 0) = \mathsf{P}(\varepsilon(0) < 0) = \beta,$$

the model of observations (1) can be viewed as a nonlinear quantile regression. Indeed, $\hat\theta_T$ is an estimator of the unknown parameter θ for the β-quantiles $g(t,\theta)$ of the observations $X(t)$, $t \in [0,T]$.

Let $g(t,\tau)$ be a continuously differentiable function with respect to $\tau \in \Theta$ and let

$$g_i(t,\tau) = \frac{\partial}{\partial \tau_i} g(t,\tau),$$

$$d_T^2(\theta) = \operatorname{diag}\left(d_{iT}^2(\theta)\right)_{i=1}^q, \qquad d_{iT}^2(\theta) = \int_0^T g_i^2(t,\theta)\,dt,$$

$$\liminf_{T\to\infty} T^{-1/2} d_{iT}(\theta) > 0, \qquad i = 1, \dots, q.$$

The latter lower limit can even be equal to infinity.

Now we change the variables in the regression function as follows: we let $u = T^{-1/2} d_T(\theta)(\tau - \theta)$ and put

$$h(t,u) = g\left(t, \theta + T^{1/2} d_T^{-1}(\theta) u\right),$$

where θ is the true value of the parameter. The image of the parametric set Θ under this transformation is $\tilde U_T(\theta) = T^{-1/2} U_T(\theta)$, where $U_T(\theta) = d_T(\theta)(\Theta - \theta)$. The idea behind this change of variables is that the Koenker–Bassett estimator $\hat\theta_T$ is transformed into the normalized vector $\bar u_T = T^{-1/2} d_T(\theta)(\hat\theta_T-\theta)$.

Denote $Q_T^*(u) = Q_T\left(\theta + T^{1/2} d_T^{-1}(\theta) u\right)$, $u\in\tilde U_T^c(\theta)$,

$$\Phi_{kT}(u_1,u_2) = \int_0^T |h(t,u_1)-h(t,u_2)|^k\,dt, \qquad k=1,2,$$
$$\Psi_T(u_1,u_2) = \int_0^T \rho_\beta\left(h(t,u_1)-h(t,u_2)\right)dt, \quad u_1,u_2\in\tilde U_T^c(\theta),$$
$$\varepsilon^+(t) = \max\left(\varepsilon(t),0\right).$$

3. Some properties of the loss function

Let $\underline{\beta} = \min\{\beta,1-\beta\}$ and $\overline{\beta} = \max\{\beta,1-\beta\}$. We list below some useful properties of the loss function ρ_β (the proofs of these properties are straightforward).

I. $\rho_\beta(ax) = a\rho_\beta(x)$, $a\ge 0$.
II. $\rho_\beta(x)+\rho_\beta(-x) = |x|$.
III. $\underline{\beta}|x| \le \rho_\beta(x) \le \overline{\beta}|x|$.
IV. $\rho_\beta(x+y)\le\rho_\beta(x)+\rho_\beta(y)$.
V. $|\rho_\beta(x)-\rho_\beta(y)| \le \max\{\rho_\beta(x-y),\rho_\beta(y-x)\} \le \overline{\beta}|x-y|$.
VI. If the first moment of a random variable ξ exists, then

$$\mathsf{E}\,\rho_\beta(\xi) = \mathsf{E}\,\rho_{1-\beta}(-\xi).$$

VII. If the second moment of a random variable ξ exists, then

$$\operatorname{Var}\rho_\beta(\xi) = \operatorname{Var}\rho_{1-\beta}(-\xi).$$

It follows from property **IV** that

$$\rho_\beta(x-y)\ge\rho_\beta(x)-\rho_\beta(y). \tag{3}$$

If we additionally assume that $\mathsf{E}\,\xi=0$, then

$$\mathsf{E}\,\rho_\beta(\xi) = \mathsf{E}\,\rho_\beta(-\xi) = \mathsf{E}\,\rho_{1-\beta}(\xi) = \mathsf{E}\,\rho_{1-\beta}(-\xi) = \mathsf{E}\,\xi^+. \tag{4}$$

Properties **VI** and $\mathsf{E}\,\varepsilon(t)=0$, $t\in\mathbb{R}$, imply that

$$\mathsf{E}\,\rho_\beta(\varepsilon(t)) = \mathsf{E}\,\rho_\beta(\varepsilon(0)) = \mathsf{E}\,\varepsilon^+(0).$$

4. Main result

In what follows we need the following conditions.

B. (i) For all $\varepsilon>0$ and $r>0$, there exists $\delta=\delta(r,\varepsilon)>0$ such that

$$\sup_{\substack{u_1,u_2\in V^c(r)\cap\tilde U_T^c(\theta),\\ \|u_1-u_2\|\le\delta}} T^{-1}\Phi_{1T}(u_1,u_2)\le\varepsilon, \tag{5}$$

where $V^c(r) = \{u\in\mathbb{R}^q\colon \|u\|\le r\}$.

(ii) For all $r>0$, there exists a constant $\sigma=\sigma(r)<\infty$ such that

$$\sup_{u\in V^c(r)\cap\tilde U_T^c(\theta)} T^{-1}\Phi_{2T}(u,0)\le\sigma. \tag{6}$$

C. For all $r>0$, there exists $\Delta(r)>0$ such that

$$\inf_{u\in\tilde U_T^c(\theta)\setminus V^c(r)} T^{-1}\,\mathsf{E}\,Q_T^*(u) \ge \mathsf{E}\,\varepsilon^+(0)+\Delta(r) \tag{7}$$

for sufficiently large $T > T_0$, and moreover

$$\Delta(r_0) = a_0 \,\mathsf{E}\, \varepsilon^+(0) + \Delta_0$$

for some $r_0 > 0$, where $a_0 > 2$ and $\Delta_0 > 0$ are some numbers.

Theorem 1. *Let $\varepsilon(t)$ satisfy conditions* **A1**, **A2**, *and* **A3**. *If conditions* **B** *and* **C** *hold, then*

$$\mathsf{P}(\|\bar{u}_T\| \geq r) = O\left(B\left(T\right)\right) \quad \text{as } T \to \infty$$

for all $r > 0$.

Proof. Put $\delta_T(\theta, u) = Q_T^*(u) - \mathsf{E}\, Q_T^*(u)$,

$$\delta_T(\theta, 0) = Q_T^*(0) - \mathsf{E}\, Q_T^*(0) = Q_T(\theta) - \mathsf{E}\, Q_T(\theta) = \int_0^T \rho_\beta(\varepsilon(t))\, dt - \int_0^T \mathsf{E}\, \rho_\beta(\varepsilon(t))\, dt$$
$$= \int_0^T \rho_\beta(\varepsilon(t))\, dt - T\,\mathsf{E}\, \varepsilon^+(0).$$

According to the definition of the Koenker–Bassett estimator,

$$Q_T^*(\bar{u}_T) = \inf_{u \in \tilde{U}_T^c(\theta)} Q_T^*(u), \qquad Q_T^*(u) = \int_0^T \rho_\beta(X(t) - h(t, u))\, dt,$$
$$Q_T^*(\bar{u}_T) \leq Q_T^*(0) = \delta_T(\theta, 0) + T\,\mathsf{E}\, \varepsilon^+(0) \quad \text{a.s.}$$

If $\gamma \in (0, 1)$, then condition **C** implies that

$$\begin{aligned}
&\mathsf{P}(\|\bar{u}_T\| \geq r) \\
&\quad \leq \mathsf{P}\left(\{\|\bar{u}_T\| \geq r\} \cap \left\{Q_T^*(\bar{u}_T) \leq \delta_T(\theta, 0) + T\,\mathsf{E}\, \varepsilon^+(0)\right\}\right) \\
&\quad \leq \mathsf{P}\left(\inf_{u \in \tilde{U}_T^c(\theta) \setminus V(r)} T^{-1} Q_T^*(u) \leq T^{-1} \delta_T(\theta, 0) + \mathsf{E}\, \varepsilon^+(0)\right) \\
&\quad \leq \mathsf{P}\Bigg(\inf_{u \in \tilde{U}_T^c(\theta) \setminus V(r)} T^{-1} Q_T^*(u) \\
&\qquad\qquad \leq T^{-1} \delta_T(\theta, 0) + \inf_{u \in \tilde{U}_T^c(\theta) \setminus V(r)} T^{-1}\,\mathsf{E}\, Q_T^*(u) - \Delta(r)\Bigg) \qquad (8)\\
&\quad \leq \mathsf{P}\left(T^{-1} \delta_T(\theta, 0) \geq (1 - \gamma)\Delta(r)\right) \\
&\qquad + \mathsf{P}\left(\inf_{u \in \tilde{U}_T^c(\theta) \setminus V(r)} T^{-1} Q_T^*(u) - \inf_{u \in \tilde{U}_T^c(\theta) \setminus V(r)} T^{-1}\,\mathsf{E}\, Q_T^*(u) \leq -\gamma \Delta(r)\right) \\
&\quad = \mathsf{P}_1 + \mathsf{P}_2\,.
\end{aligned}$$

By Chebyshev's inequality,

$$\mathsf{P}_1 \leq \frac{T^{-2}\,\mathsf{E}\, \delta_T^2(\theta, 0)}{((1 - \gamma)\Delta(r))^2},$$

$$T^{-2}\,\mathsf{E}\, \delta_T^2(\theta, 0) = T^{-2}\,\mathsf{E}\left(\int_0^T \rho_\beta(\varepsilon(t))\, dt - T\,\mathsf{E}\, \varepsilon^+(0)\right)^2$$
$$= T^{-2} \int_0^T \int_0^T \mathsf{E}\, \rho_\beta(\varepsilon(t)) \rho_\beta(\varepsilon(s))\, dt\, ds - (\mathsf{E}\, \varepsilon^+(0))^2.$$

Property **III** of the function ρ_β implies that

$$\mathsf{E}\, \rho_\beta^2(\varepsilon(t)) \leq \bar{\beta}^2\,\mathsf{E}\, \varepsilon^2(0) = K < \infty.$$

This means that the decomposition

$$\rho_\beta(G(u)) = \sum_{m=0}^{\infty} \frac{c_m}{m!} H_m(u), \qquad c_m = \int_{\mathbb{R}} \rho_\beta(G(u)) H_m(u) \varphi(u)\, du, \qquad m \ge 0,$$

is well defined in the Hilbert space $L_2(\mathbb{R}, \varphi(u)\, du)$, where

$$H_m(u) = (-1)^m e^{u^2/2} \frac{d^m}{du^m} e^{-u^2/2}, \qquad m \ge 0,$$

are the Chebyshev–Hermite polynomials and where $\varphi(u) = (2\pi)^{-1/2} e^{-u^2/2}$ is the standard Gaussian density. Note that

$$\mathsf{E}\, H_m(\xi(t)) H_k(\xi(s)) = \delta_m^k m!\, B^m(t-s).$$

Hence

$$\rho_\beta(G(\xi(t))) = \sum_{m=0}^{\infty} \frac{c_m}{m!} H_m(\xi(t)),$$

$$\mathsf{E}\, \rho_\beta(\varepsilon(t)) \rho_\beta(\varepsilon(s)) = \sum_{m=0}^{\infty} \frac{c_m^2}{m!} B^m(t-s).$$

In particular,

$$\mathsf{E}\, \rho_\beta^2(\varepsilon(0)) = \sum_{m=0}^{\infty} \frac{c_m^2}{m!} \le K < \infty.$$

Since $\mathsf{E}\, \rho_\beta(\varepsilon(0)) = \mathsf{E}\, \varepsilon^+(0) = c_0$, we have

$$\begin{aligned} T^{-2}\, \mathsf{E}\, \delta_T^2(\theta, 0) &= T^{-2} \int_0^T \int_0^T \left(\left(\sum_{m=0}^{\infty} \frac{c_m^2}{m!} B^m(t-s) \right) - \left(\mathsf{E}\, \varepsilon^+(0) \right)^2 \right) dt\, ds \\ &= T^{-2} \int_0^T \int_0^T \sum_{m=1}^{\infty} \frac{c_m^2}{m!} B^m(t-s)\, dt\, ds \le K T^{-2} \int_0^T \int_0^T B(t-s)\, dt\, ds. \end{aligned}$$

We need the following result of [9] to estimate the latter integral.

Lemma 1. *Let $\eta \ge 0$ be a number and let $f(t,s)$ be a measurable function defined on $(0,\infty) \times (0,\infty)$. Assume that the integral*

$$\int_0^\beta \int_0^\beta \frac{f(t,s)}{|t-s|^\eta}\, dt\, ds$$

converges for some $0 < \beta < \infty$. Let a slowly varying function L be bounded on every bounded interval of $\mathbb{R}_+$. If $\eta > 0$, then

$$\int_0^\beta \int_0^\beta f(t,s) \frac{L(T|t-s|)}{L(T)}\, dt\, ds \underset{T\to\infty}{\longrightarrow} \int_0^\beta \int_0^\beta f(t,s)\, dt\, ds.$$

If $\eta = 0$, then the same conclusion holds under the additional assumption that the function L is nondecreasing on the axis $(0,\infty)$.

Lemma 1 with $f(t,s) = |t-s|^{-\alpha}$ and $\eta \ge 0$ such that $\alpha + \eta < 1$ implies that

$$\begin{aligned} \int_0^T \int_0^T B(t-s)\, dt\, ds &= T^2 \int_0^1 \int_0^1 B(T(t-s))\, dt\, ds = T^2 \int_0^1 \int_0^1 \frac{L(T|t-s|)}{T^\alpha |t-s|^\alpha}\, dt\, ds \\ &\sim \left(\int_0^1 \int_0^1 \frac{1}{|t-s|^\alpha}\, dt\, ds \right) \cdot \frac{L(T)}{T^{\alpha-2}} = \frac{2}{(1-\alpha)(2-\alpha)} \cdot \frac{L(T)}{T^{\alpha-2}} \end{aligned}$$

as $T \to \infty$, where the relation $a(T) \sim b(T)$ as $T \to \infty$ means that

$$\lim_{T\to\infty} \frac{a(T)}{b(T)} = 1.$$

Therefore

$$\begin{aligned} \mathsf{P}_1 &\le \frac{KT^{-2}\int_0^T \int_0^T B(t-s)\,dt\,ds}{((1-\gamma)\Delta(r))^2} \\ &\sim \frac{2K}{(1-\alpha)(2-\alpha)((1-\gamma)\Delta(r))^2} \cdot \frac{L(T)}{T^\alpha} = O\left(B(T)\right) \end{aligned} \tag{9}$$

as $T \to \infty$.

On the other hand,

$$\mathsf{P}_2 \le \mathsf{P}\left(\inf_{u \in \tilde U_T^c(\theta)\setminus V(r)} T^{-1}\delta_T(\theta, u) \le -\gamma\Delta(r)\right);$$

$$\begin{aligned} Q_T^*(u) &= \int_0^T \rho_\beta(X(t) - h(t,u))\,dt \le \int_0^T \rho_\beta(\varepsilon(t))\,dt + \int_0^T \rho_\beta(h(t,0) - h(t,u))\,dt \\ &= \int_0^T \rho_\beta(\varepsilon(t))\,dt + \Psi_T(0,u) \end{aligned}$$

in view of property **IV**. Using bound (3) we prove the inequality

$$\Psi_T(0,u) - \int_0^T \rho_\beta(-\varepsilon(t))\,dt \le Q_T^*(u) \le \Psi_T(0,u) + \int_0^T \rho_\beta(\varepsilon(t))\,dt.$$

Further

$$\begin{aligned} T^{-1}\delta_T(\theta,u) &= T^{-1}\left(Q_T^*(u) - \mathsf{E}\,Q_T^*(u)\right) \\ &\ge T^{-1}\left(\Psi_T(0,u) - \int_0^T \rho_\beta(-\varepsilon(t))\,dt - \mathsf{E}\,Q_T^*(u)\right) \\ &= T^{-1}\left(-\int_0^T \rho_\beta(-\varepsilon(t))\,dt + \Psi_T(0,u)\right) \\ &\quad - T^{-1}\left(\mathsf{E}\int_0^T \rho_\beta(X(t) - h(t,u))\,dt\right) \\ &= T^{-1}\left(-\int_0^T \rho_\beta(-\varepsilon(t))\,dt - T\,\mathsf{E}\,\varepsilon^+(0)\right). \end{aligned}$$

Hence

$$T^{-1}\delta_T(\theta,u) \ge -T^{-1}\int_0^T \rho_\beta(-\varepsilon(t))\,dt - \mathsf{E}\,\varepsilon^+(0),$$

whence

$$\mathsf{P}_2 \le \mathsf{P}\left(T^{-1}\int_0^T \rho_\beta(-\varepsilon(t))\,dt + \mathsf{E}\,\varepsilon^+(0) \ge \gamma\Delta(r)\right). \tag{10}$$

Let $r = r_0$ and $\gamma = 2/a_0$, where the numbers r_0 and a_0 are defined in condition **C**. Then (10) together with Chebyshev's inequality implies that

$$\begin{aligned}
\mathsf{P}_2 &\le \mathsf{P}\left(T^{-1}\int_0^T \rho_\beta(-\varepsilon(t))\,dt + \mathsf{E}\,\varepsilon^+(0) \ge \gamma\Delta(r)\right)\\
&= \mathsf{P}\left(T^{-1}\int_0^T \rho_\beta(-\varepsilon(t))\,dt + \mathsf{E}\,\varepsilon^+(0) \ge \frac{2}{a_0}\Delta(r_0)\right)\\
&= \mathsf{P}\left(T^{-1}\int_0^T \rho_\beta(-\varepsilon(t))\,dt + \mathsf{E}\,\varepsilon^+(0) \ge \frac{2}{a_0}\left(a_0\,\mathsf{E}\,\varepsilon^+(0) + \Delta_0\right)\right) \qquad (11)\\
&= \mathsf{P}\left(T^{-1}\int_0^T \left(\rho_\beta(-\varepsilon(t)) - \mathsf{E}\,\varepsilon^+(0)\right)\,dt \ge \frac{2\Delta_0}{a_0}\right)\\
&\le \left(\frac{a_0}{2\Delta_0}\right)^2 T^{-2}\,\mathsf{E}\left(\int_0^T \left(\rho_\beta(-\varepsilon(t)) - \mathsf{E}\,\varepsilon^+(0)\right)\,dt\right)^2.
\end{aligned}$$

Following the method used to estimate the probability P_1, we consider the representation in the Hilbert space $L_2(\mathbb{R}, \varphi(u)du)$:

$$\rho_\beta(-G(u)) = \sum_{m=0}^\infty \frac{d_m}{m!}H_m(u),$$

where

$$d_m = \int_{\mathbb{R}} \rho_\beta(-G(u))H_m(u)\varphi(u)\,du, \qquad m \ge 0,$$

and

$$\mathsf{E}\,\rho_\beta(-\varepsilon(t))\rho_\beta(-\varepsilon(s)) = \sum_{m=0}^\infty \frac{d_m^2}{m!}B^m(t-s),$$

$$\mathsf{E}\,\rho_\beta^2(-\varepsilon(0)) = \sum_{m=0}^\infty \frac{d_m^2}{m!} \le K < \infty.$$

Since $c_0 = d_0$ by condition (4), we get

$$T^{-2}\,\mathsf{E}\left(\int_0^T \left(\rho_\beta(-\varepsilon(t)) - \mathsf{E}\,\varepsilon^+(0)\right)\,dt\right)^2 \le KT^{-2}\int_0^T\int_0^T B(t-s)\,dt\,ds.$$

As above, the right-hand side of (11) is $O\left(B(T)\right)$ as $T \to \infty$.

It remains to estimate the probability

$$\begin{aligned}
\mathsf{P}(r_0 > \|\bar{u}_T\| \ge r) &\le \mathsf{P}\left(T^{-1}\delta_T(\theta, 0) \ge (1-\gamma')\Delta(r)\right)\\
&\quad + \mathsf{P}\left(\inf_{u\in(V^c(r_0)\setminus V(r))\cap\tilde{U}_T^c(\theta)} T^{-1}\delta_T(\theta, u) \le -\gamma'\Delta(r)\right)\\
&\le \mathsf{P}\left(\sup_{V^c(r_0)\cap\tilde{U}_T^c(\theta)} T^{-1}|\delta_T(\theta, u)| \ge \gamma'\Delta(r)\right) + O\left(B(T)\right)
\end{aligned}$$

for $\gamma' \in (0, 1)$.

Let $F^{(1)}, \dots, F^{(l)} \subset V^c(r_0) = \{u \in \mathbb{R}^q : \|u\| \le r_0\}$ be closed sets, whose diameters do not exceed δ defined by (5) with $r = r_0$. Let $\varepsilon = \nu\Delta(r)\gamma'/(2\bar{\beta})$, where $\nu \in (0, 1)$ is a certain number, and let

$$\bigcup_{i=1}^{l} F^{(i)} = V^c(r_0).$$

Fix $u_i \in F^{(i)} \cap \tilde{U}_T^c(\theta)$, $i = 1, \dots, l$. Then

$$\begin{aligned}\mathsf{P}_3 &= \mathsf{P}\left(\sup_{u \in \bigcup_{i=1}^{l}\left(F^{(i)} \cap \tilde{U}_T^c(\theta)\right)} T^{-1}|\delta_T(\theta, u)| \geq \gamma' \Delta(r)\right) \\ &= \mathsf{P}\left(\bigcup_{i=1}^{l}\left(\sup_{u \in \left(F^{(i)} \cap \tilde{U}_T^c(\theta)\right)} T^{-1}|\delta_T(\theta, u)| \geq \gamma' \Delta(r)\right)\right) \\ &\leq \sum_{i=1}^{l} \mathsf{P}\left(\sup_{u', u'' \in \left(F^{(i)} \cap \tilde{U}_T^c(\theta)\right)} T^{-1}|\delta_T(\theta, u') - \delta_T(\theta, u'')| + T^{-1}|\delta_T(\theta, u_i)| \geq \gamma' \Delta(r)\right).\end{aligned}$$

According to property **V** we have

$$\begin{aligned}|\delta_T(\theta, u') - \delta_T(\theta, u'')| &\leq |Q_T^*(u') - Q_T^*(u'')| + \mathsf{E}\,|Q_T^*(u') - Q_T^*(u'')| \\ &\leq \int_0^T |\rho_\beta(X(t) - h(t, u')) - \rho_\beta(X(t) - h(t, u''))|\,dt \\ &\quad + \mathsf{E}\int_0^T |\rho_\beta(X(t) - h(t, u')) - \rho_\beta(X(t) - h(t, u''))|\,dt \\ &\leq 2\bar{\beta}\Phi_{1T}(u', u'').\end{aligned}$$

Considering condition (5), we conclude that

$$\sup_{u', u'' \in \left(F^{(i)} \cap \tilde{U}_T^c(\theta)\right)} 2T^{-1}\bar{\beta}\Phi_{1T}(u', u'') \leq 2\bar{\beta}\varepsilon = \nu\gamma'\Delta(r).$$

Thus

$$\mathsf{P}_3 \leq \sum_{i=1}^{l} \mathsf{P}\left(T^{-1}|\delta_T(\theta, u_i)| \geq (1 - \nu)\gamma'\Delta(r)\right). \tag{12}$$

Every term of the latter sum is estimated separately:

$$\mathsf{P}\left(T^{-1}|\delta_T(\theta, u_i)| \geq (1 - \nu)\gamma'\Delta(r)\right) \leq \frac{T^{-2}\,\mathsf{E}\,\delta_T^2(\theta, u_i)}{((1 - \nu)\gamma'\Delta(r))^2},$$

where

$$\begin{aligned}\delta_T(\theta, u_i) &= Q_T^*(u_i) - \mathsf{E}\,Q_T^*(u_i) \\ &= \int_0^T \rho_\beta(X(t) - h(t, u_i))\,dt - \mathsf{E}\int_0^T \rho_\beta(X(t) - h(t, u_i))\,dt.\end{aligned}$$

Put $\Delta h(t, u_i) = h(t, u_i) - h(t, 0)$. Then $\rho_\beta(X(t) - h(t, u_i)) = \rho_\beta(\varepsilon(t) - \Delta h(t, u_i))$ and

$$\begin{aligned}\mathsf{E}\,\delta_T^2(\theta, u_i) &= \int_0^T\int_0^T \mathsf{E}\,\rho_\beta(\varepsilon(t) - \Delta h(t, u_i))\rho_\beta(\varepsilon(s) - \Delta h(s, u_i))\,dt\,ds \\ &\quad - \left(\int_0^T \mathsf{E}\,\rho_\beta(\varepsilon(t) - \Delta h(t, u_i))\,dt\right)^2.\end{aligned}$$

Let

$$\rho_\beta(\varepsilon(t) - \Delta h(t, u_i)) = Z(\varepsilon(t)) = Z(G(\xi(t))).$$

Since, for every fixed $t \in [0,T]$,

$$\begin{aligned} \mathsf{E}\, Z^2(\varepsilon(0)) &= \mathsf{E}\, \rho_\beta^2(\varepsilon(t) - \Delta h(t,u_i)) \le \bar\beta^2\, \mathsf{E}(\varepsilon(0) - \Delta h(t,u_i))^2 \\ &= \bar\beta^2 \left(\mathsf{E}\, \varepsilon^2(0) + \Delta h^2(t,u_i)\right) < \infty, \end{aligned}$$

the function $Z(G(\cdot))$ can be decomposed in the Hilbert space $L_2(\mathbb{R}, \varphi(x)dx)$ with respect to the Chebyshev–Hermite polynomials, namely

$$Z(G(x)) = \rho_\beta(G(x) - \Delta h(t,u_i)) = \sum_{m=0}^{\infty} \frac{C_m(t,u_i)}{m!} H_m(x),$$

$$C_m(t,u_i) = \int_{\mathbb{R}} \rho_\beta(G(x) - \Delta h(t,u_i)) H_m(x)\varphi(x)\, dx, \qquad m \ge 0,$$

$$C_0(t,u_i) = \mathsf{E}\, Z(\varepsilon(0)).$$

Further

$$\mathsf{E}\, Z(G(\xi(t)))Z(G(\xi(s))) = \sum_{m=0}^{\infty} \frac{C_m(t,u_i)C_m(s,u_i)}{m!} B^m(t-s).$$

Then

$$\begin{aligned} &T^{-2}\, \mathsf{E}\, \delta_T^2(\theta,u_i) \\ &\quad = T^{-2} \int_0^T\!\!\int_0^T \left(\sum_{m=0}^{\infty} \frac{C_m(t,u_i)C_m(s,u_i)}{m!} B^m(t-s) - C_0(t,u_i)C_0(s,u_i)\right) dt\, ds \\ &\quad = T^{-2} \int_0^T\!\!\int_0^T \sum_{m=1}^{\infty} \frac{C_m(t,u_i)C_m(s,u_i)}{m!} B^m(t-s)\, dt\, ds \\ &\quad \le T^{-2} \int_0^T\!\!\int_0^T \sum_{m=1}^{\infty} \frac{C_m^2(t,u_i)}{m!} B^m(t-s)\, dt\, ds \\ &\quad \le T^{-2} \int_0^T\!\!\int_0^T \left(\sum_{m=1}^{\infty} \frac{C_m^2(t,u_i)}{m!}\right) B(t-s)\, dt\, ds \\ &\quad \le T^{-2} \int_0^T\!\!\int_0^T \mathsf{E}\, Z^2(\varepsilon(0)) B(t-s)\, dt\, ds \\ &\quad \le T^{-2} \int_0^T\!\!\int_0^T \bar\beta^2 \left(\mathsf{E}(\varepsilon^2(0) + \Delta h^2(t,u_i)\right) B(t-s)\, dt\, ds \\ &\quad \le \bar\beta^2 T^{-2} \int_0^T\!\!\int_0^T \Delta h^2(t,u_i) B(t-s)\, dt\, ds + O\left(B(T)\right) \end{aligned}$$

as $T \to \infty$. Since

$$T^{-1} \int_0^T B(s)\, ds = O\left(B(T)\right)$$

(see [7, Chapter 2, Theorem 2.7]), we estimate the latter integral as follows:

$$\begin{aligned} T^{-2} \int_0^T\!\!\int_0^T \Delta h^2(t,u_i) B(t-s)\, dt\, ds &= T^{-1} \int_0^T \Delta h^2(t,u_i) \left(T^{-1} \int_0^T B(t-s)\, ds\right) dt \\ &\le T^{-1} \Phi_{2T}(u_i, 0) T^{-1} \int_{-T}^T B(s)\, ds \\ &\le 2\sigma(r) T^{-1} \int_0^T B(s)\, ds = O\left(B(T)\right). \end{aligned}$$

This relation implies that

$$T^{-2}\,\mathsf{E}\,\delta_T^2(\theta,u_i)=O\left(B(T)\right)$$

as $T\to\infty$. Thus

$$\mathsf{P}\left(T^{-1}|\delta_T(\theta,u_i)|\ge(1-\nu)\gamma'\Delta(r)\right)=O\left(B(T)\right),\qquad i=1,\dots,l. \tag{13}$$

Now we obtain from (12) and (13) that

$$\mathsf{P}_3=O\left(B(T)\right). \qquad \square$$

5. Concluding remarks

Sufficient conditions are found in the paper for weak consistency of the Koenker–Bassett estimator in the quantile regression model with long-range memory noise and continuous time. The proof of the asymptotic normality of the estimator will be published elsewhere.

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