

# A BOUND FOR THE DISTANCE BETWEEN FRACTIONAL BROWNIAN MOTION AND THE SPACE OF GAUSSIAN MARTINGALES ON AN INTERVAL

UDC 519.21

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ABSTRACT. We obtain a lower bound for the distance between fractional Brownian motion and the space of Gaussian martingales on an interval. The distances between fractional Brownian motion and some subspaces of Gaussian martingales are compared. The upper and lower bounds are obtained for the constant in the representation of a fractional Brownian motion in terms of the Wiener process.

## 1. INTRODUCTION

A zero mean Gaussian process  $\{B_t^H, t \geq 0\}$ ,  $B_0^H = 0$ , with the covariance function

$$\mathbb{E} B_t^H B_s^H = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H})$$

is called a fractional Brownian motion with the Hurst index  $H \in (0, 1)$ . In what follows we consider the case where  $H \in (\frac{1}{2}, 1)$ . It is known that the fractional Brownian motion  $\{B_t^H, t \in [0, T]\}$  admits the representation

$$B_t^H = \int_0^t z(t, s) dW_s,$$

where  $\{W_t, t \in [0, T]\}$  is a Wiener process,

$$(1.1) \quad \begin{aligned} z(t, s) &= \left(H - \frac{1}{2}\right) c_H s^{1/2-H} \int_s^t u^{H-1/2} (u-s)^{H-3/2} du, \\ c_H &= \left(\frac{2H \cdot \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)}\right)^{1/2}, \end{aligned}$$

and where  $\Gamma(x)$ ,  $x > 0$ , is the Gamma function (see [7]). Throughout the paper we use the notation  $\alpha = H - \frac{1}{2}$ .

The best approximation in the space  $L_\infty([0, T]; L_2(\Omega))$  of fractional Brownian motion by martingales of the form

$$\int_0^t a(s) dW_s$$

is found in the papers [2]–[4] and [6], where  $W$  is a Wiener process and  $a$  is a function possessing one of the following properties:

- 1)  $a(s)$  is a constant, that is,  $a(s) = a$ ,  $s \in [0, T]$ ;

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2010 *Mathematics Subject Classification.* Primary 60G15; Secondary 60G44.

*Key words and phrases.* Wiener process, fractional Brownian motion, Gaussian martingale, approximation in a class of functions.

- 2)  $a(s)$  is a power function of the form  $a(s) = k \cdot s^\alpha$ , where  $k \in \mathbb{R}$ ,  $\alpha = H - 1/2$ , and  $H$  is the Hurst index of the fractional Brownian motion;
- 3)  $a(s)$  is a power function of the form  $a(s) = k \cdot s^\gamma$ , where  $k \in \mathbb{R}$  and  $\gamma > 0$ ;
- 4)  $a(s)$  is a power function with a negative index, namely  $a(s) = k \cdot s^{-\alpha}$ , where  $k > 0$ ,  $\alpha = H - 1/2$ , and  $H$  is the Hurst index of the fractional Brownian motion;
- 5)  $a(s)$  is a function of the form  $a(s) = k_1 + k_2 \cdot s^\alpha$ , where  $k_1, k_2 \in \mathbb{R}$  and  $s \in [0, T]$ .

For each case listed above, a function is known at which the square of the distance

$$\rho_T := \min_{a \in L_2[0, T]} \max_{0 \leq t \leq T} \mathbf{E} \left( B_t^H - \int_0^t a(s) dW_s \right)^2$$

is minimal; the precise value of the minimum is evaluated in [2]–[4] and [6]. To be specific, below we write the corresponding functions and minimal values for all five cases listed above.

1)

$$\min_a \max_{0 \leq t \leq T} \mathbf{E} (B_t^H - M_t)^2 = T^{2H} (1 - c_1^2),$$

where

$$c_1 = c_1(H) = \alpha c_H \cdot \frac{1}{\alpha + 1} \cdot B(1 - \alpha, \alpha)$$

(see [3]). In this case,  $a_{\min} = c_1(H) \cdot T^\alpha$ .

2)

$$\min_k \max_{0 \leq t \leq T} \mathbf{E} (B_t^H - M_t)^2 = T^{2H} \left( 1 - \frac{c_H^2}{2H} \right).$$

In this case,  $k_{\min} = c_H$  (see [3]).

3)

$$\min_\gamma \min_k \max_{0 \leq t \leq T} \mathbf{E} (B_t^H - M_t)^2 = T^{2H} (1 - c_1^2(H)).$$

In this case, the minimum is attained for  $\gamma = 0$  and  $k_{\min} = c_1(H) \cdot T^\alpha$  (see [3]).

4)

$$\min_k \max_{0 \leq t \leq T} \mathbf{E} (B_t^H - M_t)^2 = \varphi_1(k^*).$$

The minimum is attained at the function  $a(s) = k^* s^{1/2-H}$ , where the coefficient  $k^*$  is the minimum argument of two intersection points of the functions

$$\varphi_1(k) = k^{\frac{2H}{2H-1}} c_1(H) \quad \text{and} \quad \varphi_2(k) = T^{2H} - \frac{4kH}{c_H} T + k^2 \frac{T^{2-2H}}{2-2H}$$

(see [4]); here

$$c_1(H) = (p(H))^{-\frac{2H}{2H-1}} \frac{2H-1}{1-H} \left( \frac{2H}{c_H} p(H) - 1 \right).$$

5)

$$\begin{aligned} & \min_{k_1, k_2} \max_{0 \leq t \leq T} \mathbf{E} (B_t^H - M_t)^2 \\ & = T^{2\alpha+1} - T^{2\alpha+1} c_H^2 \left( B^2(1 - \alpha, \alpha) - \frac{2B(1 - \alpha, \alpha)}{\alpha} + \frac{(\alpha + 1)^2}{\alpha^2(2\alpha + 1)} \right). \end{aligned}$$

The minimum is attained for  $k_1 = k_1^*$  and  $k_2 = k_2^*$  (the precise values of the constants are given in [2] and [6]).

In this paper, we obtain a nonzero lower bound for  $\rho_T$ .

It is proved in the paper [3] that the best approximation among the functions possessing the properties 1)–3) above is given by the constant function mentioned in the case 1). Below, we compare the minimal values for the cases 1) and 5), as well.

The paper is organized as follows. In Section 2, we recall the properties of a function that minimizes the square of the distance  $\rho_T$  under the assumption that such a function exists (the explicit form of this function as well as conditions for its existence are not yet known). A lower bound for the best approximation of the fractional Brownian motion by Gaussian martingales is obtained in Section 3. In Section 4, we compare the best approximations of a fractional Brownian motion by corresponding Gaussian martingales in some classes of functions. A new upper bound for the square of the distance  $\rho_T$  as well as a new estimate for the constant  $c_H$  are obtained in Section 5; these results are based on the properties established in Section 2. Various bounds for the constant  $c_H$  are compared numerically and graphically in Section 6.

## 2. AUXILIARY RESULTS

Let  $\{W_t, t \in [0, T]\}$  be a Wiener process and let  $B_t^H = \int_0^t z(t, s) dW_s$  be a fractional Brownian motion with the Hurst index  $H > 1/2$ . Set  $\alpha = H - 1/2$ . For a locally square integrable function  $a: \mathbb{R}_+ \rightarrow \mathbb{R}$ , let

$$F_a(t) = \mathbb{E} \left[ \left( B_t^H - \int_0^t a(s) dW_s \right)^2 \right] = \int_0^t (a(s) - z(t, s))^2 ds.$$

If  $c > 0$  is arbitrary, then

$$(2.1) \quad F_a(t) = c^{2H} F_{c^{-\alpha} a(c \cdot)}(t/c).$$

This result follows from the equality of distributions

$$\left( c^{-1/2} W_{ct}, c^{-H} B_{ct}^H \right) \stackrel{d}{=} (W_t, B_t^H).$$

Consider the optimization problem

$$(2.2) \quad M_a := \max_{t \in [0, 1]} F_a(t) \rightarrow \min$$

with respect to the function  $a$ .

Let a function  $a^* \in C(\mathbb{R}_+)$  minimize the functional  $M_a$ .

**Lemma 2.1.**  $M_{a^*} = F_{a^*}(1)$ .

*Proof.* Assume that  $F_{a^*}(1) < M_{a^*}$ . Since  $F_{a^*}(t)$  is continuous with respect to  $t$ , a number  $c > 1$  exists such that  $F_{a^*}(t) < M_{a^*}$  for  $t \in [1, c]$ . Thus  $\max_{t \in [0, c]} F_{a^*}(t) = M_{a^*}$ . Put  $b(t) = c^{-\alpha} a^*(tc)$ . Equality (2.1) implies that  $F_b(t) = c^{-2H} F_{a^*}(tc)$ ,  $t \in [0, 1]$ . This yields  $M_b = c^{-2H} M_{a^*} < M_{a^*}$  which contradicts the assumption.  $\square$

Now we assume that the function  $F_{a^*}(t)$  is continuously differentiable.

**Lemma 2.2.**  $F'_{a^*}(1) \geq 2H F_{a^*}(1)$ .

*Proof.* The preceding lemma implies that  $F'_{a^*}(1) \geq 0$ . For  $c > 1$ , let  $b(t) = c^{-\alpha} a^*(tc)$ ,  $t \in [0, 1]$ . Equality (2.1) implies  $F_b(t) = c^{-2H} F_{a^*}(tc)$ ,  $t \in [0, 1/c]$ . Since  $M_b \geq M_{a^*}$ , a point  $t_c \in [0, 1]$  exists such that  $F_b(t_c) \geq M_{a^*} = F_{a^*}(1)$ . Hence  $t_c \in (1/c, 1]$ . On the other hand,

$$F_{a^*}(1) \leq F_b(t_c) = c^{-2H} F_{a^*}(ct_c) \leq (ct_c)^{-2H} F_{a^*}(ct_c).$$

Since  $ct_c \rightarrow 1+$  as  $c \rightarrow 1+$ , we get

$$(x^{-2H} F_{a^*}(x))'_x \Big|_{x=1} \geq 0,$$

and this is equivalent to the statement of the lemma.  $\square$

The latter lemma implies, in particular, that  $F'_{a^*}(1) > 0$ , since  $F_{a^*}(1) = M_{a^*} > 0$ .

### 3. A LOWER BOUND FOR THE BEST APPROXIMATION OF FRACTIONAL BROWNIAN MOTION BY MARTINGALES OF THE FORM $\int_0^t a(s) dW_s$

**Theorem 3.1.** *The square of the distance*

$$\rho_T := \min_{a \in L_2[0, T]} \max_{0 \leq t \leq T} \mathbb{E} \left( B_t^H - \int_0^t a(s) dW_s \right)^2$$

admits the following estimate:

$$(3.1) \quad \rho_T \geq \max_{0 \leq t_1 \leq 1} \frac{(1 - t_1^{2H} - (1 - t_1)^{2H})^2}{16t_1^{2H}} \cdot T^{2H} > 0.$$

*Proof.* In view of property (2.1),  $\rho_T$  can be rewritten via  $\rho_1$ , namely  $\rho_T = T^{2H} \cdot \rho_1$ . This implies that one can restrict the consideration to the case of  $\rho_T$  with  $T = 1$ . Now we construct a lower bound for

$$\max_{0 \leq t \leq 1} \mathbb{E} \left( B_t^H - \int_0^t a(s) dW_s \right)^2 = \max_{0 \leq t \leq 1} \int_0^t (z(t, s) - a(s))^2 ds.$$

Let  $0 < t_1 \leq 1$ . Consider the random variable  $\int_0^1 a(s) dW_s =: B$ . Then

$$\begin{aligned} & \max_{0 \leq t \leq 1} \mathbb{E} \left( B_t^H - \int_0^t a(s) dW_s \right)^2 \\ &= \max_{0 \leq t \leq 1} \mathbb{E} (B_t^H - \mathbb{E}[B | \mathcal{F}_t])^2 \\ &\geq \max \left( \mathbb{E} (B_{t_1}^H - \mathbb{E}[B | \mathcal{F}_{t_1}])^2, \mathbb{E}(B_1^H - B)^2 \right) \\ &\geq \max \left( \mathbb{E} (B_{t_1}^H - \mathbb{E}[B | B_{t_1}^H])^2, \mathbb{E} (\mathbb{E}[B_1^H | B_{t_1}^H] - \mathbb{E}[B | B_{t_1}^H])^2 \right) \\ &\geq \frac{1}{2} \left( \mathbb{E} (B_{t_1}^H - \mathbb{E}[B | B_{t_1}^H])^2 + \mathbb{E} (\mathbb{E}[B_1^H | B_{t_1}^H] - \mathbb{E}[B | B_{t_1}^H])^2 \right) \\ &\geq \frac{1}{4} \mathbb{E} (B_{t_1}^H - \mathbb{E}[B_1^H | B_{t_1}^H])^2 \\ &= \frac{1}{4} \mathbb{E} \left( B_{t_1}^H - \frac{\mathbb{E}(B_1^H B_{t_1}^H)}{\mathbb{E}((B_{t_1}^H)^2)} B_{t_1}^H \right)^2 = \frac{1}{4} \mathbb{E} \left( B_{t_1}^H \cdot \left( 1 - \frac{1 + t_1^{2H} - (1 - t_1)^{2H}}{2t_1^{2H}} \right) \right)^2 \\ &= \frac{1}{4} t_1^{2H} \left( 1 - \frac{1 + t_1^{2H} - (1 - t_1)^{2H}}{2t_1^{2H}} \right)^2 = \frac{(1 - t_1^{2H} - (1 - t_1)^{2H})^2}{16t_1^{2H}}. \end{aligned}$$

The theorem is proved.  $\square$

*Remark 3.2.* Using the inequality  $\frac{1}{2}(a^2 + b^2) \geq \left(\frac{a+b}{2}\right)^2$ , we obtain

$$\frac{t_1^{2H} + (1 - t_1)^{2H}}{2} \geq \left( \frac{t_1 + (1 - t_1)}{2} \right)^{2H} = \frac{1}{2^{2H}}.$$

Moreover, the equality holds if and only if  $t_1 = 1 - t_1$ , that is, if  $t_1 = \frac{1}{2}$ . Thus

$$\rho_T \geq \frac{(2^{2H} - 2)^2}{16 \cdot 2^{2H}} \cdot T^{2H}$$

if  $t_1 = \frac{1}{2}$  in Theorem 3.1.

#### 4. A COMPARISON OF THE BEST APPROXIMATIONS OF FRACTIONAL BROWNIAN MOTION IN SOME CLASSES OF FUNCTIONS

Let  $\{B_t^H, t \geq 0\}$  be a fractional Brownian motion with the Hurst index  $H \in (\frac{1}{2}, 1)$ ,  $T > 0$  be fixed,  $a: [0, T] \rightarrow \mathbb{R}$  be a measurable function such that  $a \in L_2[0, T]$ , a square integrable martingale  $\{M_t, t \in [0, T]\}$  be of the form  $M_t = \int_0^t a(s) dW_s$ , where  $\{W_t, t \in [0, T]\}$  is a Wiener process related to the fractional Brownian motion  $B^H$  as follows: for all  $t \in [0, T]$ ,

$$(4.1) \quad B_t^H = \int_0^t z(t, s) dW_s,$$

where

$$z(t, s) = c_H (H - 1/2) s^{1/2-H} \int_s^t u^{H-1/2} (u - s)^{H-3/2} du.$$

Representation (4.1) is proved in the paper [7]. Here

$$c_H = \left( \frac{2H \cdot \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \right)^{1/2}.$$

Class 5) in the list given in Section 1 is wider than the class of constant functions; namely class 5) consists of the functions  $a(s) = k_1 + k_2 \cdot s^\alpha$ ,  $s \in [0, T]$ , for  $k_1, k_2 \in \mathbb{R}$ . Thus the best approximation by the elements of this class is not worse than that by elements of the class 1), that is,

$$\min_k \max_{0 \leq t \leq T} \mathbb{E} \left( B_t^H - \int_0^t k dW_s \right)^2 \geq \min_{k_1, k_2} \max_{0 \leq t \leq T} \mathbb{E} \left( B_t^H - \int_0^t (k_1 + k_2 s^\alpha) dW_s \right)^2.$$

We prove that the inequality is strict; in fact,

$$(4.2) \quad \min_k \max_{0 \leq t \leq T} \mathbb{E} (B_t^H - k W_t)^2 > \min_{k_1, k_2} \max_{0 \leq t \leq T} \mathbb{E} \left( B_t^H - \int_0^t (k_1 + k_2 s^\alpha) dW_s \right)^2.$$

Put  $f(t, k_1, k_2) := \mathbb{E} (B_t^H - \int_0^t (k_1 + k_2 s^\alpha) dW_s)^2$ .

It is known that  $k_2^0 < 0$  and  $k_2^* < 0$ . Note that  $f(T, k_1, k_2)$  is a polynomial of the second order of variables  $k_1$  and  $k_2$ ; moreover, the polynomial is concave with respect to  $k_1$  and  $k_2$  and attains its minimum at the point  $(k_1^*, k_2^*)$ . For  $k_2 \geq k_2^0$ , the function  $f(t, k_1, k_2)$  increases with respect to  $t$  in the interval  $[0, +\infty)$ . It is proved in [3] that

$$(4.3) \quad \min_k \max_{0 \leq t \leq T} f(t, k, 0) = T^{2H} (1 - c_1^2(H)),$$

where

$$c_1(H) = \frac{\alpha}{\alpha + 1} c_H B(1 - \alpha, \alpha).$$

The minimum is attained at  $k = c_1(H) \cdot T^\alpha$ , while the maximum

$$\max_{0 \leq t \leq T} f(t, c_1(H) T^\alpha, 0)$$

is attained at  $t = T$ .

We have  $(c_1(H)T^\alpha, 0) \neq (k_1^*, k_2^*)$ , since  $k_2^* < 0$ . Thus  $f(T, k_1, k_2)$  does not have a local minimum at the point  $(c_1(H)T^\alpha, 0)$ . This implies that there exists a point  $(k_1^{(2)}, k_2^{(2)})$ ,  $k_2^{(2)} > k_2^0$ , such that  $f(T, c_1(H)T^\alpha, 0) > f(T, k_1^{(2)}, k_2^{(2)})$ .

The minimum of (4.3) is attained at  $k = c_1(H) \cdot T^\alpha$ . Taking into account the inequality  $k_2^{(2)} > k_2^0$  and that the function  $f(t, k_1^{(2)}, k_2^{(2)})$  increases, we obtain the following bound:

$$\begin{aligned} \min_k \max_{0 \leq t \leq T} f(t, k, 0) &= f(T, c_1(H)T^\alpha, 0) > f\left(T, k_1^{(2)}, k_2^{(2)}\right) = \max_{0 \leq t \leq T} f\left(t, k_1^{(2)}, k_2^{(2)}\right) \\ &\geq \min_{k_1, k_2} \max_{0 \leq t \leq T} f(t, k_1, k_2). \end{aligned}$$

To show that the best approximation by functions of the class 5) is better than that by functions of the class 1), we need to prove that

$$\begin{aligned} T^{2\alpha+1} \left( 1 - \left( \frac{\alpha c_H}{\alpha+1} \cdot B(1-\alpha, \alpha) \right)^2 \right) \\ > T^{2\alpha+1} \left( 1 - c_H^2 \left( B^2(1-\alpha, \alpha) - \frac{2B(1-\alpha, \alpha)}{\alpha} + \frac{(\alpha+1)^2}{\alpha^2(2\alpha+1)} \right) \right) \end{aligned}$$

or, in other words,

$$(4.4) \quad B^2(1-\alpha, \alpha) - \frac{2B(1-\alpha, \alpha)}{\alpha} + \frac{(\alpha+1)^2}{\alpha^2(2\alpha+1)} > \left( \frac{\alpha c_H}{\alpha+1} \cdot B(1-\alpha, \alpha) \right)^2.$$

Consider the difference between the left and right hand sides of (4.4):

$$\begin{aligned} (4.5) \quad & B^2(1-\alpha, \alpha) - \frac{2B(1-\alpha, \alpha)}{\alpha} + \frac{(\alpha+1)^2}{\alpha^2(2\alpha+1)} - \left( \frac{\alpha}{\alpha+1} \cdot B(1-\alpha, \alpha) \right)^2 \\ &= \frac{2\alpha+1}{(\alpha+1)^2} B^2(1-\alpha, \alpha) - \frac{2B(1-\alpha, \alpha)}{\alpha} + \frac{(\alpha+1)^2}{\alpha^2(2\alpha+1)} \\ &= (2\alpha+1) \left( \frac{B(1-\alpha, \alpha)}{\alpha+1} - \frac{\alpha+1}{\alpha(2\alpha+1)} \right)^2 \geq 0. \end{aligned}$$

Now we show that the right hand side of (4.5) does not equal zero, that is,

$$f(\alpha) := (\alpha+1)^2 \cdot \sin \pi \alpha - \pi \alpha (2\alpha+1) \neq 0.$$

In fact, we will prove that  $f(\alpha) < 0$  for  $0 < \alpha < \frac{1}{2}$ .

We use the Taylor expansion with a remainder term written in the Lagrange form:

$$f(\alpha) = f(0) + \alpha f'(0) + \frac{\alpha^2 f''(0)}{2} + \frac{\alpha^3 f'''(\theta\alpha)}{3!}, \quad 0 < \theta < 1.$$

It is obvious that  $f'(0) = 0$ ,  $f''(0) = 0$ , and  $f'''(\theta\alpha) < 0$ . Indeed,

$$f'''(\alpha) = \pi \cos \pi \alpha (6 - (\alpha+1)^2 \pi^2) - 6(\alpha+1) \pi^2 \sin \pi \alpha < 0.$$

Since  $\alpha^3 f'''(\theta\alpha)/3! < 0$ , we obtain  $f(\alpha) < 0$  for  $0 < \alpha < \frac{1}{2}$ .

This proves that (4.4) holds, indeed.

## 5. AN UPPER BOUND FOR THE SQUARE OF THE DISTANCE $\rho_T$ IN THE CLASS OF FUNCTIONS $a(s) = a_0 s^\alpha + a_1 s^{\alpha+1}$

Let  $M_t = \int_0^t a(s) dW_s$ ,  $B_t^H = \int_0^s z(t, s) dW_s$ , and let

$$z(t, s) = c_H \alpha s^{-\alpha} \int_s^t u^\alpha (u-s)^{\alpha-1} du.$$

We apply Theorem 2 of the paper [3] and obtain

$$\int_0^t (z(t, s) - c_H s^\alpha)^2 ds = t^{2H} \left( 1 - \frac{c_H^2}{2H} \right).$$

According to this result,

$$\begin{aligned} \mathbb{E}(B_t^H - M_t)^2 &= \int_0^t (z(t, s) - a(s))^2 ds \\ &= \int_0^t ((z(t, s) - c_H s^\alpha) + (c_H s^\alpha - a(s)))^2 ds \\ &= \int_0^t (z(t, s) - c_H s^\alpha)^2 ds \\ &\quad + 2 \int_0^t (z(t, s) - c_H s^\alpha) \cdot (c_H s^\alpha - a(s)) ds \\ &\quad + \int_0^t (c_H s^\alpha - a(s))^2 ds \\ &= t^{2H} \left( 1 - \frac{c_H^2}{2H} \right) \\ &\quad - 2\alpha c_H \int_0^t u^\alpha \int_0^u (u-s)^{\alpha-1} \cdot (a(s)s^{-\alpha} - a(u)u^{-\alpha}) ds du \\ &\quad + \int_0^t (c_H s^\alpha - a(s))^2 ds. \end{aligned} \tag{5.1}$$

Making the change of variables  $b(u) = a(u)u^{-\alpha}$  and  $c(u) = b(u) - c_H$  we obtain

$$\begin{aligned} \mathbb{E}(B_t^H - M_t)^2 &= t^{2H} \left( 1 - \frac{c_H^2}{2H} \right) - 2\alpha c_H \int_0^t u^\alpha \int_0^u (u-s)^{\alpha-1} \cdot (b(s) - b(u)) ds du \\ &\quad + \int_0^t s^{2\alpha} (b(s) - c_H)^2 ds \\ &= t^{2H} \left( 1 - \frac{c_H^2}{2H} \right) - 2\alpha c_H \int_0^t u^\alpha \int_0^u (u-s)^{\alpha-1} \cdot (c(s) - c(u)) ds du \\ &\quad + \int_0^t s^{2\alpha} c^2(s) ds. \end{aligned} \tag{5.2}$$

Put  $c(u) = a_0 + a_1 u$  on the right hand side of (5.2), which means that

$$a(u) = (a_0 + c_H + a_1 u)u^\alpha$$

in (5.1). The optimization with respect to both coefficients  $a_0$  and  $a_1$  requires a cumbersome calculation; thus we minimize the distance by using Lemmas 2.1 and 2.2. In doing so, we consider the following class of functions:

$$\mathcal{A} = \{a \in L_2[0, 1]: F'_a(1) = 2HF_a(1)\}.$$

**Theorem 5.1.** *Let  $\frac{1}{2} < H < 1$  and  $\alpha = H - \frac{1}{2}$ . Then the minimum*

$$\min_{a_0, a_1} \max_{0 \leq t \leq T} \mathbb{E} \left( B_t^H - \int_0^t (a_0 s^\alpha + a_1 s^{\alpha+1}) dW_s \right)^2$$

in the class of functions  $c(u) = a_0 + a_1 u \in \mathcal{A}$  is attained for the coefficients  $a_0 = a_0^*$  and  $a_1 = a_1^*$ , where

$$a_0^* = \frac{\alpha(2\alpha + 1)(2\alpha + 3)}{(\alpha + 1)} c_H,$$

$$a_1^* = -\frac{2\alpha + 3}{T(2\alpha + 2)} \cdot \left( a_0 + \frac{\alpha c_H}{\alpha + 1} \right).$$

The minimum is equal to

$$T^{2\alpha+1} \left[ 1 - \frac{c_H^2}{2\alpha + 1} - \frac{\alpha^2(2\alpha + 3)c_H^2}{(\alpha + 1)^2} \right].$$

*Proof.* Substituting  $c(u) = a_0 + a_1 u$  in (5.2) we obtain

$$\begin{aligned} F_a(t) &= \mathbf{E}(B_t^H - M_t)^2 \\ &= t^{2H} \left( 1 - \frac{c_H^2}{2H} \right) + 2\alpha c_H \int_0^t u^\alpha \int_0^u (u-s)^\alpha \cdot a_1 ds du \\ &\quad + \int_0^t s^{2\alpha} (a_0^2 + 2a_0 a_1 s + a_1^2 s^2) ds \\ (5.3) \quad &= t^{2\alpha+1} \left( 1 - \frac{c_H^2}{2\alpha + 1} + \frac{a_0^2}{2\alpha + 1} \right) + t^{2\alpha+2} \frac{a_1}{\alpha + 1} \left( a_0 + \frac{\alpha c_H}{\alpha + 1} \right) \\ &\quad + t^{2\alpha+3} \cdot \frac{a_1^2}{2\alpha + 3}. \end{aligned}$$

Passing to the derivatives with respect to  $t$  on both sides of (5.3) we get

$$(5.4) \quad F'_a(t) = t^{2\alpha} (\kappa_0 + \kappa_1 t + \kappa_2 t^2),$$

where

$$(5.5) \quad \kappa_0 = 2\alpha + 1 - c_H^2 + a_0^2, \quad \kappa_1 = 2a_1 \left( a_0 + \frac{\alpha c_H}{\alpha + 1} \right), \quad \kappa_2 = a_1^2.$$

Recalling that the functions under consideration belong to the class  $\mathcal{A}$ , we evaluate at the point  $T$  that

$$T^{2\alpha} (\kappa_0 + \kappa_1 T + \kappa_2 T^2) = \frac{2\alpha + 1}{T} \cdot T^{2\alpha+1} \cdot \left( \frac{\kappa_0}{2\alpha + 1} + T \cdot \frac{\kappa_1}{2\alpha + 2} + T^2 \cdot \frac{\kappa_2}{2\alpha + 3} \right),$$

whence

$$(5.6) \quad \kappa_2 = -\kappa_1 \cdot \frac{2\alpha + 3}{2T \cdot (2\alpha + 2)}.$$

The latter equality is equivalent to the following one:

$$(5.7) \quad a_1 = -\frac{2\alpha + 3}{T \cdot (2\alpha + 2)} \cdot \left( a_0 + \frac{\alpha c_H}{\alpha + 1} \right),$$

whence

$$(5.8) \quad \begin{aligned} F_a(T) &= T^{2\alpha+1} \left[ 1 - \frac{c_H^2}{2\alpha + 1} \right. \\ &\quad \left. + \frac{1}{(2\alpha + 2)^2} \left( a_0^2 \cdot \frac{1}{2\alpha + 1} - a_0 \cdot \frac{2(2\alpha + 3)\alpha c_H}{\alpha + 1} - \frac{(2\alpha + 3)\alpha^2 \cdot c_H^2}{(\alpha + 1)^2} \right) \right]. \end{aligned}$$



The minimum of the expression on the right hand side of (5.8) with respect to  $a_0$  is attained at

$$a_0 = \frac{\alpha(2\alpha + 1)(2\alpha + 3)}{(\alpha + 1)} \cdot c_H$$

and is equal to

$$\begin{aligned} \min_{a_0} F_a(T) &= T^{2\alpha+1} \left[ 1 - \frac{c_H^2}{2\alpha + 1} + \frac{1}{(2\alpha + 2)^2} \left( -\frac{\alpha^2(2\alpha + 3)^2(2\alpha + 1)c_H^2}{(\alpha + 1)^2} \right. \right. \\ (5.9) \quad &\quad \left. \left. - \frac{(2\alpha + 3)\alpha^2 c_H^2}{(\alpha + 1)^2} \right) \right] \\ &= T^{2\alpha+1} \left[ 1 - \frac{c_H^2}{2\alpha + 1} - \frac{\alpha^2(2\alpha + 3)c_H^2}{(\alpha + 1)^2} \right]. \end{aligned}$$

It remains to prove that the function  $F_a(t)$  increases with respect to  $t$  if  $a_0 = a_0^*$  and  $a_1 = a_1^*$ . Then

$$\min_{a_0, a_1} \max_{0 \leq t \leq T} F_a(T) \leq F_{a^*}(T), \quad a^* = (a_0^*, a_1^*).$$

Taking into account (5.4), we need to prove that the least root of the equation

$$\kappa_0 + \kappa_1 t + \kappa_2 t^2 = 0$$

exceeds  $T$ . Without loss of generality we restrict the consideration to the case of  $T = 1$ . Then this root is equal to

$$t_1 = \frac{-\kappa_1 - \sqrt{\kappa_1^2 - 4\kappa_0\kappa_2}}{2\kappa_2} = \alpha + 1 - \frac{1}{\alpha c_H} \sqrt{c_H^2 \alpha^2 (\alpha + 1)^2 - \kappa_0}.$$

The inequality  $t_1 > 1$  is equivalent to

$$\alpha^2 > (\alpha + 1)^2 - \frac{\kappa_0}{\alpha^2 c_H^2} \quad \text{or} \quad 2\alpha + 1 < \frac{\kappa_0}{\alpha^2 c_H^2}.$$

On the other hand,

$$\begin{aligned} \frac{\kappa_0}{\alpha^2 c_H^2} &= \left( 2H - c_H^2 + \frac{\alpha^2 c_H^2 (2\alpha + 1)^2 (2\alpha + 3)^2}{(\alpha + 1)^2} \right) \cdot \frac{1}{\alpha^2 c_H^2} \\ &= \frac{2H - c_H^2}{(\alpha + 1)^2} + \frac{(2\alpha + 1)^2 (2\alpha + 3)^2}{(\alpha + 1)^2} > \frac{(2\alpha + 1)^2 (2\alpha + 3)^2}{(\alpha + 1)^2} > (2\alpha + 1)^2 > 2\alpha + 1, \end{aligned}$$

that is,  $t_1 > 1$ , indeed. □

**Corollary 5.2.** *Equality (5.9) implies*

$$1 - \frac{c_H^2}{2\alpha + 1} - \frac{\alpha^2(2\alpha + 3)c_H^2}{(\alpha + 1)^2} > 0,$$

whence we obtain a bound for  $c_H$ :

$$(5.10) \quad c_H < \sqrt{\frac{(2\alpha + 1)(\alpha + 1)^2}{(\alpha + 1)^2 + \alpha^2(2\alpha + 1)(2\alpha + 3)}} =: C_H^{(c)}.$$

Estimate (5.10) obviously improves the estimate  $c_H^2 \leq 2H$  obtained in the paper [2].

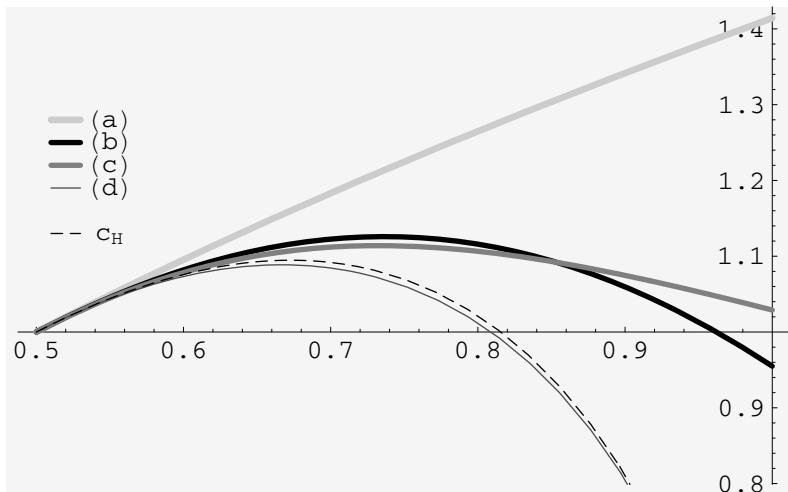


FIGURE 1

6. A COMPARISON OF SOME LOWER AND UPPER BOUNDS FOR  $c_H$ 

Several upper bounds for the constant  $c_H$  are obtained in the papers [2] and [3]. Below are two of these bounds:

$$(6.1) \quad c_H < \sqrt{2\alpha + 1} =: C_H^{(a)}$$

([2], Corollary 1; Figure 1, graph (a));

$$(6.2) \quad c_H < \frac{(\alpha + 1) \sin \pi \alpha}{\pi \alpha} =: C_H^{(b)}$$

([3], Remark 1; Figure 1, graph (b)). The latter bound is obtained by transforming the distance

$$(6.3) \quad \rho_T = T^{2H} (1 - c_1^2),$$

where

$$c_1 = c_1(H) = \alpha c_H \cdot \frac{1}{\alpha + 1} \cdot B(1 - \alpha, \alpha).$$

Relation (5.10) provides another upper bound for  $c_H$ , namely  $C_H^{(c)}$ .

The paper [4] (Lemma 3.1; Figure 1, graph (d)) contains a lower bound for the constant  $c_H$ , namely

$$(6.4) \quad c_H > \sqrt{8H^2(1 - H)} =: C_H^{(d)}.$$

Below we compare these four bounds with the help of *Mathematica*.

It is seen from Figure 1 that  $C_H^{(b)}$  approximates  $c_H$  better than  $C_H^{(a)}$  does in the whole interval  $(\frac{1}{2}, 1)$ . It is also clear that  $C_H^{(c)}$  approximates  $c_H$  better than  $C_H^{(b)}$  does in the interval  $H \in (\frac{1}{2}, 0.853497)$  (the upper end point of this interval is found numerically with the help of *Mathematica*). On the other hand,  $C_H^{(b)}$  approximates  $c_H$  better than  $C_H^{(c)}$  does in the interval  $H \in (0.853497, 1)$ . The graphs of the differences  $C_H^{(a)} - C_H^{(b)}$  and  $C_H^{(b)} - C_H^{(c)}$  depicted in Figure 2 and Figure 3 confirm this conclusion.

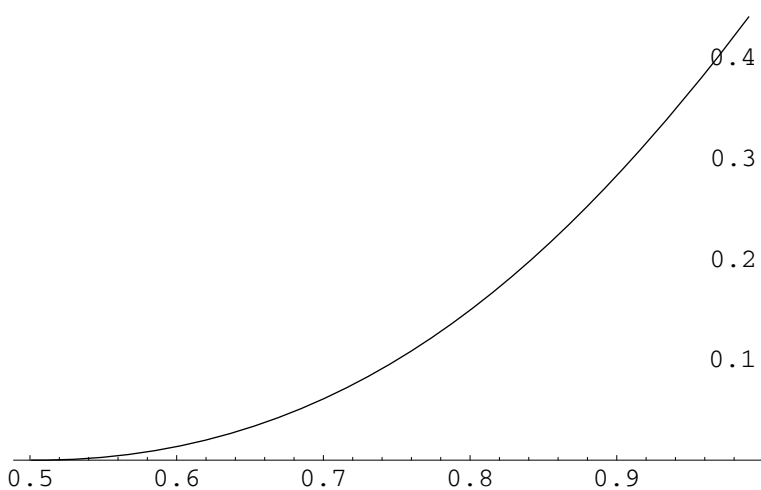


FIGURE 2

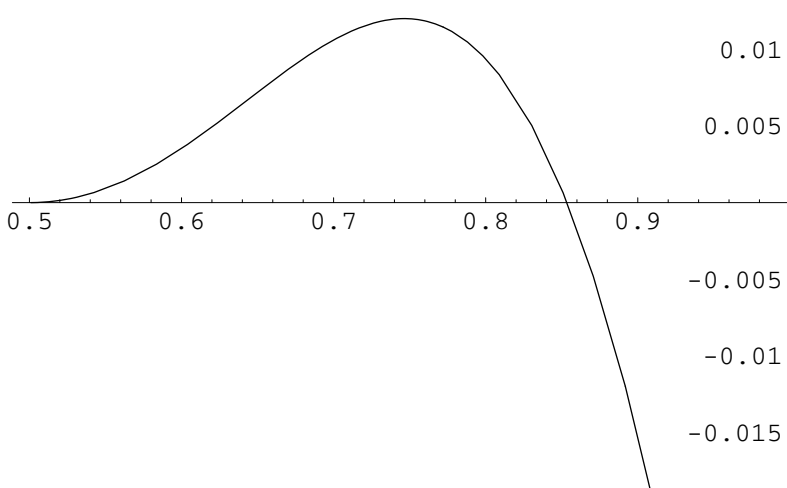


FIGURE 3

## 7. CONCLUDING REMARKS

We obtained some upper bounds for the constant  $c_H$ . The bounds  $C_H^{(b)}$  and  $C_H^{(c)}$  are the best among them in the corresponding intervals mentioned above. This allows one to determine the best upper bound for  $\rho_T$ .

We compare the upper bound for  $\rho_T$  and lower bound (3.1) (see Figure 4, curve (a)). We substitute the expression for  $c_H$  given by (1.1) to (6.3) and to (5.9) (see Figure 4, curve (c) and Figure 4, curve (b)). Thus we obtain the best upper bounds for  $\rho_T$ . The difference between the curves (b) and (c) presented on Figure 4 is depicted on Figure 5. We see that the best upper bound for  $\rho_T$  is given by the approximation obtained from (5.9) in the interval  $(\frac{1}{2}, 0.720536)$  and by the approximation obtained from (6.3) in the interval  $(0.720536, 1)$ .

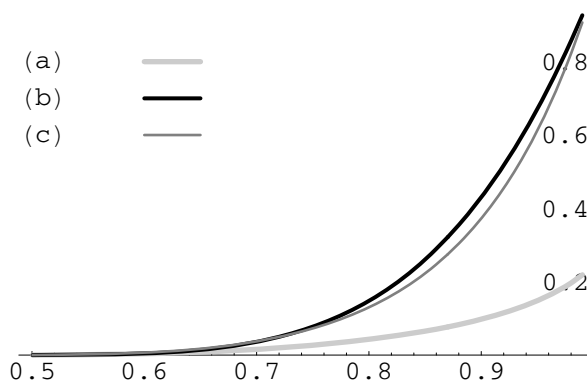


FIGURE 4

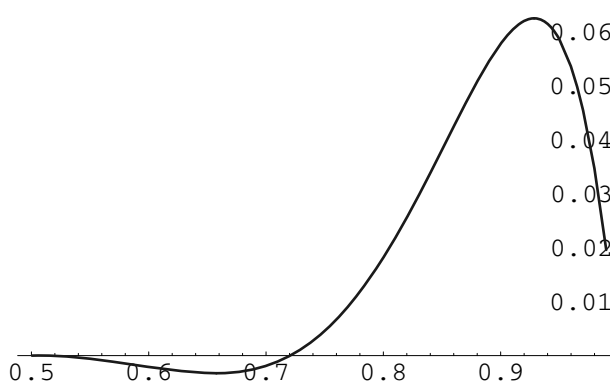


FIGURE 5

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Received 7/APR/2010  
Translated by N. SEMENOV