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WEAK CONVERGENCE OF SEQUENCES FROM FRACTIONAL PARTS OF RANDOM VARIABLES AND APPLICATIONS

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RITA GIULIANO

Abstract. We prove results concerning the weak convergence to the uniform distribution on $[0,1]$ of sequences $(Z_n)_{n\geq 1}$ of the form $Z_n = Y_n \pmod 1 = \{Y_n\}$, where $(Y_n)_{n\geq 1}$ is a general sequence of real random variables. Applications are given: (i) to the case of partial sums of (i.i.d.) random variables having a distribution belonging to the domain of attraction of a stable law; (ii) to the case of sample maxima of i.i.d. random variables.

1. Introduction

Let $(Y_n)_{n\geq 1}$ be a sequence of real random variables; we are interested in the sequence $(Z_n)_{n\geq 1}$ where, for every integer $n \geq 1$, $Z_n = Y_n \pmod 1$ $(= \{Y_n\})$. A variety of results concerning the convergence in distribution of the sequence $(Z_n)_{n\geq 1}$ have been proved in past years in the case in which Y_n is the n-th partial sum of i.i.d. random variables $(X_n)_{n\geq 1}$ taking values in $[0,1]$ (or, more generally, in a compact Hausdorff group). Roughly speaking, such types of results state that the partial sums of "nice" random variables $(X_n)_{n\geq 1}$ converge in distribution to the uniform distribution on $[0,1]$ (in the more general case, to the Haar measure of the group). See for instance the paper [10] for an exhaustive list of references on the subject.

On the other hand, little attention seems to have been paid to other kinds of sequences $(Y_n)_{n\geq 1}$ (for instance to partial sums of independent random variables $(X_n)_{n\geq 1}$ not identically distributed, or not taking values in $[0,1]$, or to partial sums of dependent random variables $(X_n)_{n\geq 1}$, or even to more general sequences).

The present paper is an attempt to fill in the gap: here we consider the case of a general real random sequence $(Y_n)_{n\geq 1}$, and our aim is twofold:

(i) to give necessary and sufficient conditions for the weak convergence of $(Z_n)_{n\geq 1}$ to the uniform distribution on $[0,1]$ using some Fourier analysis; we obtain a "Weyl criterion for probability laws on $\mathbb{R}$", i.e., Theorem 2.1 of Section 2. Section 3 contains some applications, enlightening the fact that our result can be applied to more general cases than the classic one of partial sums of i.i.d. $(X_n)_{n\geq 1}$.

(ii) to give sufficient conditions for the weak convergence of $(Z_n)_{n\geq 1}$ to the uniform distribution on $[0,1]$ in terms of the densities of the involved variables $(Y_n)_{n\geq 1}$ (assumed

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to be absolutely continuous). This is done in Section 4; our first result in this section (Theorem 4.1) relies once more on the Weyl criterion of Section 2, while the second result (Theorem 4.5) is somehow of a different nature, the main feature used there being a sort of "generalized unimodality" of the densities.

Section 5 contains some applications of the results of Section 4 (i) to the case of partial sums of (i.i.d.) random variables having a distribution belonging to the domain of attraction of a stable law; (ii) to the case of running maxima of i.i.d. random variables.

As is well known, the problem of the convergence of partial sums of random variables to the uniform distribution on $[0,1]$ is equivalent to the problem of the convergence of partial products of random variables to the so-called Benford's Law (see the introduction of [10] for more details on the history and the practical relevance of this topic). Our wider point of view sheds some new light on the subject, showing that the phenomenon of the convergence to Benford's Law is more general than expected.

2. The Weyl criterion

Theorem 2.1 (Weyl criterion for probability laws on $\mathbb{R}$)**.** *For every integer $n \geq 1$, let Y_n be a random variable on the probability space $(\Omega, \mathcal{A}, \mathsf{P})$, having law μ_n. For every integer $h \in \mathbb{Z}$, let $\hat{\mu}_n(h)$ be the h-th Fourier coefficient*

$$\hat{\mu}_n(h) = \int_{-\infty}^{+\infty} e^{2i\pi hx}\, \mu_n(dx).$$

Then the sequence $(Z_n)_{n\geq 1}$, where $Z_n = Y_n \pmod 1 = \{Y_n\}$ converges in distribution to the uniform distribution on $[0,1]$ if and only if for every integer $h \in \mathbb{Z}$, $h \neq 0$, we have

$$\lim_{n\to\infty} \hat{\mu}_n(h) = 0.$$

Proof. Let ν_n be the law of Z_n (having support in the interval $[0,1]$). For every integer $k \in \mathbb{Z}$, denote by $\mu_n^{(k)}$ the law of $Y_n - k$. The obvious relation

$$\mathsf{P}(Z_n \leq x) = \sum_{k=-\infty}^{+\infty} \mathsf{P}(0 \leq Y_n - k \leq x)$$

can be written as

$$\nu_n = \sum_{k=-\infty}^{+\infty} \mu_n^{(k)},$$

which implies, for every integer $h \in \mathbb{Z}$,

$$\begin{aligned}
\hat{\nu}_n(h) &= \int_0^1 e^{2i\pi hx}\, \nu_n(dx) \\
&= \sum_{k=-\infty}^{+\infty} \int_0^1 e^{2i\pi hx}\, \mu_n^{(k)}(dx) \\
&= \sum_{k=-\infty}^{+\infty} \int_k^{k+1} e^{2i\pi h(y-k)}\, \mu_n(dy) \\
&= \sum_{k=-\infty}^{+\infty} \int_k^{k+1} e^{2i\pi hy}\, \mu_n(dy) \\
&= \int_{-\infty}^{+\infty} e^{2i\pi hy}\, \mu_n(dy) = \hat{\mu}_n(h).
\end{aligned}$$

Hence the statement follows from the following proposition. $\square$

Proposition 2.2 (Weyl criterion for probability laws on $[0,1]$)**.** *For every integer $n \geq 1$, let Z_n be a random variable on the probability space $(\Omega, \mathcal{A}, \mathsf{P})$, having law ν_n with support in $[0,1]$. For every integer $h \in \mathbb{Z}$, let $\hat{\nu}_n(h)$ be the h-th Fourier coefficient*

$$\hat{\nu}_n(h) = \int_0^1 e^{2i\pi hx}\, \nu_n(dx).$$

Then the sequence $(Z_n)_{n\geq 1}$ converges in distribution to the uniform distribution on $[0,1]$ if and only if for every integer $h \in \mathbb{Z}$, $h \neq 0$, we have

$$\lim_{n\to\infty} \hat{\nu}_n(h) = 0. \tag{1}$$

Proof. The proof is sketched in [1, p. 50]. We detail it for the reader's convenience. The convergence in distribution of ν_n to the uniform distribution on $[0,1]$ is equivalent to the following condition:

($\mathcal{H}$) for every complex-valued continuous function f defined on $\mathbb{R}$ we have

$$\lim_{n\to\infty} \int_0^1 f(x)\, \nu_n(dx) = \int_0^1 f(x)\, dx.$$

For $f(x) = e^{2i\pi hx}$, condition $\mathcal{H}$ becomes (1).

Conversely, assume that (1) holds for every $h \in \mathbb{Z} \setminus \{0\}$ and let us prove $\mathcal{H}$. Let f be a fixed complex-valued continuous function. We can repeat almost verbatim the proof of the sufficiency for the classical Weyl criterion for uniform distributed sequences of numbers (following [9, p. 7], for instance): let $\varepsilon > 0$ be an arbitrary positive number; by the Weierstrass approximation theorem, there exists a trigonometric polynomial $\Psi(x)$, i.e., a finite linear combination of functions of the type $e^{2i\pi hx}$, $h \in \mathbb{Z}$, such that

$$\sup_{0\leq x\leq 1} |f(x) - \Psi(x)| \leq \varepsilon.$$

We may assume that

$$\int_0^1 \Psi(x)\, dx = 0; \tag{2}$$

hence, by (1),

$$\int_0^1 \Psi(x)\, \nu_n(dx) \to 0, \qquad n \to \infty. \tag{3}$$

Since, for $n \geq n_0$,

$$\begin{aligned}
&\left|\int_0^1 f(x)dx - \int_0^1 f(x)\, \nu_n(dx)\right| \\
&\quad \leq \left|\int_0^1 \big(f(x) - \Psi(x)\big)dx\right| + \left|\int_0^1 \Psi(x)dx - \int_0^1 \Psi(x)\nu_n(dx)\right| \\
&\qquad + \left|\int_0^1 \big(f(x) - \Psi(x)\big)\nu_n(dx)\right| \\
&\quad \leq 2\varepsilon + \left|\int_0^1 \Psi(x)dx - \int_0^1 \Psi(x)\, \nu_n(dx)\right|,
\end{aligned}$$

the result follows from (2), (3) and the arbitrariness of ε. □

3. Some applications of the Weyl criterion

(a) A particular case of Theorem 2.1 is the following result.

Theorem 3.1. *Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables defined on $(\Omega, \mathcal{A}, \mathsf{P})$, and let λ_n be the law of X_n (we do not assume that the (X_n) are identically distributed). Put $Y_n = X_1+\cdots+X_n$. Then the sequence $(Z_n)_{n\geq 1}$ converges in distribution to the uniform distribution on $[0,1]$ if and only if for every integer $h \in \mathbb{Z}$, $h \neq 0$, we have*

$$\lim_{n\to\infty} \hat{\lambda}_1(h)\cdots\hat{\lambda}_n(h) = 0.$$

Remark 3.2. The interest in Theorem 3.1 relies in the fact that the variables of the basic sequence $(X_n)_{n\geq 1}$ are not assumed to be identically distributed with values in $[0,1]$: this particular case is considered for instance in [4, Theorem 3, p. 274].

Remark 3.3. For every function f defined and integrable on $[0,1]$ and for every $h \in \mathbb{Z}$, put

$$\hat{f}(h) = \int_0^1 e^{2i\pi hx} f(x)\,dx.$$

In the paper [12] the following theorems are proved:

Theorem 3.3.1 (Central Limit Theorem modulo 1)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of independent absolutely continuous random variables defined on $(\Omega, \mathcal{A}, \mathsf{P})$ and with values in $[0,1]$ (not necessarily identically distributed); for each n, let f_n be a density of X_n. Put $Y_n = X_1+\cdots+X_n$. A necessary and sufficient condition for the sequence of the densities of $(Z_n)_{n\geq 1}$ to converge in $L^1([0,1])$ to the uniform density in $[0,1]$ as $n\to\infty$ is that, for each $h \in \mathbb{Z}$, $h \neq 0$, we have $\lim_{n\to\infty} \hat{f}_1(h)\cdots\hat{f}_n(h) = 0$.*

Theorem 3.3.2. *Let $(X_n)_{n\geq 1}$ be a sequence of independent discrete random variables defined on $(\Omega, \mathcal{A}, \mathsf{P})$ and with values in $[0,1]$ (not necessarily identically distributed), with densities*

$$f_n(x) = \sum_{k=1}^{r_n} w_{k,n}\delta_{\alpha_{k,n}}(x), \qquad w_{k,n} > 0, \quad \sum_{k=1}^{r_n} w_{k,n} = 1 \tag{4}$$

(where $\delta_\alpha(x)$ denotes the Dirac measure in α). Assume that there is a finite set $A \subset [0,1]$ such that all $\alpha_{k,n} \in A$. Put $Y_n = X_1+\cdots+X_n$ as above. A necessary and sufficient condition for the sequence $(Z_n)_{n\geq 1}$ to converge in distribution to the uniform distribution as $n\to\infty$ is that, for each $h \in \mathbb{Z}$, $h \neq 0$, we have $\lim_{n\to\infty} \hat{f}_1(h)\cdots\hat{f}_n(h) = 0$.

Notice that $\hat{f}_1(h)\cdots\hat{f}_n(h)$ is the h-th Fourier coefficient of the law of Y_n.

Theorem 3.1 concerns convergence in distribution only (and not L^1-convergence, as Theorem 3.3.1); hence from this point of view (the sufficiency part of) Theorem 3.3.1 gives more information than Theorem 3.1.

On the other hand, Theorem 3.1 is more general than Theorem 3.3.1 in the sense that we need not assume that the involved variables X_n are absolutely continuous and with values in $[0,1]$; hence Theorem 3.3.2 is just a particular case of Theorem 3.1: no assumptions on the densities f_n (such as (4)) are needed.

(b) Let $\alpha \in (0,2]$ be fixed. For every integer n, let Y_n be a random variable having stable density with exponent α. We recall (see for instance [4, p. 570]) that the characteristic function of the law μ_n of Y_n has the form

$$\phi_n(t) = \int_{-\infty}^{+\infty} e^{itx}\,\mu_n(dx) = e^{\psi_n(t)},$$

where

$$\psi_n(t) = |t|^\alpha C_n \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\left[\cos\frac{\pi\alpha}{2} \pm i(p_n - q_n)\sin\frac{\pi\alpha}{2}\right] \tag{5}$$

if $0 < \alpha < 1$ or $1 < \alpha \leq 2$, while for $\alpha = 1$,

$$\psi_n(t) = -|t|C_n\left[\frac{\pi}{2} \pm i(p_n - q_n)\log|t|\right]. \tag{6}$$

In both of the above formulas the upper sign applies for $t > 0$, the lower for $t < 0$; C_n are positive constants, while $p_n \geq 0$, $q_n \geq 0$ and $p_n + q_n = 1$.

Noticing that $\hat{\mu}_n(h) = \phi_n(2\pi h) = e^{\psi_n(2\pi h)}$ and that $\mathrm{Re}(\psi_n(t)) < 0$ for $t \neq 0$, from Theorem 3.1 we deduce the following.

Corollary 3.4. *Let $(Y_n)_{n\geq 1}$ be a sequence of random variables having stable density with the same exponent α. Then $(Z_n)_{n\geq 1}$ converges in distribution to the uniform distribution on $[0,1]$ if and only if $\lim_{n\to\infty} C_n = +\infty$, where the C_n are the norming constants defined in (5) and (6).*

(c) Let $\alpha \in (0,2]$, $\beta > 0$ be fixed. For every integer n, let Y_n be a random variable having characteristic function

$$\phi_n(t) = \frac{\lambda_n^{\alpha/\beta}}{(\lambda_n^\alpha + |t|^\alpha)^{1/\beta}}. \tag{7}$$

For the case $\lambda_n = 1$ this characteristic function has been studied in [2] and the corresponding distribution is a generalization of the so-called *Linnik's distribution* (which is the case $\beta = 1$ in (7); see [10]). From (7) we have, for $h \in \mathbb{Z}$,

$$\hat{\mu}_n(h) = \phi_n(2\pi h) = \frac{\lambda_n^{\alpha/\beta}}{(\lambda_n^\alpha + |2\pi h|^\alpha)^{1/\beta}},$$

and, for $h \neq 0$, $\hat{\mu}_n(h) \to 0$ as $n \to \infty$ if and only if $\lambda_n \to 0$ as $n \to \infty$.

Corollary 3.5. *Let $(Y_n)_{n\geq 1}$ be a sequence of random variables such that, for each n, Y_n has its law determined by the characteristic function (7) (with parameters α and β not depending on n). Then the sequence $(Z_n)_{n\geq 1}$ converges in distribution to the uniform distribution on $[0,1]$ if and only if $\lim_{n\to\infty} \lambda_n = 0$.*

4. More sufficient conditions

In the present section we prove two results (Theorem 4.1 and Theorem 4.5) which provide some sufficient conditions for the weak convergence of fractional parts of a sequence $(Y_n)_{n\geq 1}$. Theorem 4.1 relies once more on the Weyl criterion, while the hypotheses used in Theorem 4.5 are mostly on the set of the densities of $(Y_n)_{n\geq 1}$; the Weyl criterion seems to be of no utility in this situation.

Theorem 4.1. *Let $(Y_n)_{n\geq 1}$ be a sequence of absolutely continuous random variables defined on $(\Omega, \mathcal{A}, \mathsf{P})$. Assume that there exists a sequence of positive numbers $(a_n)_{n\geq 1}$ with $\lim_{n\to\infty} a_n = +\infty$ and such that, denoting by f_n the density of Y_n/a_n, the sequence $(f_n)_{n\geq 1}$ converges uniformly to a density f, as $n \to \infty$. Then $(Z_n)_{n\geq 1}$ converges in distribution to the uniform distribution on $[0,1]$.*

Proof. We start by proving the particular case in which $f_n = f$ for every integer n.

Lemma 4.2. *Let $(Y_n)_{n\geq 1}$ be a sequence of absolutely continuous random variables defined on $(\Omega, \mathcal{A}, \mathsf{P})$. Assume that there exists a sequence of positive numbers $(a_n)_{n\geq 1}$ with $\lim_{n\to\infty} a_n = +\infty$ and such that, for every n, Y_n/a_n has the same density f. Then $(Z_n)_{n\geq 1}$ converges in distribution to the uniform distribution on $[0,1]$.*

Proof of Lemma 4.2. Y_n has the density

$$g_n(x) = \frac{1}{a_n} f\left(\frac{x}{a_n}\right). \tag{8}$$

For $h \in \mathbb{Z}$, the h-th Fourier coefficients of g_n are given by

$$\begin{aligned}\hat{g}_n(h) &= \int_{-\infty}^{+\infty} e^{2i\pi hx} g_n(x)\,dx \\ &= \int_{-\infty}^{+\infty} e^{2i\pi hx} \frac{1}{a_n} f\left(\frac{x}{a_n}\right)\,dx \\ &= \int_{-\infty}^{+\infty} e^{2i\pi hta_n} f(t)\,dt = \phi(2\pi h a_n),\end{aligned}$$

where ϕ denotes the characteristic function of f. Since f is integrable, the Riemann–Lebesgue Theorem (see [4, Lemma 3, p. 513]) assures that, for $h \neq 0$,

$$\lim_{n\to\infty} \hat{g}_n(h) = \lim_{t\to\pm\infty} \phi(t) = 0,$$

and the conclusion follows from the Weyl criterion of Section 2. □

Now the proof of the general case carries over easily; in fact, denote by h_n the density of Y_n/a_n, i.e.,

$$h_n(x) = \frac{1}{a_n} f_n\left(\frac{x}{a_n}\right). \tag{9}$$

Then, as above,

$$\begin{aligned}\hat{h}_n(h) &= \int_{-\infty}^{+\infty} e^{2i\pi hta_n} f_n(t)\,dt \\ &\leq |\hat{g}_n(h)| + \int_{-\infty}^{+\infty} |f_n(t) - f(t)|\,dt,\end{aligned}$$

and the conclusion follows from Scheffé's Theorem (see [2, p. 218]) and Lemma 4.2, by means of the Weyl criterion (Theorem 2.1). □

Remark 4.3. Assume that f is bounded and put

$$\Delta_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)|.$$

From the relation

$$0 \leq h_n(x) \leq \frac{1}{a_n}\left|f_n\Big(\frac{x}{a_n}\Big) - f\Big(\frac{x}{a_n}\Big)\right| + \frac{1}{a_n} f\Big(\frac{x}{a_n}\Big) \leq \frac{\|f\|_\infty + \Delta_n}{a_n} \quad \text{for all } x \in \mathbb{R},$$

it follows that the sequence $(h_n)_{n\geq 1}$ converges uniformly to 0.

Let $(Y_n)_{n\geq 1}$ be a sequence of absolutely continuous random variables, such that the corresponding densities $(h_n)_{n\geq 1}$ converge uniformly to 0. In view of Remark 4.3, some natural questions arise:

(i) Does the sole condition of uniform convergence to 0 of the densities $(h_n)_{n\geq 1}$ suffice to guarantee that $(Z_n)_{n\geq 1}$ converges weakly? (Notice that the Weyl criterion is not useful in this case, since a priori on $\mathbb{R}$ we have no dominated convergence to 0 for the functions $x \mapsto e^{2i\pi hx} h_n(x)$). The answer is negative, as can be expected and as Example 4.4(b) here below shows.

(ii) If $(Z_n)_{n\geq 1}$ converges weakly, is the limit the uniform distribution on $[0, 1]$ in every case? Once more, the answer is negative, as proved by Example 4.4(a).

(iii) Besides those of Theorem 4.1, are there any other sets of assumptions that, along with the condition of uniform convergence to 0 of the densities $(h_n)_{n\geq 1}$, guarantee that

$(Z_n)_{n\geq 1}$ converges weakly to the uniform distribution on $[0,1]$? We give an answer to this question in Theorem 4.5.

Example 4.4. (a) For every integer $n \geq 1$, put

$$g_n(t) = \frac{2}{n}\sum_{k=1}^{2n}\left\{(t-k)1_{[k,k+1/2]}(t) - (t-k-1)1_{(k+1/2,k+1]}(t)\right\}.$$

It is easy to check that g_n is a density. Moreover, for $0 \leq x \leq \frac{1}{2}$,

$$\sum_{k=-\infty}^{+\infty}\int_k^{k+x} g_n(t)\,dt = \sum_{k=1}^{2n}\int_k^{k+x}\frac{2}{n}(t-k)\,dt = \sum_{k=1}^{2n}\frac{2}{n}\frac{x^2}{2} = 2x^2,$$

while, for $\frac{1}{2} \leq x \leq 1$,

$$\begin{aligned}\sum_{k=-\infty}^{+\infty}\int_k^{k+x} g_n(t)\,dt &= \sum_{k=1}^{2n}\left(\int_k^{k+1/2}\frac{2}{n}(t-k)\,dt - \int_{k+1/2}^{k+x}\frac{2}{n}(t-k-1)\,dt\right)\\ &= \sum_{k=1}^{2n}\frac{2}{n}\left(\frac{1}{4} - \frac{(x-1)^2}{2}\right) = 1-2(x-1)^2.\end{aligned}$$

Hence we have obtained the limit distribution function

$$F(x) = 2x^2 \quad \text{for } 0 \leq x \leq \frac{1}{2}, \qquad F(x) = 1-2(x-1)^2 \quad \text{for } \frac{1}{2} \leq x \leq 1.$$

(b) It is now easy to construct a sequence $(h_n)_{n\geq 1}$ of densities such that $(Z_n)_{n\geq 1}$ does not converge. Let g_n be the density of point (a) and denote by f_n the Cauchy density with parameter n, i.e.,

$$f_n(t) = \frac{n}{\pi(n^2+t^2)}, \qquad t \in \mathbb{R}.$$

We know from Theorem 2.1 (or Theorem 4.1) that

$$\lim_{n\to\infty}\sum_{k=-\infty}^{+\infty}\int_k^{k+x} f_n(t)\,dt = x,$$

while, from the previous point (a),

$$\lim_{n\to\infty}\sum_{k=-\infty}^{+\infty}\int_k^{k+x} g_n(t)\,dt = F(x).$$

Hence we can take

$$h_n(t) = g_n(t) \quad \text{for even } n, \qquad f_n(t) = h_n(t) \quad \text{for odd } n.$$

Example 4.4 suggests that we may obtain the uniform distribution on $[0,1]$ as the weak limit of $(Z_n)_{n\geq 1}$ if the corresponding densities $(h_n)_{n\geq 1}$ have some property not too far from unimodality. In fact, we are proving the following result.

Theorem 4.5. *Let $(Y_n)_{n\geq 1}$ be a sequence of absolutely continuous random variables defined on $(\Omega, \mathcal{A}, \mathsf{P})$ such that, for every integer n, Y_n is absolutely continuous with density g_n. Assume that g_n is continuous on its support (supposed to be a bounded or unbounded interval in $\mathbb{R}$) and that, for every integer n, there exists an interval $[a_n, b_n]$ such that g_n is increasing on $(-\infty, a_n]$ and decreasing on $[b_n, +\infty)$. Then*

$$\sup_{0\leq x\leq 1}|\,\mathsf{P}(Z_n \leq z) - x| \leq \int_{a_n-1}^{b_n+2} g_n(t)\,dt.$$

If in addition

$$\lim_{n\to+\infty}\int_{a_n-1}^{b_n+2} g_n(t)\,dt = 0, \tag{10}$$

then the sequence $(Z_n)_{n\geq 1}$ *converges in distribution to the uniform distribution on* $[0,1]$.

Remark 4.6. (i) Assumption (10) is not valid for the sequence $(g_n)_{n\geq 1}$ of Example 4.4.

(ii) Assumption (10) is verified if $(g_n)_{n\geq 1}$ converges uniformly to 0 as $n\to\infty$ and $C \doteq \sup_{n\in\mathbb{N}}(b_n - a_n) < +\infty$ (in particular the second condition is trivial if g_n is unimodal for each n).

Remark 4.7. The idea of using the property of unimodality of the densities is not new. It traces back to Feller (see [4], pp. 62–63 (b), Poincaré roulette problem). Also, it has been recently used in [6]. Our Theorem 4.5 is more general than Theorem 1 of [6] since our bound is in terms of the integral of g_n (on a suitable interval), so that g_n need not be unimodal (as assumed in Theorem 1 of [6]), or even have a finite number of modes.

Proof. With no loss of generality, we may assume that a_n and b_n are integers, with $a_n + 1 \leq b_n$, for every n. We can write

$$\begin{aligned}\mathsf{P}(Z_n \leq x) &= \sum_{k=-\infty}^{+\infty}\int_k^{k+x} g_n(t)\,dt\\ &= x\left\{\sum_{k=b_n+1}^{+\infty} g_n(\xi_k) + \sum_{k=-\infty}^{a_n-1} g_n(\xi_k)\right\} + \sum_{k=a_n}^{b_n}\int_k^{k+x} g_n(t)\,dt\end{aligned}$$

for a suitable sequence of numbers $(\xi_k)_{k\in\mathbb{Z}\setminus[a_n,b_n]}$, where, for each integer k, $\xi_k \in [k, k+x]$ (recall that g_n is continuous). The function g_n is decreasing on the half-line $x \geq b_n$ and increasing on the half-line $x \leq a_n$. For $k \geq b_n+1$ we have $b_n+1 \leq k \leq \xi_k \leq k+1$; hence

$$\sum_{k=b_n+2}^{+\infty} g_n(k) = \sum_{k=b_n+1}^{+\infty} g_n(k+1) \leq \sum_{k=b_n+1}^{+\infty} g_n(\xi_k) \leq \sum_{k=b_n+1}^{+\infty} g_n(k). \tag{11}$$

On the other hand, for $k \leq a_n - 1$ we have $k \leq \xi_k \leq k + x \leq k+1 \leq a_n$, so that

$$\sum_{k=-\infty}^{a_n-1} g_n(k) \leq \sum_{k=-\infty}^{a_n-1} g_n(\xi_k) \leq \sum_{k=-\infty}^{a_n-1} g_n(k+1) = \sum_{k=-\infty}^{a_n} g_n(k). \tag{12}$$

It is easy to see that

$$\sum_{k=b_n+1}^{+\infty} g_n(k) \leq \int_{b_n}^{+\infty} g_n(t)\,dt, \qquad \sum_{k=-\infty}^{a_n} g_n(k) \leq \int_{-\infty}^{a_n+1} g_n(t)\,dt, \tag{13}$$

$$\sum_{k=-\infty}^{a_n-1} g_n(k) \geq \int_{-\infty}^{a_n-1} g_n(t)\,dt, \qquad \sum_{k=b_n+2}^{+\infty} g_n(k) \geq \int_{b_n+2}^{+\infty} g_n(t)\,dt; \tag{14}$$

from relations (11), (12), (13) we obtain

$$\begin{aligned}\mathsf{P}(Z_n \leq x) &\leq x\left(\int_{-\infty}^{a_n+1} g_n(t)\,dt + \int_{b_n}^{+\infty} g_n(t)\,dt\right) + \sum_{k=a_n}^{b_n}\int_k^{k+x} g_n(t)\,dt\\ &= x\left(1 - \int_{a_n+1}^{b_n} g_n(t)\,dt\right) + \sum_{k=a_n}^{b_n}\int_k^{k+x} g_n(t)\,dt;\end{aligned} \tag{15}$$

from relations (11), (12), (14) we obtain

$$
\begin{aligned}
\mathsf{P}(Z_n \le x) &\ge x\left(\int_{-\infty}^{a_n-1} g_n(t)\,dt + \int_{b_n+2}^{+\infty} g_n(t)\,dt\right) + \sum_{k=a_n}^{b_n}\int_k^{k+x} g_n(t)\,dt \\
&= x\left(1 - \int_{a_n-1}^{b_n+2} g_n(t)\,dt\right) + \sum_{k=a_n}^{b_n}\int_k^{k+x} g_n(t)\,dt.
\end{aligned} \tag{16}
$$

Since

$$0 \le \sum_{k=a_n}^{b_n}\int_k^{k+x} g_n(t)\,dt \le \int_{a_n}^{b_n+1} g_n(t)\,dt,$$

the statement follows from (15) and (16). □

5. Applications to partial sums and sample maxima

(a) Let $(X_n)_{n\ge1}$ be a sequence of i.i.d. random variables, having mean $\mu = 0$ and belonging to the domain of attraction of a stable distribution F of index α, $1 < \alpha \le 2$. This means that there exists a constant $a > 0$ such that, putting $Y_n = X_1 + \cdots + X_n$, the random variables $(an^r)^{-1}Y_n$ converge in distribution to F, where $r = \alpha^{-1}$. Assume that the law of X_1 has a characteristic function ϕ such that $|\phi|^m$ is integrable for some integer m. From Theorem 2 of [8, p. 227] we know that $(an^r)^{-1}Y_n$ has a density f_n and that the sequence $(f_n)_{n\ge1}$ converges uniformly to the density f of F. Hence

Corollary 5.1. *Let $(X_n)_{n\ge1}$ be a sequence of centered random variables, belonging to the domain of attraction of a stable law with index $\alpha \in (1,2]$ and such that the law of X_1 has a characteristic function ϕ such that $|\phi|^m$ is integrable for some integer m. Put $Y_n = X_1 + \cdots + X_n$. Then $(Z_n)_{n\ge1}$ converges weakly to the uniform distribution on $[0,1]$.*

Remark 5.2. In the recent paper [10] it has been proved that, provided some further assumptions hold, the densities (f_n) of $(an^r)^{-1}Y_n$ converge uniformly to the density f of F even for $\alpha \in (0,1]$. Hence our corollary is in force also in these cases.

Remark 5.3. In the paper [11, p. 10] the authors remark that their proof of the Central Limit Theorem modulo 1 doesn't require the finiteness of the variance of X_1. In the light of our application to distributions belonging to the domain of attraction of a stable law, this is quite clear: apart from the case $\alpha = 2$ (i.e., the case of the Central Limit Theorem), such distributions don't have finite variance (see the lemma in [4, p. 578]).

(b) Let $(X_n)_{n\ge1}$ be a sequence of centered random variables, with finite variances. Again let $Y_n = X_1+\cdots+X_n$. It may happen that, though no assumption of independence for the basic sequence $(X_n)_{n\ge1}$ is made, nevertheless one can show, under some suitable assumptions, that there exists a sequence (a_n) with $\lim_{n\to\infty} a_n = +\infty$, such that the random variable Y_n/a_n has a density f_n which converges uniformly to the Gaussian $\mathcal{N}(0,1)$ density

$$\eta(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

(see for instance the paper [13]). Thus we have the following result.

Corollary 5.4. *Let $(X_n)_{n\ge1}$ be a sequence of random variables, centered and with finite variances. Assume that there exists a sequence (a_n) with $\lim_{n\to\infty} a_n = +\infty$, such that the random variable Y_n/a_n has a density f_n which converges uniformly to the Gaussian $\mathcal{N}(0,1)$ density. Then $(Z_n)_{n\ge1}$ converges weakly to the uniform distribution on $[0,1]$.*

(c) Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables, with common distribution F. Put $Y_n = \max(X_1, \dots, X_n)$. If, for some choice of a_n and b_n, we have

$$\lim_{n\to\infty} \mathsf{P}\left(a_n^{-1} Y_n + b_n \leq x\right) = G(x) \quad \text{for all } x \in \mathbb{R},$$

then F is said to be in the max domain of attraction of G. It is a classical fact (see [6]) that, if this is the case, G must be one of the following three extreme value types:

$$\begin{aligned} \phi_\alpha(x) &= \exp(-x^{-\alpha}), \qquad & x \geq 0,\ \alpha > 0, \\ \Psi_\alpha(x) &= \exp\big(-(-x)^\alpha\big), \qquad & x \leq 0,\ \alpha > 0, \\ \Lambda(x) &= \exp\big(-e^{-x}\big), \qquad & x \in \mathbb{R}. \end{aligned}$$

In [5] the following result is proved (Theorem 2, (a); see also [12], Lemma 1, from which we take the present formulation):

Theorem 5.5. *Suppose F is absolutely continuous with bounded density f, which is assumed to be positive for all sufficiently large x. Let f_n denote the density of Y_n/a_n, where a_n is defined by $n^{-1} = -\log F(a_n)$. If for some $\alpha > 0$,*

$$\lim_{x\to+\infty} \frac{xf(x)}{1-F(x)} = \alpha,$$

then as $n \to \infty$, $f_n(x) \to \phi'_\alpha(x)$ uniformly in x.

It is not difficult to verify that $\lim_{n\to\infty} a_n = +\infty$. Thus we have

Corollary 5.6. *Under the hypotheses of the above Theorem* 5.5*, the sequence* $(Z_n)_{n\geq 1}$ *converges in distribution to the uniform distribution on* $[0,1]$.

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Dipartimento di Matematica, "L. Tonelli", Largo B. Pontecorvo 5, Pisa 56100, Italy
E-mail address: giuliano@dm.unipi.it

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