Теорія Ймовір. та Матем. Статист.
Вип. 83, 2010

Theor. Probability and Math. Statist.
No. 83, 2011, Pages 71–82
S 0094-9000(2012)00842-9
Article electronically published on February 2, 2012

# AN APPROXIMATION OF $L_p(\Omega)$ PROCESSES

UDC 519.21

O. E. KAMENSHCHIKOVA AND T. O. YANEVICH

Abstract. Bounds for the increments of stochastic processes belonging to some classes of the space $L_p(\Omega)$ are obtained in the $L_q[a,b]$ metric. An approximation of such processes by trigonometric sums is studied in the space $L_q[0,2\pi]$.

## 1. Introduction

Conditions for stochastic processes to belong to the spaces $L_p(\Omega)$ (being a subclass of Orlicz spaces) as well as bounds for the norms of processes of the spaces $L_p(\Omega)$ are studied in the paper [1]. Properties of increments of stochastic processes belonging to Orlicz spaces and, as a particular case, those of increments of processes belonging to the spaces $L_p(\Omega)$ are studied in the paper [2].

An approximation of $\mathrm{SSub}_\varphi$ stochastic processes by cubic splines in the norm of the space $L_p(T)$ is considered in the paper [3]. The same problem but for the approximation by interpolation lines is considered in [4].

In the current paper, we consider a $2\pi$-periodic stochastic process

$$X = \{X(t), t \in \mathbf{R}\}$$

belonging to the space $L_p(\Omega)$. We study the approximation of such processes by trigonometric sums in the space $L_q[0,2\pi]$ for various relations between the numbers $p$ and $q$. For all cases considered in the paper, we obtain a bound for the best approximation with respect to accuracy and reliability. We also obtain bounds for the increments of $L_p(\Omega)$ processes in the metric of the space $L_q[a,b]$.

First we recall some definitions and results needed in what follows.

Let $(\Omega, B, \mathsf{P})$ be a standard probability space. Consider a stochastic process

$$X = \{X(t), t \in T\},$$

where $T = [a,b]$ is an interval.

**Theorem 1.1** ([5])**.** *Let $\{X(t), t \in T\}$ be a bounded and separable $L_p(\Omega)$ process. Assume that there exists an increasing continuous function $\sigma(h)$, $h > 0$, such that $\sigma(h) \to 0$ as $h \to 0$ and*

$$\sup_{\rho(t,s) \le h} \|(X(t) - X(s))\|_{L_p(\Omega)} \le \sigma(h). \tag{1}$$

*Let $\rho_X(t,s) = \|X(t) - X(s)\|_{L_p(\Omega)}$, $t,s \in T$, and let $N(\varepsilon) = N_{\rho_X}(T,\varepsilon)$, $\varepsilon > 0$, denote the metric entropy of the set of parameters $T$ with respect to the pseudometric $\rho_X$. Recall that $N(\varepsilon)$ is the minimal number of closed balls (defined with respect to the pseudometric $\rho_X$) of radius $\varepsilon$ that cover $T$. Let $\varepsilon_0 = \sup_{t,s\in T} \rho_X(t,s)$.*

---

2010 *Mathematics Subject Classification.* Primary 60G07, 41A25; Secondary 42A10.

*Key words and phrases.* The forward problem of harmonic approximation, $L_p$ processes, increments, accuracy of approximation, reliability of approximation.

*Assume that*

$$\int_0^{\varepsilon_0} N^{1/p}(\varepsilon)\, d\varepsilon < \infty.$$

*Then*

$$\left(\mathsf{E}\left(\sup_{t\in T}|X(t)|\right)^p\right)^{1/p} \le B_p$$

*and*

$$\mathsf{P}\left(\sup_{t\in T}|X(t)| \ge \varepsilon\right) \le \frac{B_p^p}{\varepsilon^p}$$

*for all* $\varepsilon > 0$*, where*

$$B_p = \inf_{t\in T}\left(\mathsf{E}\,|X(t)|^p\right)^{1/p} + \inf_{0<\theta<1}\frac{1}{\theta(1-\theta)}\int_0^{\theta\varepsilon_0} N^{1/p}(\varepsilon)\, d\varepsilon.$$

**Definition 1.1.** We say that $\widetilde{X}(t)$ approximates a process $X(t)$ with given accuracy $\varepsilon > 0$ and reliability $1-\delta$, $0 < \delta < 1$, in the space $L_q(T)$ if

$$\mathsf{P}\left\{\left(\int_T \left|X(t) - \widetilde{X}(t)\right|^q dt\right)^{1/q} > \varepsilon\right\} \le \delta.$$

**Definition 1.2** ([6])**.** Let $q \ge 1$ and let $f \in L_q[0,2\pi]$. The function

$$\omega_q(\delta) = \omega_q(\delta; f) = \sup_{0<h\le\delta}\left\{\int_0^{2\pi} |f(x+h) - f(x)|^q\, dx\right\}^{1/q}, \qquad \delta > 0,$$

is called the modulus of continuity of the function $f$.

**Definition 1.3** ([6])**.** Let $f \in L_q[0,2\pi]$ be a $2\pi$-periodic function. We approximate the function $f$ with its trigonometric sums $S_{n-1}(t)$ of an order that does not exceed $n-1$ and with period $2\pi$. Recall that the trigonometric sum of order $n$ and with period $2\pi$ is any linear combination of the following form:

$$S_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}.$$

Put

$$I_n^{(q)}[f] = \inf_{S_{n-1}\in L_q[0,2\pi]} \|f - S_{n-1}\|_{L_q[0,2\pi]}.$$

The finding of $I_n^{(q)}[f]$ is called the forward problem of the harmonic approximation in the metric of the space $L_q[0,2\pi]$.

**Theorem 1.2** ([6])**.** *Let* $q \ge 1$ *and let* $f \in L_q[0,2\pi]$ *be a* $2\pi$*-periodic function. For every natural number* $n$*, there exists a trigonometric sum* $S_{n-1}(x) = S_{n-1}(x; f)$ *of an order that does not exceed* $n-1$ *such that*

$$I_n^{(q)}[f] \le \|f - S_{n-1}\|_{L_q[0,2\pi]} \le \frac{3}{n}\omega_q\left(\frac{1}{n}; f\right).$$

*Remark* 1.1. The above definition and theorem are given in [6]. We use these results for stochastic processes for which they are valid almost surely.

## 2. The approximation of $L_p(\Omega)$ processes in the metric of the space $L_q[0, 2\pi]$. Case of $p > q > 1$

**Lemma 2.1.** *Let $p > q > 1$ and let $X = \{X(t), t \in [a, b]\}$ be a $(b-a)$-periodic stochastic process. We assume that $X$ is bounded, separable, and belongs to the space $L_p(\Omega)$. We further assume that inequality* (1) *holds for the process $X$.*

*Let $T_\delta = [0, \delta]$ and put*

$$Y(w) := \|X(t+w) - X(t)\|_{L_q[a,b]}$$

*for $w \in T_\delta$. Then $Y(w) \in L_p(\Omega)$ and moreover*

$$\sup_{w \in T_\delta} \|Y(w+h) - Y(w)\|_{L_p(\Omega)} \le |b-a|^{1/q} \cdot \sigma(h) \tag{2}$$

*for all $h > 0$.*

*Proof.* We use the Hölder inequality:

$$\int_a^b |f(t)g(t)|\, dt \le \left( \int_a^b |f(t)|^m dt \right)^{1/m} \left( \int_a^b |g(t)|^n\, dt \right)^{1/n}$$

for $m > 1$ and $n > 1$ such that

$$\frac{1}{m} + \frac{1}{n} = 1.$$

Putting $m = p/q$, $n = p/(p-q)$, $f(t) = X(t+w) - X(t)$, and $g(t) \equiv 1$, $t \in [a, b]$, in this inequality we obtain

$$\left( \int_a^b |X(t+w) - X(t)|^q\, dt \right)^{1/q} \le \left( \int_a^b |X(t+w) - X(t)|^p\, dt \right)^{1/p} \cdot c_1^{1/q},$$

where $c_1 = |b-a|^{1-q/p}$. Thus

$$\mathsf{E} \left( \int_a^b |X(t+w) - X(t)|^q\, dt \right)^{p/q} \le c_1^{p/q} \cdot \mathsf{E} \left( \int_a^b |X(t+w) - X(t)|^p\, dt \right) \tag{3}$$

and

$$\begin{aligned} \|Y(w)\|_{L_p(\Omega)} &= \left( \mathsf{E} \left( \int_a^b |X(t+w) - X(t)|^q\, dt \right)^{p/q} \right)^{1/p} \\ &\le c_1^{1/q} \left( \int_a^b \mathsf{E}\, |X(t+w) - X(t)|^p\, dt \right)^{1/p} \\ &\le c_1^{1/q} \cdot (b-a)^{1/p} \cdot \sigma(w) = (b-a)^{1/q} \cdot \sigma(w), \end{aligned}$$

that is, $Y(w) \in L_p(\Omega)$.

Using inequality (3), we get

$$\begin{aligned}
&\sup_{w\in T_\delta}\|Y(w+h)-Y(w)\|^p_{L_p(\Omega)}\\
&\quad=\sup_{w\in T_\delta}\mathsf{E}\left|\left(\int_a^b|X(t+w+h)-X(t)|^q\,dt\right)^{1/q}-\left(\int_a^b|X(t+w)-X(t)|^q\,dt\right)^{1/q}\right|^p\\
&\quad\le\sup_{w\in T_\delta}\mathsf{E}\left(\int_a^b|X(t+w+h)-X(t+w)|^q\,dt\right)^{p/q}\\
&\quad\le\sup_{w\in T_\delta}|b-a|^{p/q-1}\cdot\mathsf{E}\left(\int_a^b|X(t+w+h)-X(t+w)|^p\,dt\right)\\
&\quad\le|b-a|^{p/q}\cdot(\sigma(h))^p,
\end{aligned}$$

whence $\sup_{w\in T_\delta}\|Y(w+h)-Y(w)\|_{L_p(\Omega)}\le|b-a|^{1/q}\cdot\sigma(h)$. □

**Theorem 2.1.** *Let $\{X(t),t\in[0,2\pi]\}$ be a $2\pi$-periodic stochastic process. Assume that $X$ is bounded, separable, and belongs to the space $L_p(\Omega)$. We further assume that*

$$\sup_{t,s\in T}\|X(t)-X(s)\|_{L_p(\Omega)}\le c|t-s|^\alpha,\qquad c>0,\ 0<\alpha<1, \tag{4}$$

*where $p>q>1$.*

*Then there exists a trigonometric sum $S_{n-1}$ of an order that does not exceed $n-1$ such that*

$$\mathsf{P}\left\{I_n^{(q)}[X]>\varepsilon\right\}\le\mathsf{P}\left\{\|X(t)-S_{n-1}(t)\|_{L_q[0,2\pi]}>\varepsilon\right\}\le\frac{3}{n^{\alpha p}}\,\frac{(2\pi)^{p/q}\cdot c^p(1+\alpha p)^{1/\alpha+p}}{\varepsilon^p\cdot(\alpha p-1)^p}.$$

*Proof.* Since $Y(w)\in L_p(\Omega)$, one can use Theorem 1.1 for this process.

Put $\widetilde\sigma(h)=\widetilde ch^\alpha$ and $\widetilde c=|b-a|^{1/q}\cdot c$. Since

$$N(\varepsilon)\le\frac{\delta}{2\widetilde\sigma^{(-1)}(\varepsilon)}+1=\frac{\delta\widetilde c^{1/\alpha}}{2\varepsilon^{1/\alpha}}+1,\qquad\varepsilon_0\le\widetilde\sigma(\delta),$$

we derive from Theorem 1.1 that

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{w\in T_\delta}\left|\left(\int_a^b|X(t+w)-X(t)|^q\,dt\right)^{1/q}\right|>\varepsilon\right\}\\
&\quad=\mathsf{P}\left\{\sup_{w\in T_\delta}|Y(w)|>\varepsilon\right\}\\
&\quad\le\frac{1}{\varepsilon^p}\left(\inf_{0<\theta<1}\frac{1}{\theta(1-\theta)}\int_0^{\theta\widetilde c\delta^\alpha}\left(\frac{\delta\widetilde c^{1/\alpha}}{2\varepsilon^{1/\alpha}}+1\right)^{1/p}d\varepsilon\right)^p\\
&\quad\le\frac{1}{\varepsilon^p}\left(\inf_{0<\theta<1}\frac{1}{\theta(1-\theta)}\int_0^{\theta\widetilde c\delta^\alpha}\left(\frac{\delta\widetilde c^{1/\alpha}}{\varepsilon^{1/\alpha}}\right)^{1/p}d\varepsilon\right)^p\\
&\quad=\frac{1}{\varepsilon^p}\left(\frac{\widetilde c\delta^\alpha}{\left(1-\frac{1}{\alpha p}\right)}\inf_{0<\theta<1}\frac{1}{\theta^{1/(\alpha p)}(1-\theta)}\right)^p=\frac{1}{\varepsilon^p}\left(\frac{\widetilde c\delta^\alpha}{\left(1-\frac{1}{\alpha p}\right)}\frac{(1+\alpha p)^{1/(\alpha p)}}{1-\frac{1}{1+\alpha p}}\right)^p\\
&\quad=\frac{|b-a|^{p/q}\cdot c^p\delta^{\alpha p}(1+\alpha p)^{1/\alpha+p}}{\varepsilon^p\cdot(\alpha p-1)^p}.
\end{aligned}\tag{5}$$

Substituting $a=0$, $b=2\pi$, and $\delta=1/n$ in (5) we deduce from Theorem 1.2 that

$$\mathsf{P}\left\{I_n^{(q)}[X]>\varepsilon\right\}\le \mathsf{P}\left\{\|X(t)-S_{n-1}(t)\|_{L_q[0,2\pi]}>\varepsilon\right\}$$
$$\le \frac{3}{n^{\alpha p}}\frac{(2\pi)^{p/q}\cdot c^p(1+\alpha p)^{1/\alpha+p}}{\varepsilon^p\cdot(\alpha p-1)^p}. \qquad \square$$

## 3. The approximation of $L_p(\Omega)$ processes in the metric of the space $L_q[0,2\pi]$. Case of $p=q>1$

**Lemma 3.1.** *Let $p=q>1$ and let $X=\{X(t),t\in[a,b]\}$ be a $(b-a)$-periodic, bounded, and separable stochastic process belonging to the space $L_p(\Omega)$. We further assume that inequality* (1) *holds.*

*Set $T_\delta=[0,\delta]$ and denote $Y(w):=\|X(t+w)-X(t)\|_{L_p[a,b]}$ for $w\in T_\delta$. Then $Y(w)\in L_p(\Omega)$ and moreover*

$$\sup_{w\in T_\delta}\|Y(w+h)-Y(w)\|_{L_p(\Omega)}\le|b-a|^{1/p}\cdot\sigma(h).$$

*Proof.* We have

$$\mathsf{E}\,|Y(w)|^p=\mathsf{E}\left(\int_a^b|X(t+w)-X(t)|^p\,dt\right)$$
$$\le|b-a|\cdot\sup_{t\in[a,b]}\mathsf{E}\,|X(t+w)-X(t)|^p\le|b-a|\cdot(\sigma(w))^p.$$

We further show that $\sup_{w\in T_\delta}\|Y(w+h)-Y(w)\|_{L_p}\le|b-a|^{1/p}\cdot\sigma(h)$. Indeed,

$$\sup_{w\in T_\delta}\|Y(w+h)-Y(w)\|^p_{L_p(\Omega)}$$
$$=\sup_{w\in T_\delta}\mathsf{E}\left|\left(\int_a^b|X(t+w+h)-X(t)|^p\,dt\right)^{1/p}-\left(\int_a^b|X(t+w)-X(t)|^p\,dt\right)^{1/p}\right|^p$$
$$\le\sup_{w\in T_\delta}\mathsf{E}\left(\int_a^b|X(t+w+h)-X(t+w)|^p\,dt\right)\le|b-a|\cdot(\sigma(h))^p,$$

whence

$$\sup_{w\in T_\delta}\|Y(w+h)-Y(w)\|_{L_p}\le|b-a|^{1/p}\cdot\sigma(h). \tag{6}$$

$\square$

**Theorem 3.1.** *Let $\{X(t),t\in[0,2\pi]\}$ be a $2\pi$-periodic, bounded, and separable stochastic process belonging to the space $L_p(\Omega)$. We further assume that inequality* (4) *holds for the process $X$ and that $p=q>1$. Then there exists a trigonometric sum $S_{n-1}$ of an order that does not exceed $n-1$ such that*

$$\mathsf{P}\left\{I_n^{(p)}[f]>\varepsilon\right\}\le\mathsf{P}\left\{\|X(t)-S_{n-1}(t)\|_{L_p[0,2\pi]}>\varepsilon\right\}\le\frac{6\pi}{n^{\alpha p}}\frac{c^p(1+\alpha p)^{1/\alpha+p}}{\varepsilon^p\cdot\alpha^p p^p}.$$

*Proof.* The proof follows from Lemma 3.1 and is completely identical to that of the case $p>q>1$. $\square$

## 4. The approximation of $L_p(\Omega)$ processes in the metric of the space $L_q[0,2\pi]$. Case of $q > p > 1$

**Theorem 4.1.** *Let $\{X(t), t \in [a,b]\}$ be a $(b-a)$-periodic, separable, and bounded stochastic process belonging to the space $L_p(\Omega)$. We further assume that inequality (1) holds for this process and that $q > p > 1$. By*

$$\Delta_h X(t) = X(t+h) - X(t), \qquad t \in [a,b],$$

*we denote the increments of the process $X$.*

*Then there exists a number $m = m(h) \in \{1,2,\dots\}$ such that*

$$\big\| \|\Delta_h X(\cdot)\|_{L_q[a,b]} \big\|_{L_p(\Omega)} \le 2(b-a)^{1/p} \sum_{k=m}^{\infty} (\varepsilon_{k+1})^{1/q-1/p} [\sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k)] + 2\varepsilon_m^{1/q-1/p} (b-a)^{1/p} \sigma(2\varepsilon_m) =: B_m, \tag{7}$$

*where the sequence $\{\varepsilon_k\}_{k\ge 0}$ is such that*

1) $\varepsilon_k \ge \varepsilon_{k+1}$ *for all* $k \ge 0$,
2) $\varepsilon_k \to 0$ *as* $k \to \infty$,
3) *there is a sequence* $\{\alpha_k\}_{k\ge 0}$ *such that* $0 < \alpha_k < 1$ *and*

$$\frac{\sigma(\varepsilon_k)}{\alpha_k} \to 0 \quad \text{as} \quad k \to \infty \qquad \text{and} \qquad \sum_{k=0}^{\infty} \alpha_k < \infty.$$

*Proof.* Let $\varepsilon_k > 0$, $k \ge 0$, be a sequence such that $\varepsilon_k > \varepsilon_{k+1}$ and $\varepsilon_k \to 0$ as $k \to \infty$. Let $I_{\varepsilon_k}$ be a partition of the set $[a,b]$. Assume that the elements of this partition $B_k^r$, $1 \le r \le N(k)$, are measurable sets such that

1) $B_k^u \cap B_k^r = \varnothing$ for $u \ne r$ and $\bigcup_{r=1}^{N(k)} B_k^r = [a,b]$;
2) for every $B_k^r$, there exists a point $t_k^r$ such that $|t - t_k^r| < \varepsilon_k$ for all $t \in B_k^r$.

Note that one can always find a partition for a given $k$ such that the length of each segment does not exceed $\varepsilon_k$. In what follows we consider partitions satisfying this property.

Consider the following stochastic process:

$$X_k(t) = \sum_{r=1}^{N(\varepsilon_k)} X(t_k^r)\chi_{B_k^r}(t), \ k \ge 0, \ t \in [a,b], \quad \text{where } \chi_{B_k^r}(t) = \begin{cases} 1, & t \in B_k^r, \\ 0, & t \notin B_k^r. \end{cases}$$

Let $m$ be a number such that $2\varepsilon_{m+1} < h \le 2\varepsilon_m$. Then

$$\begin{aligned} |\Delta_h X(t)| &\le |X(t+h) - X_m(t+h)| + |X_m(t+h) - X_m(t)| + |X_m(t) - X(t)| \\ &\le \sum_{k=m}^{n-1} [|X_{k+1}(t+h) - X_k(t+h)| + |X_{k+1}(t) - X_k|] \\ &\quad + |X_m(t+h) - X_m(t)| + |X(t+h) - X_n(t+h)| + |X(t) - X_n(t)|. \end{aligned}$$

Passing to the limit as $n \to \infty$, we get

$$\begin{aligned} |\Delta_h X(t)| &\le |X_m(t+h) - X_m(t)| + \sum_{k=m}^{\infty} [|X_{k+1}(t+h) - X_k(t+h)| + |X_{k+1}(t) - X_k|] \\ &=: I_1 + I_2 \end{aligned}$$

for all $t \in [a,b]$. The properties of the partition mentioned above imply that

$$\begin{aligned}
&\|X_{k+1}(t+h) - X_k(t+h)\|_{L_p(\Omega)} \\
&\quad \le \|X_{k+1}(t+h) - X(t+h)\|_{L_p(\Omega)} + \|X(t+h) - X_k(t+h)\|_{L_p(\Omega)} \\
&\quad \le \sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k)
\end{aligned}$$

and similarly

$$\begin{aligned}
\|X_{k+1}(t) - X_k(t)\|_{L_p(\Omega)} &\le \|X_{k+1}(t) - X(t)\|_{L_p(\Omega)} + \|X(t) - X_k(t)\|_{L_p(\Omega)} \\
&\le \sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k).
\end{aligned}$$

Assuming that a point $t+h$ belongs to the interval $B_m^{r+1}$ and that $t$ belongs to $B_m^r$, where $\{t_m^{k,1}, 0 \le k \le N(\varepsilon_m)\}$ are the points of the partition

$$B_m^k = \left[t_m^{k-1}, t_m^k\right), \qquad 1 \le k \le N(\varepsilon_m),$$

we get

$$\begin{aligned}
I_1 &= |X_m(t+h) - X_m(t)| \\
&\le \sum_{r=1}^{N(\varepsilon_m)} \left|X\left(t_m^{r+1}\right) - X\left(t_m^r\right)\right| \chi_{B_m^r}(t)\chi_{B_m^{r+1}}(t+h) \\
&\le \sum_{k=1}^{N(\varepsilon_m)-1} \left|X\left(t_m^{k+1}\right) - X\left(t_m^k\right)\right| \chi_{\left[t_m^{k-1,1}, t_m^{k,1}\right)}(t) \\
&\quad + \left|X\left(t_m^{N(\varepsilon_m)+1}\right) - X\left(t_m^{N(\varepsilon_m)}\right)\right| \chi_{\left[t_m^{N(\varepsilon_m)-1,1}, t_m^{N(\varepsilon_m),1}\right]}(t),
\end{aligned}$$

where $X\left(t_m^{N(\varepsilon_m)+1}\right) = X(t_m^1)$.

We need the following result proved in [7].

**Lemma 4.1** ([7])**.** *Let $(T,\rho)$ be a compact pseudometric space, let $B$ be the $\sigma$-algebra of Borel sets in $(T,\rho)$, and let $\mu(\cdot)$ be a finite measure in the measurable space $(T,B)$. Assume that $A_k \in B$, $k = 1,2,\dots,n$, are some sets such that $A_k \cap A_l = \varnothing$ for $k \ne l$ and $\bigcup_{k=1}^n A_k = T$. Further we assume that $Y^{(n)} = \{Y^{(n)}(t), t \in T\}$ is a function such that*

$$Y^{(n)}(t) = \sum_{k=1}^n c_k \chi_{A_k}(t),$$

*where*

$$\chi_{A_k}(t) = \begin{cases} 1, & t \in A_k, \\ 0, & t \notin A_k. \end{cases}$$

*Then*

$$\|Y^{(n)}(t)\|_{L_q(T)} \le \left(\frac{1}{r_n}\right)^{1/p-1/q} \|Y^n(t)\|_{L_p(T)}, \tag{8}$$

*where $r_n = \inf_{1\le k\le n} \mu(A_k)$.*

Set

$$A_k = \left[t_m^{k-1,1}, t_m^{k,1}\right), \qquad 1 \le k \le N(\varepsilon_m) - 1,$$
$$A_{N(\varepsilon_m)} = \left[t_m^{N(\varepsilon_m)-1,1}, t_m^{N(\varepsilon_m),1}\right],$$
$$Y^{(n)}(t) = \sum_{k=1}^{N(\varepsilon_m)-1} \left|X\left(t_m^{k+1}\right) - X\left(t_m^k\right)\right| \chi_{\left[t_m^{k-1,1}, t_m^{k,1}\right)}(t) + \left|X\left(t_m^{N(\varepsilon_m)+1}\right) - X\left(t_m^{N(\varepsilon_m)}\right)\right| \chi_{\left[t_m^{N(\varepsilon_m)-1,1}, t_m^{N(\varepsilon_m),1}\right]}(t)$$

in Lemma 4.1. Then we conclude that

$$\inf_{k=1,\dots,N(\varepsilon_m)} \mu(A_k) \ge \varepsilon_m$$

and

$$\|I_1\|_{L_q[a,b]} = \|X_m(t+h) - X_m(t)\|_{L_q[a,b]} \le (\varepsilon_m)^{1/q-1/p} (2\varepsilon_m)^{1/p} \left(\sum_{k=1}^{N(\varepsilon_m)} \left|X\left(t_m^{k+1}\right) - X\left(t_m^k\right)\right|^p\right)^{1/p}.$$

Hence

$$\begin{aligned}
\left\|\|I_1\|_{L_q[a,b]}\right\|_{L_p(\Omega)} &\le 2^{1/p} \varepsilon_m^{1/q} \left(\mathsf{E} \sum_{k=1}^{N(\varepsilon_m)} \left|X\left(t_m^{k+1}\right) - X\left(t_m^k\right)\right|^p\right)^{1/p} \\
&= 2^{1/p} \varepsilon_m^{1/q} \left(\sum_{k=1}^{N(\varepsilon_m)} \mathsf{E}\left|X\left(t_m^{k+1}\right) - X\left(t_m^k\right)\right|^p\right)^{1/p} \\
&\le 2^{1/p} \varepsilon_m^{1/q} N^{1/p}(\varepsilon_m) \sigma(2\varepsilon_m) \le 2^{1/p} \varepsilon_m^{1/q} \left(\frac{b-a}{2\varepsilon_m} + 1\right)^{1/p} \sigma(2\varepsilon_m) \\
&\le 2\varepsilon_m^{1/q-1/p} (b-a)^{1/p} \sigma(2\varepsilon_m).
\end{aligned}$$

Further

$$\begin{aligned}
I_2^* &= \left\|\|I_2\|_{L_p(\Omega)}\right\|_{L_q[a,b]} \\
&= \Bigg\| \Bigg\| \sum_{k=m}^{\infty} \Bigg\{ \frac{|X_{k+1}(t) - X_k(t)|}{\|X_{k+1}(t) - X_k(t)\|_{L_p(\Omega)}} \|X_{k+1}(t) - X_k(t)\|_{L_p(\Omega)} \\
&\qquad + \frac{|X_{k+1}(t+h) - X_k(t+h)|}{\|X_{k+1}(t+h) - X_k(t+h)\|_{L_p(\Omega)}} \\
&\qquad\qquad \times \|X_{k+1}(t+h) - X_k(t+h)\|_{L_p(\Omega)} \Bigg\} \Bigg\|_{L_q[a,b]} \Bigg\|_{L_p(\Omega)} \\
&\le \Bigg\| \sum_{k=m}^{\infty} \Bigg\{ \left\| \frac{|X_{k+1}(t) - X_k(t)|}{\|X_{k+1}(t) - X_k(t)\|_{L_p(\Omega)}} \right\|_{L_q[a,b]} \\
&\qquad + \left\| \frac{|X_{k+1}(t+h) - X_k(t+h)|}{\|X_{k+1}(t+h) - X_k(t+h)\|_{L_p(\Omega)}} \right\|_{L_q[a,b]} \Bigg\} \Bigg\|_{L_p(\Omega)} \cdot [\sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k)]
\end{aligned}$$

$$
\begin{aligned}
&\le \left\| \sum_{k=m}^{\infty} \left\{ \left\| \frac{|X_{k+1}(t) - X_k(t)|}{\|X_{k+1}(t) - X_k(t)\|_{L_p(\Omega)}} \right\|_{L_p[a,b]} \right.\right. \\
&\qquad\qquad \left.\left. + \left\| \frac{|X_{k+1}(t+h) - X_k(t+h)|}{\|X_{k+1}(t+h) - X_k(t+h)\|_{L_p(\Omega)}} \right\|_{L_p[a,b]} \right\} \right\|_{L_p(\Omega)} \\
&\quad \times (\varepsilon_{k+1})^{1/q-1/p}[\sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k)] \\
&\le \sum_{k=m}^{\infty} \left\{ (b-a)^{1/p} + (b-a)^{1/p} \right\} (\varepsilon_{k+1})^{1/q-1/p}[\sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k)] \\
&= 2(b-a)^{1/p} \sum_{k=m}^{\infty} (\varepsilon_{k+1})^{1/q-1/p}[\sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k)].
\end{aligned}
$$

The latter two inequalities are proved in Lemmas 3.5 and 4.1 of [7].

Therefore,

$$
\begin{aligned}
\left\| \|\Delta_h X(\cdot)\|_{L_q[a,b]} \right\|_{L_p(\Omega)} &\le 2(b-a)^{1/p} \sum_{k=m}^{\infty} (\varepsilon_{k+1})^{1/q-1/p}[\sigma(\varepsilon_{k+1}) + \sigma(\varepsilon_k)] \\
&\quad + 2\varepsilon_m^{1/q-1/p} (b-a)^{1/p} \sigma(2\varepsilon_m).
\end{aligned}
$$

□

**Corollary 4.1.** *Let a stochastic process $\{X = X(t), t \in [a,b]\}$ be $(b-a)$-periodic, bounded, and separable. Assume further that inequality* (4) *holds for this process and that $q > p > 1$. Then*

$$
\left\| \|\Delta_h X(\cdot)\|_{L_q[a,b]} \right\|_{L_p(\Omega)} \le \frac{36 \cdot 2^{-\alpha+1/p-1/q}(b-a)^{1/p} c h^{\alpha-1/p+1/q}}{1 + \frac{1}{\alpha}\left(\frac{1}{q} - \frac{1}{p}\right)}. \tag{9}
$$

*Proof.* We choose a sequence $\{\varepsilon_k\}_{k\ge 0}$ such that

$$
\varepsilon_0 = \frac{b-a}{2}, \qquad \gamma_0 = \sigma(\varepsilon_0), \qquad \varepsilon_k = \sigma^{(-1)}\left(\theta^k \gamma_0\right),
$$

where $0 < \theta < 1$ is an arbitrary number. It is obvious that such a sequence satisfies the assumptions of Theorem 4.1. Put

$$
\begin{aligned}
B_m := 2(b-a)^{1/p} &\sum_{k=m}^{\infty} \left(\sigma^{(-1)}\left(\theta^{k+1}\gamma_0\right)\right)^{1/q-1/p} \left[\theta^{k+1}\gamma_0 + \theta^k\gamma_0\right] \\
&+ 2\varepsilon_m^{1/q-1/p}(b-a)^{1/p}\sigma(2\varepsilon_m).
\end{aligned}
$$

Then

$$
\begin{aligned}
B_m &\le 2(b-a)^{1/p} \sum_{k=m}^{\infty} \frac{\theta^{k+1}\gamma_0 + \theta^k\gamma_0}{\theta^{k+1}\gamma_0 - \theta^{k+2}\gamma_0} \int_{\theta^{k+2}\gamma_0}^{\theta^{k+1}\gamma_0} \left(\sigma^{(-1)}(u)\right)^{1/q-1/p} du \\
&\quad + 2\varepsilon_m^{1/q-1/p} (b-a)^{1/p} \sigma(2\varepsilon_m) \\
&\le 2(b-a)^{1/p} \frac{1+\theta}{\theta(1-\theta)} \int_0^{\theta\sigma(\varepsilon_m)} \left(\sigma^{(-1)}(u)\right)^{1/q-1/p} du + 2\varepsilon_m^{1/q-1/p} (b-a)^{1/p} \sigma(2\varepsilon_m).
\end{aligned}
$$

If $\sigma(h) = ch^\alpha$, then

$$B_m \le 2(b-a)^{1/p}\frac{1+\theta}{\theta(1-\theta)}\int_0^{\theta c\varepsilon_m^\alpha}\left(\frac{u}{c}\right)^{(1/q-1/p)/\alpha}du + 4c\varepsilon_m^{\alpha+1/q-1/p}(b-a)^{1/p}$$
$$= 2(b-a)^{1/p}c\frac{1+\theta}{(1-\theta)}\frac{\theta^{(1/q-1/p)/\alpha}(\varepsilon_m)^{1/q-1/p+\alpha}}{\left(\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)+1\right)} + 4c\varepsilon_m^{\alpha+1/q-1/p}(b-a)^{1/p}$$
$$= 2(b-a)^{1/p}c(\varepsilon_m)^{1/q-1/p+\alpha}\left(\frac{\theta^{(1/q-1/p)/\alpha}(1+\theta)}{\left(\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)+1\right)(1-\theta)}+2\right).$$

Since $0<\theta<1$, we have

$$2 < \frac{2\theta^{(1/q-1/p)/\alpha}(1+\theta)}{\left(\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)+1\right)(1-\theta)},$$

that is,

$$B_m \le 6(b-a)^{1/p}c(\varepsilon_m)^{1/q-1/p+\alpha}\frac{\theta^{(1/q-1/p)/\alpha}(1+\theta)}{\left(\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)+1\right)(1-\theta)}.$$

Since $2\varepsilon_{m+1} < h \le 2\varepsilon_m$, the properties of the sequence $\varepsilon_m$ and those of the function $\sigma(h)$ imply that

$$\sigma(\varepsilon_{m+1}) < \sigma(h/2) \le \sigma(\varepsilon_m),$$
$$\varepsilon_m < \frac{h}{2\theta^{1/\alpha}},$$

whence

$$B_m \le \frac{3\cdot 2^{1-\alpha+1/p-1/q}(b-a)^{1/p}ch^{1/q-1/p+\alpha}(1+\theta)}{\left(\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)+1\right)\theta(1-\theta)}.$$

For $\theta = 1/2$, the latter bound transforms to

$$B_m \le \frac{36\cdot 2^{-\alpha+1/p-1/q}(b-a)^{1/p}ch^{1/q-1/p+\alpha}}{\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)+1}.$$

Therefore

$$\left\|\|\Delta_h X(\cdot)\|_{L_q[a,b]}\right\|_{L_p(\Omega)} \le \frac{36\cdot 2^{-\alpha+1/p-1/q}(b-a)^{1/p}ch^{\alpha-1/p+1/q}}{1+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}. \qquad \square$$

**Theorem 4.2.** *Let $\{X(t), t\in T=[0,2\pi]\}$ be a $2\pi$-periodic, bounded, and separable stochastic process belonging to the space $L_p(\Omega)$. Assume further that inequality* (4) *holds for this process and that $q>p>1$. Then there exists a trigonometric sum $S_{n-1}$ of an order that does not exceed $n-1$ and such that*

$$\mathsf{P}\left\{I_n^{(q)}[X] > \varepsilon\right\} \le \mathsf{P}\left\{\|X(t)-S_{n-1}(t)\|_{L_q[0,2\pi]} > \varepsilon\right\}$$
$$\le \frac{12\pi\cdot c^p 36^p\left(\alpha p+\frac{p}{q}\right)^{p+\frac{1}{\alpha-1/p+1/q}}}{\left(1+\frac{1}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)\right)^p 2^{p\alpha+p/q}n^{\alpha p+p/q}\varepsilon^p p^p\left(p\left(\alpha-\frac{1}{p}+\frac{1}{q}\right)-1\right)^p}.$$

*Proof.* Similarly to the proof of Lemma 2.1 and Theorem 2.1 for $p>q>1$, we apply Theorem 1.1 for the stochastic process

$$Y(w) := \|X(t+w)-X(t)\|_{L_q[a,b]} \in L_p(\Omega), \qquad q>p>1.$$

Let $T_\delta = [0, \delta]$. Then

$$\sup_{w\in T_\delta} \|Y(w+h) - Y(w)\|_{L_p}^p \le \sup_{w\in T_\delta} \mathsf{E}\left(\int_a^b |X(t+w+h) - X(t+w)|^q\, dt\right)^{p/q}$$
$$\le \frac{36 \cdot 2^{-\alpha+1/p-1/q}(b-a)^{1/p} c h^{\alpha-1/p+1/q}}{1 + \frac{1}{\alpha}\left(\frac{1}{q} - \frac{1}{p}\right)},$$

whence

$$\sup_{w\in T_\delta} \|Y(w+h) - Y(w)\|_{L_p} \le \widehat{c} h^{\alpha-1/p+1/q} = \widehat{\sigma}(h),$$
$$\widehat{c} = \frac{36 \cdot 2^{-\alpha+1/p-1/q}(b-a)^{1/p} c}{1 + \frac{1}{\alpha}\left(\frac{1}{q} - \frac{1}{p}\right)}. \tag{10}$$

Hence

$$N(\varepsilon) \le \frac{\delta}{2\widehat{\sigma}^{(-1)}(\varepsilon)} + 1 = \frac{\delta \widehat{c}^{\frac{1}{\alpha-1/p+1/q}}}{2\varepsilon^{\frac{1}{\alpha-1/p+1/q}}} + 1,$$
$$\varepsilon_0 \le \widehat{\sigma}(\delta) = \widehat{c}\delta^{\alpha-1/p+1/q}.$$

Thus

$$I := \mathsf{P}\left\{\sup_{w\in T_\delta}\left|\left(\int_a^b |X(t+w) - X(t)|^q\, dt\right)^{1/q}\right| > \varepsilon\right\} = \mathsf{P}\left\{\sup_{w\in T_\delta} |Y(w)| > \varepsilon\right\}$$
$$\le \frac{1}{\varepsilon^p}\left(\inf_{0<\theta<1} \frac{1}{\theta(1-\theta)} \int_0^{\theta\widehat{c}\delta^{\alpha-1/p+1/q}} \left(\frac{\delta\widehat{c}^{\frac{1}{\alpha-1/p+1/q}}}{2\varepsilon^{\frac{1}{\alpha-1/p+1/q}}} + 1\right)^{1/p} d\varepsilon\right)^p$$
$$\le \frac{1}{\varepsilon^p}\left(\inf_{0<\theta<1} \frac{1}{\theta(1-\theta)} \int_0^{\theta\widehat{c}\delta^{\alpha-1/p+1/q}} \left(\frac{\delta\widehat{c}^{\frac{1}{\alpha-1/p+1/q}}}{\varepsilon^{\frac{1}{\alpha-1/p+1/q}}}\right)^{1/p} d\varepsilon\right)^p$$
$$= \frac{1}{\varepsilon^p}\left(\inf_{0<\theta<1} \frac{1}{\theta(1-\theta)} \left(\delta\widehat{c}^{\frac{1}{\alpha-1/p+1/q}}\right)^{1/p} \cdot \frac{(\theta\widehat{c}\delta^{\alpha-1/p+1/q})^{-\frac{1}{p(\alpha-1/p+1/q)}+1}}{1 - \frac{1}{p(\alpha-1/p+1/q)}}\right)^p$$
$$= \frac{\widehat{c}^p \delta^{p(\alpha-1/p+1/q)}}{\varepsilon^p\left(1 - \frac{1}{p(\alpha-1/p+1/q)}\right)^p} \cdot \inf_{0<\theta<1} \frac{1}{(1-\theta)\theta^{\frac{1}{p(\alpha-1/p+1/q)}}}.$$

Note that the infimum in the latter relation is attained at the point

$$\theta = \frac{1}{1 + p\left(\alpha - \frac{1}{p} + \frac{1}{q}\right)},$$

that is,

$$I \le \frac{\widehat{c}^p \delta^{p(\alpha-1/p+1/q)}\left(1 + p\left(\alpha - \frac{1}{p} + \frac{1}{q}\right)\right)^{p+\frac{1}{\alpha-1/p+1/q}}}{\varepsilon^p\left(p\left(\alpha - \frac{1}{p} + \frac{1}{q}\right) - 1\right)^p p^p}. \tag{11}$$

Using Theorem 1.2 and substituting $a = 0$, $b = 2\pi$, and $\delta = 1/n$ in inequality (11), we complete the proof. □

## Bibliography


1. T. O. Yakovenko, *Conditions for the belonging of stochastic processes to some Orlicz spaces of functions*, Visnyk Kyiv University, Ser. fiz-mat. nauk (2002), no. 5, 64–74. (Ukrainian)
2. T. O. Yakovenko, *Properties of increments of processes belonging to Orlicz spaces*, Visnyk Kyiv University, Ser. Matematika, Mekhanika (2003), no. 9–10, 142–147. (Ukrainian).
3. O. Kamenshchykova, *Approximation of random processes by cubic splines*, Theory Stoch. Processes **14(30)** (2008), no. 3–4, 53–66. MR2498604 (2010h:65007)
4. Yu. V. Kozachenko and O. E. Kamenshchikova, *Approximation of* $\mathrm{SSub}_\varphi(\Omega)$ *stochastic processes in the space* $L_p(\mathbb{T})$, Teor. Imovirnost. ta Mat. Statist. **79** (2008), 73–78; English transl. in Theor. Probability and Math. Statist. **79** (2009), 83–88. MR2494537 (2010d:60097)
5. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, TViMS, Kiev, 1998; English transl., American Mathematical Society, Providence, RI, 2000. MR1743716 (2001g:60089)
6. N. I. Akhiezer [Achieser], *Theory of Approximation*, Nauka, Moscow, 1965; English transl. of the 1st edition: Frederick Ungar Publishing, New York, 1956. MR0095369 (20:1872)
7. Yu. V. Kozachenko, *Stochastic processes in Orlicz function spaces*, Teor. Imovirnost. i Mat. Statist. **60** (1999), 64–76; English transl. in Theor. Probability and Math. Statist. **60** (2000), 73–85. MR1826143



Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue 2, Kiev 03127, Ukraine

*E-mail address*: kamalev@gmail.com

Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue 2, Kiev 03127, Ukraine

*E-mail address*: yata452@univ.kiev.ua



Received 10/JUN/2010

Translated by O. KLESOV