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LIMIT THEOREMS FOR DIFFERENCE ADDITIVE FUNCTIONALS

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Abstract. We consider additive functionals defined on Markov chains that approximate a Markov process. Sufficient conditions are obtained for the convergence of the functionals. These conditions are expressed in terms of convergence of some conditional expectations (called the characteristics of the functionals) under general assumptions on the convergence of processes. Sufficient conditions for the uniform convergence of additive functionals are also given.

1. Introduction

We study the limit behavior of the following functionals:

$$\phi_n^{s,t} \stackrel{\text{def}}{=} \sum_{k:s\le t_{n,k}<t} F_{n,k}\left(X_n\left(t_{n,k}\right)\right), \quad 0\le s<t, \tag{1.1}$$

where $\lambda_n \stackrel{\text{def}}{=} \{t_{n,k}, n, k \ge 1\}$ is a sequence of partitions of $\mathbb{R}^+$, X_n, $n \ge 1$ is a sequence of processes assuming values in a locally compact metric space $\mathfrak{X}$, and where $F_{n,k}$, $n, k \ge 1$, are nonnegative Borel functions defined on $\mathfrak{X}$. Let X_n have the Markov property at points $t_{n,k}$ and let X_n weakly converge to a Markov process X.

An approach is proposed in the papers [1]–[3] to study the limit behavior of the above functionals by using the limit behavior of their characteristics (conditional expectations). General results obtained in these papers imply the weak convergence of functionals under the assumption that the characteristics ϕ_n converge uniformly. This idea is an extension of the Dynkin approach [9] to the proof of the convergence of W-functionals (nonnegative, continuous, homogeneous functionals of Markov processes with bounded characteristics). Our contribution to this approach is that we study functionals of the form (1.1) and that we weaken the conditions for the uniform convergence (see [3]).

The results of [1], [2], and [3] are based on a certain type of convergence of the processes X_n, namely on the so-called Markov approximation (see [4] for the definition and main examples). The Markov approximation holds for many well-known cases, for example if X_n are random walks converging to a Wiener process (or, more generally, to a stable process) or if X_n are difference approximations of diffusions. However, the proof of the Markov approximation becomes too complicated for more advanced models.

This paper allows one to obtain analogous results on the convergence of ϕ_n expressed in terms of characteristics if the Markov approximation not necessarily holds. Instead, we propose a scheme suitable to deduce the convergence of ϕ_n from the convergence of their smoothed modifications. Such a result is expected in the case where the ϕ_n are integral sums for functions that uniformly converge to a continuous function. In such a

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case, an analogue of the Donsker invariance principle implies the convergence of ϕ_n to the integral functional of X. On the other hand, the distance between two smoothed functionals of the form of (1.1) (of the same process X_n) can be estimated by the distance between their characteristics by using the Markov property of X_n and following an idea of [1].

Another part of this paper is devoted to the proof of the uniform convergence of functionals. This result is motivated by the paper of Bass and Khoshnevisan [7], where the uniform convergence of families of functionals of the form (1.1) is obtained for the case where the prelimit process X_n is defined via a random walk approximating a multidimensional Wiener process. The main result of the current paper (see Theorem 2 below) can also be applied to prove the uniform convergence of approximations of additive functionals for a wide class of stochastic processes. In doing so, we are able to drop the assumption that the increments of prelimit processes are independent (this assumption is crucial for the method used in the paper [7]).

The construction of an example of a situation where the results of Section 2 apply but methods of [1]–[3] do not work leads to a nontrivial case where the Markov approximation does not hold or, at least, cannot be proved in a simple way. One of the possible examples here is a random walk in a fractal set approximating a diffusion process. The convergence of functionals can be proved in this case by using the results of Section 3 and a reasoning in [7] (see the proof of Theorem 7.1 therein). This program requires an extended treatment and will be published elsewhere.

2. Results needed to weaken the conditions on the convergence of processes

Let the trajectories of processes X_n belong to the space $\mathbb{D}$ (which, in particular, means that the trajectories have left limits at all points) and let them have the Markov property at the points of a partition $\lambda_n \stackrel{\text{def}}{=} \{t_{n,k}\}$ such that $|\lambda_n| \to 0$ as $n \to \infty$. Assume the weak convergence in distribution in the Skorokhod space $D(\mathbb{R}^+)$; namely, we assume that $X_n \stackrel{\text{w}}{\to} X$ as $n \to \infty$.

Let

$$G_{n,k}(\cdot) = \frac{F_{n,k}(\cdot)}{\Delta t_{n,k}}, \qquad \Delta t_{n,k} \stackrel{\text{def}}{=} t_{n,k} - t_{n,k-1}.$$

Using this notation, the functionals ϕ_n can be rewritten in the form of partial integral sums as follows:

$$\phi_n^{s,t} = \sum_{k:\ s\le t_{n,k}<t} G_{n,k}\left(X_n\left(t_{n,k}\right)\right)\Delta t_{n,k}. \tag{2.1}$$

According to Donsker's invariance principle, the functionals of the above type converge to the integral functional

$$\phi^{s,t} \stackrel{\text{def}}{=} \int_s^t G(r, X(r))\, dr$$

if some mild conditions are imposed on $G_{n,k}$, where G is the limit of $G_{n,k}$ and if G is a smooth function.

If the limit of $G_{n,k}$ is "essentially" discontinuous (or if the limit does not exist at all in the class of usual functions), an approach based on the convergence of functionals constructed from smoothed functions $G_{n,k}^{\epsilon}$, i.e. from

$$\phi_{n,\epsilon}^{s,t} = \sum_{k:\ s\le t_{n,k}<t} G_{n,k}^{\epsilon}\left(X_n\left(t_{n,k}\right)\right)\Delta t_{n,k},$$

can be useful. The following functions

$$f_{n,\epsilon}^{s,t}(x) \stackrel{\text{def}}{=} M_x\left(\phi_{n,\epsilon}^{s,t}\right) = \mathsf{E}\left(\phi_{n,\epsilon}^{s,t} | X(s) = x\right), \qquad f_n^{s,t}(x) \stackrel{\text{def}}{=} M_x\left(\phi_n^{s,t}\right) \tag{2.2}$$

are called the characteristics of the above functionals.

The latter are defined as integrals of measurable functions of transition probabilities existing at points $t_{n,k}$ in view of the Markov property of the processes X_n with respect to the natural flow, since these functionals are determined by values of the processes at a finite number of arguments.

For the sake of simplicity, we use the following notation:

$$G_{n,k}^0 \stackrel{\text{def}}{=} G_{n,k}, \qquad \phi_{n,0} \stackrel{\text{def}}{=} \phi_n, \quad \text{and} \quad f_{n,0} \stackrel{\text{def}}{=} f_n.$$

Consider the following random polygonal lines that are "linearizations" of discontinuous functionals ϕ_n:

$$\begin{gathered}\psi_n^{s,t} = \phi_n^{t_{n,j-1},t_{n,k-1}} - (ns - j + 1)\phi_n^{t_{n,j-1},t_{n,j}} + (nt - k + 1)\phi_n^{t_{n,k-1},t_{n,k}}, \\ s \in [t_{n,j-1}, t_{n,j}), \qquad t \in [t_{k-1}, t_{n,k}).\end{gathered} \tag{2.3}$$

The random polygonal lines ψ_n and functional ϕ can be treated as random elements with values in $C(\mathbb{T})$, where $\mathbb{T} = \{(s,t) \colon 0 \le s < t < T\}$.

We derive the limit behavior of functionals with the help of their characteristics and combine this approach with the smoothing procedure that allows us to weaken conditions on the convergence of the processes X_n used in Theorem 1 of [1].

Put $\|g\| \stackrel{\text{def}}{=} \sup_{x\in\mathfrak{X}} |g(x)|$.

Theorem 1. *Consider a family of functions* $\left\{G_{n,k}^\epsilon\right\} \in C(\mathfrak{X}, \mathbb{R}^+)$, $k \ge 0$, $n \ge 1$, $\epsilon > 0$, *and the corresponding family of limit functions* $\{G^\epsilon \in C(\mathbb{R}^+ \times \mathfrak{X}, \mathbb{R}^+)\}$. *We assume that the limit functions are bounded with respect to the second coordinate and are such that the following conditions hold for all* $T > 0$:

(1) *for all* $\epsilon > 0$,

$$\varkappa_{n,\epsilon} \stackrel{\text{def}}{=} \sup_{t_{n,k}<T} \sup_{x\in\mathfrak{X}} \left|G_{n,k}^\epsilon(x) - G^\epsilon(t_{n,k}, x)\right| \to 0, \qquad n \to \infty;$$

(2) *for all* $x \in \mathfrak{X}$ *and* $n, k \colon t_{n,k} < T$,

$$G_{n,k}^\epsilon(x) \to G_{n,k}(x), \qquad \epsilon \to 0;$$

(3) $\phi_\epsilon \xrightarrow{\text{w}} \phi$ *in distribution in* $C(\mathbb{T})$ *as* $\epsilon \to 0$, *where the functional* ϕ_ϵ *is given by*

$$\phi_\epsilon^{s,t} \stackrel{\text{def}}{=} \int_s^t G^\epsilon\left(u, X(u)\right) du, \qquad (s,t) \in \mathbb{T}.$$

Moreover, we assume that the convergence of their characteristics is uniform in $\mathfrak{X}$, *namely*

$$\alpha_\epsilon \stackrel{\text{def}}{=} \sup_{(s,t)\in\mathbb{T}} \left\|f_\epsilon^{s,t} - f^{s,t}\right\| \to 0, \qquad \epsilon \to 0;$$

(4) *as* $n \to \infty$,

$$\delta_n \stackrel{\text{def}}{=} \sup_{t_{n,k}<T} \sup_{x\in\mathfrak{X}} \Delta t_{n,k} G_{n,k}(x) \to 0;$$

(5) *as* $n \to \infty$,

$$\tau_n \stackrel{\text{def}}{=} \sup_{\epsilon\ge 0} \sup_{(s,t)\in\mathbb{T}} \left\|f_{n,\epsilon}^{s,t} - f_\epsilon^{s,t}\right\| \to 0.$$

Then the random polygonal lines corresponding to the functionals ϕ_n *converge in distribution in* $C(\mathbb{T})$, *that is,* $\psi_n \to \phi$ *as* $n \to \infty$.

In this, as well as in the other sections, we assume that the initial distributions of the processes X and X_n are identical. To prove the main result, it is sufficient to show that the finite dimensional distributions of ψ_n converge to those of ϕ and are dense. The density of the distributions can be derived from the convergence by the same methods as in [1] (see the final step of the proof of Theorem 1 on page 12 therein). Thus it remains to prove that the $\phi_n^{s,t}$ weakly converge to $\phi^{s,t}$ for fixed s and t.

We prove that

$$\mathsf{E}\, g\left(\phi_{n,\epsilon}^{s,t}\right) \to \mathsf{E}\, g\left(\phi_{\epsilon}^{s,t}\right), \qquad n \to \infty, \tag{2.4}$$

for an arbitrary Lipschitz function g and for all $\epsilon > 0$. Here and throughout below the expectation corresponds to a certain fixed initial distribution of the process X.

To make the notation shorter, we put $t_k = t_{n,k}$ and $\Delta t_k = \Delta t_{n,k}$.

It is easy to check that

$$\begin{aligned}
&\left|\mathsf{E}\, g\left(\phi_{n,\epsilon}^{s,t}\right) - \mathsf{E}\, g\left(\phi_{\epsilon}^{s,t}\right)\right| \\
&\quad \le \operatorname{Lip}(g) \cdot \mathsf{E}\left(\sum_{k\colon s \le t_k < t} \left|G^{\epsilon}\left(t_k, X_n\left(t_k\right)\right) - G_{n,k}^{\epsilon}\left(X_n\left(t_k\right)\right)\right| \Delta t_k\right) \\
&\qquad + \left|\mathsf{E}\, g\left(\sum_{k\colon s \le t_k < t} G^{\epsilon}\left(t_k, X_n\left(t_k\right)\right)\Delta t_k\right) - \mathsf{E}\, g\left(\int_s^t G^{\epsilon}\left(u, X(u)\right) du\right)\right| \\
&\quad \le \operatorname{Lip}(g) \cdot T \cdot \sup_{x \in \mathfrak{X}} \sup_{k \ge 0} \left|G^{\epsilon}\left(t_k, x\right) - G_{n,k}^{\epsilon}(x)\right| \\
&\qquad + \left|\mathsf{E}\, g\left(\sum_{k\colon s \le t_k < t} G^{\epsilon}\left(t_k, X\left(t_k\right)\right)\Delta t_k\right) - \mathsf{E}\, g\left(\int_s^t G^{\epsilon}\left(u, X\left(u\right)\right) du\right)\right| \\
&\qquad + \left|\mathsf{E}\, g\left(\sum_{k\colon s \le t_k < t} G^{\epsilon}\left(t_k, X_n\left(t_k\right)\right)\Delta t_k\right) - \mathsf{E}\, g\left(\sum_{k\colon s \le t_k < t} G^{\epsilon}\left(t_k, X\left(t_k\right)\right)\Delta t_k\right)\right|.
\end{aligned}$$

The first term on the right hand side of the latter inequality approaches zero as $n \to \infty$ by the condition of the theorem.

The proof of the convergence to zero of the second term runs as follows. First, since the trajectories of X belong to the space $\mathbb{D}$, the functions $G^{\epsilon}(X(\cdot), \cdot)$ belong to the same space. This implies that the latter functions are integrable in the Riemann sense and that the corresponding integral sums converge to the Riemann integral. Note that the integrals in the Riemann and Lebesgue sense coincide in this case. Therefore

$$\sum_{k\colon s \le t_k < t} G^{\epsilon}\left(t_k, X\left(t_k\right)\right)\Delta t_k l \to \int_s^t G^{\epsilon}\left(u, X\left(u\right)\right)\, du, \qquad |\lambda_n| \to 0,$$

almost surely. This together with the Lebesgue dominated convergence theorem and continuity of g completes the proof of convergence to zero of the second term.

To prove that the third term approaches zero we use Skorokhod's common probability space principle, which allows us to assume that the X_n converge to X in the Skorokhod metric $d(\cdot, \cdot)$ almost surely. According to the proposition stated in [5, p. 112], there exists a sequence of increasing random bijections $\theta_n \colon [0, T] \to [0, T]$ such that

$$a_n' = \|\theta_n(\cdot) - \cdot\|_{C[0,T]} \to 0 \quad \text{and} \quad a_n'' = \left\|X\left(\theta_n\left(\cdot\right)\right) - X_n\left(\cdot\right)\right\|_{\infty} \to 0$$

as $n \to \infty$. Now we estimate the third term as follows:

$$\sum_{k:\, s \le t_k < t} \left| G^\epsilon\left(t_k, X_n\left(t_k\right)\right) - G^\epsilon\left(t_k, X\left(\theta_n\left(t_k\right)\right)\right)\right| \Delta t_k$$
$$+ \sum_{k:\, s \le t_k < t} \left| G^\epsilon\left(t_k, X\left(t_k\right)\right) - G^\epsilon\left(t_k, X\left(\theta_n\left(t_k\right)\right)\right)\right| \Delta t_k$$
$$\le \omega_{G_\epsilon}\left(a_n'' + |\lambda_n|\right) + \sum_{k:\, s \le t_k < t} \left| G^\epsilon\left(t_k, X\left(t_k\right)\right) - G^\epsilon\left(t_k, X\left(\theta_n\left(t_k\right)\right)\right)\right| \Delta t_k,$$

where $\omega_f(\epsilon) \stackrel{\text{def}}{=} \sup_{x_1, x_2 \in \mathfrak{X}} |f(x_1) - f(x_2)|$. The first term on the right hand side of the latter inequality approaches zero. The second term also approaches zero almost surely. To prove this result we need the following lemma.

Lemma 1. *Let $g \in \mathbb{C}([0,1] \times \mathfrak{X}, \mathbb{R})$ be bounded, $h \in \mathbb{D}([0,1], \mathfrak{X})$, and let $\lambda_n = \{t_{n,k}\}$ and θ_n be the same as above. Put $\Delta_{g,h,\theta_n}(t_{n,k}) \stackrel{\text{def}}{=} g\left(t_{n,k}, h\left(t_k\right)\right) - g\left(t_{n,k}, h\left(\theta_n\left(t_{n,k}\right)\right)\right)$. Then*

$$\sum_k \left|\Delta_{g,h,\theta_n}\left(t_{n,k}\right)\right| \Delta t_{n,k} \to 0, \qquad n \to \infty. \tag{2.5}$$

Put

$$\widetilde{\omega}_h^A(\overline{\epsilon}) = \sup_{\substack{|u-v| < \overline{\epsilon} \\ u,v \in A}} \rho\left(h(u), h(v)\right)$$

and

$$\widetilde{\omega}_h^A = \varlimsup_{\overline{\epsilon} \to 0} \widetilde{\omega}_h^A(\overline{\epsilon})$$

for $A \subset [s,t]$. Note that $\widetilde{\omega}_h^A$ does not exceed the highest jump of h in the closure of A. Let $\Upsilon = \{u_n\}$ be the points of jump of h. Since h belongs to the space $\mathbb{D}$, we may assume that these points are written in the ascending order of heights of their jumps. Fix $\delta > 0$ and put $N_\delta \stackrel{\text{def}}{=} \sup\{n \colon u_n \ge \delta\}$ and $\Upsilon_\delta = \{u_n, n = 1, \dots, N_\delta\}$. Now we fix $\epsilon > 0$ and consider $\Upsilon_\delta^\epsilon = B\left(\Upsilon_\delta, \epsilon\right)$. By construction, $\widetilde{\omega}_h^{[0,1]\setminus\Upsilon_\delta^\epsilon} \le \delta$. Finally we fix $\gamma > 0$ and let n be sufficiently large to satisfy the inequality $a_n' = \|\theta_n(\cdot) - \cdot\| < \gamma$. Thus the expression on the left hand side of (2.5) is estimated as follows:

$$\sum_{k: t_{n,k} \in \Upsilon_\delta^{2\epsilon}} \left|\Delta_{g,h,\theta_n}\left(t_{n,k}\right)\right| \Delta t_{n,k} + \sum_{k: t_{n,k} \overline{\in} \Upsilon_\delta^{2\epsilon}} \left|\Delta_{g,h,\theta_n}\left(t_{n,k}\right)\right| \Delta t_{n,k}$$
$$\le 4 N_\delta \epsilon \times 2 \|g\|_\infty + \omega_g\left(\lambda_n + \widetilde{\omega}_h^{[0,1]\setminus\Upsilon_\delta^{2\epsilon}}(\gamma)\right).$$

Since γ is arbitrary, the upper limit as $n \to \infty$ of the left hand side of (2.5) does not exceed

$$8 N_\delta \epsilon \|g\|_\infty + \omega_g\left(\widetilde{\omega}_h^{[0,1]\setminus\Upsilon_\delta^{2\epsilon}}\right) \le 8 N_\delta \epsilon \|g\|_\infty + \omega_g(\delta).$$

The latter relation proves the lemma, since δ and ϵ are arbitrary. Therefore relation (2.4) holds.

The next step of the proof of the theorem is to show that

$$\varlimsup_{\epsilon \to 0} \limsup_{n \to \infty} \left| \mathsf{E}\, g\left(\phi_{n,\epsilon}^{s,t}\right) - \mathsf{E}\, g\left(\phi_n^{s,t}\right)\right| = 0 \tag{2.6}$$

for an arbitrary function $g \in \operatorname{Lip}(\mathbb{R})$. To prove (2.6), we need an estimate for the limit behavior of $\mathsf{E}\left|\phi_{n,\epsilon_1}^{s,t} - \phi_{n,\epsilon_2}^{s,t}\right|$ as $n \to \infty$.

Lemma 2. *The following inequality holds:*

$$\mathsf{E}\left|\phi^{s,t}_{n,\epsilon_1}-\phi^{s,t}_{n,\epsilon_2}\right| \le \left[2\left(\left\|f^{0,T}\right\|+\tau_n+\alpha_{\epsilon_1}+\alpha_{\epsilon_2}\right)\right]^{\frac12}\left[\varkappa_{n,\epsilon}+\delta_n+\sup_{0\le s<t\le T}\left\|f^{s,t}_{n,\epsilon_1}-f^{s,t}_{n,\epsilon_2}\right\|\right]^{\frac12}.$$

By the Cauchy inequality,

$$\mathsf{E}\left|\phi^{s,t}_{n,\epsilon_1}-\phi^{s,t}_{n,\epsilon_2}\right| \le \left(\mathsf{E}\left(\phi^{s,t}_{n,\epsilon_1}-\phi^{s,t}_{n,\epsilon_2}\right)^2\right)^{\frac12}. \tag{2.7}$$

Put

$$\xi_n \overset{\text{def}}{=} \phi_{n,\epsilon_1}, \qquad \zeta_n \overset{\text{def}}{=} \phi_{n,\epsilon_2}, \quad \text{and} \quad \epsilon \overset{\text{def}}{=} \max(\epsilon_1,\epsilon_2).$$

We define the flow $\mathcal{F}^n_t$ as the natural filtration generated by the process X_n. Let

$$N_T = \#\left\{\lambda_n \cap [0,T]\right\},$$

where the symbol $\#$ stands for the number of elements of a set. It is sufficient to consider the case where $s=0$, $t=T$, and $t_{n,N_T+1}=T$.

We rewrite $\mathsf{E}\left(\phi^{s,t}_{n,\epsilon_1}-\phi^{s,t}_{n,\epsilon_2}\right)^2$ as follows:

$$\begin{aligned}
&\mathsf{E}\left(\sum_{i=0}^{N_T}\sum_{k=0}^{N_T}\left(\xi_n^{t_{n,i},t_{n,i+1}}-\zeta_n^{t_{n,i},t_{n,i+1}}\right)\left(\xi_n^{t_{n,k},t_{n,k+1}}-\zeta_n^{t_{n,k},t_{n,k+1}}\right)\right)\\
&\quad= 2\,\mathsf{E}\left[\sum_{i=0}^{N_T}\sum_{k=i+1}^{N_T}\left(\xi_n^{t_{n,i},t_{n,i+1}}-\zeta_n^{t_{n,i},t_{n,i+1}}\right)\left(\xi_n^{t_{n,k},t_{n,k+1}}-\zeta_n^{t_{n,k},t_{n,k+1}}\right)\right]\\
&\qquad+\sum_{i=0}^{N_T}\left(\xi_n^{t_{n,i},t_{n,i+1}}-\zeta_n^{t_{n,i},t_{n,i+1}}\right)^2\\
&\quad\le 2\,\mathsf{E}\left[\mathsf{E}\left(\sum_{i=0}^{N_T}\sum_{k=i+1}^{N_T}\xi_n^{t_{n,i},t_{n,i+1}}\times\left(\xi_n^{t_{n,k},t_{n,k+1}}-\zeta_n^{t_{n,k},t_{n,k+1}}\right)\,\middle|\,\mathcal{F}^n_{t_{n,i+1}}\right)\right]\\
&\quad\le 2\,\mathsf{E}\left[\mathsf{E}\left(\sum_{i=0}^{N_T}\sum_{k=i+1}^{N_T}\zeta_n^{t_{n,i},t_{n,i+1}}\times\left(\xi_n^{t_{n,k},t_{n,k+1}}-\zeta_n^{t_{n,k},t_{n,k+1}}\right)\,\middle|\,\mathcal{F}^n_{t_{n,i+1}}\right)\right]\\
&\qquad+\beta_{n,\epsilon}\\
&\quad\le 2\left(\left\|f^{0,T}_{n,\epsilon_1}\right\|+\left\|f^{0,T}_{n,\epsilon_2}\right\|\right)\times\sup_{0\le s<t\le T}\left\|f^{s,t}_{n,\epsilon_1}-f^{s,t}_{n,\epsilon_2}\right\|+\beta_{n,\epsilon},
\end{aligned} \tag{2.8}$$

where

$$\beta_{n,\epsilon}\overset{\text{def}}{=}\left(\left\|f^{0,T}_{n,\epsilon_1}\right\|+\left\|f^{0,T}_{n,\epsilon_2}\right\|\right)\times\left(\varkappa_{n,\epsilon}+\delta_n\right)\le 2\left(\left\|f^{0,T}\right\|+\tau_n+\alpha_{\epsilon_1}+\alpha_{\epsilon_2}\right)\times\left(\varkappa_{n,\epsilon}+\delta_n\right).$$

Now we apply Fatou's lemma as $\epsilon_2\to0$ and use condition (1) of Theorem 1:

$$\begin{aligned}
\mathsf{E}\left|\phi^{s,t}_{n,\epsilon_1}-\phi^{s,t}_{n}\right| &= \mathsf{E}\varliminf_{\epsilon_2\to0}\left|\phi^{s,t}_{n,\epsilon_1}-\phi^{s,t}_{n,\epsilon_2}\right| \le \varliminf_{\epsilon_2\to0}\mathsf{E}\left|\phi^{s,t}_{n,\epsilon_1}-\phi^{s,t}_{n,\epsilon_2}\right|\\
&\le \left(2\left\|f^{0,T}\right\|+\tau_n+\alpha_{\epsilon_1}\right)^{\frac12}\times\left[\varkappa_{n,\epsilon}+\delta_n+\alpha_{\epsilon_1}+\tau_n\right]^{\frac12}.
\end{aligned}$$

Finally we obtain

$$\limsup_{n\to\infty}\mathsf{E}\left|\phi^{s,t}_{n,\epsilon}-\phi^{s,t}_{n}\right| \le \left(2\alpha_\epsilon\left\|f^{0,T}\right\|+(\alpha_\epsilon)^2\right)^{\frac12},$$

where the right hand side approaches zero as $\epsilon\to0$.

To complete the proof of the theorem we note that

$$\begin{aligned}&\left|\mathsf{E}\, g(\phi^{s,t}) - \mathsf{E}\, g(\phi_n^{s,t})\right| \\ &\quad \le \left|\mathsf{E}\, g(\phi^{s,t}) - \mathsf{E}\, g(\phi_\epsilon^{s,t})\right| + \left|\mathsf{E}\, g(\phi_\epsilon^{s,t}) - \mathsf{E}\, g(\phi_{n,\epsilon}^{s,t})\right| + \left|\mathsf{E}\, g(\phi_n^{s,t}) - \mathsf{E}\, g(\phi_{n,\epsilon}^{s,t})\right|\end{aligned}$$

for all Lipschitz functions g. Passing to the limit as $n \to \infty$ we derive the statement of Theorem 1 from the above bounds, since ϵ is arbitrary.

3. Uniform invariance principle

In this section, we consider a family of functionals $\{\phi_{n,\mu}, n \ge 1, \mu \in \mathfrak{M}\}$ represented in the form of (1.1). In contrast to the preceding section, the set of parameters $\mathfrak{M}$ is of a general nature and may not be equal to $\mathbb{R}^+$.

The functionals $\phi_{n,\mu}$ are written in the same way as $\phi_{n,\epsilon}$ and are of the same form (1.1); however, the index ϵ for the setting of Section 1 is positive and measures a "closeness" to some fixed functional.

The corresponding functions and random polygonal lines are denoted by $F_{n,\mu}$ and $\psi_{n,\mu}$, respectively. Let the set $\mathfrak{M}$ be equipped with a sequence of premetrics $\delta_n(\cdot,\cdot)$. Recall that a function is called a premetric if it is finite, symmetric, and equals zero if its arguments coincide (see Definition 1.1, Section 4.1, Chapter 4 in [8]).

The characteristics for these functionals are defined similarly to the case considered in the preceding section, namely

$$f_{n,\mu}^{s,t}(x) \stackrel{\text{def}}{=} M_x\left(\phi_{n,\mu}^{s,t}\right). \tag{3.1}$$

In this section, we study the convergence of generalized processes $\{\phi_{n,\mu}\}$ with the set of parameters $\mathfrak{M}$ and with the phase space $C(\mathbb{T})$. This result, being interesting in its own right, helps to obtain the uniform convergence in $\mathfrak{M}$ of distributions of functionals with respect to a certain metric that metrizes the weak convergence.

The proof of the convergence of generalized processes is based on an upper bound for the exponential moment of the difference of two functionals corresponding to two different parameters. Such a bound is derived by using a technique similar to that of [7] and by an estimate of the L_2 distance between two functionals expressed in terms of the distance between their characteristics. The latter estimate is obtained by a method presented in the papers [1]–[3]. In contrast to [7], we do not impose the condition that the increments of the prelimit functionals are independent.

To apply the results of this section, one needs to prove the convergence of functionals for any given parameter. This can be done either with the help of methods presented in [1]–[3], or by using the idea of the preceding section.

Put

$$\rho_n(\mu,\nu) \stackrel{\text{def}}{=} \sup_{(s,t)\in\mathbb{T}} \left[\left\|f_{n,\mu}^{s,t} - f_{n,\nu}^{s,t}\right\| + \delta_n(\mu,\nu)\right]^{\frac{1}{2}}.$$

Let $N(n,\epsilon)$ be the minimal number of balls of radius ϵ (considered with respect to the premetrics ρ_n) needed to cover the set $\mathfrak{M}$. Put $H_n(\epsilon) \stackrel{\text{def}}{=} \ln N(n,\epsilon)$. To make the notation shorter we also put $t_k = t_{n,k}$. As in the preceding section, we consider random polygonal lines $\psi_{n,\mu}$. By C, we denote an arbitrary constant whose precise value does not matter for the reasoning, and thus we use the symbol C for different constants in the same relation.

Theorem 2. *Let, for some $T > 0$,*

(1) $\sup_{n\ge1,\mu\in\mathfrak{M}} \left\|f_{n,\mu}^{0,T}\right\| < +\infty$,
(2) $\sup_{\mu\in\mathfrak{M}} \sup_{(s,t)\in\mathbb{T}} \left\|f_{n,\mu}^{s,t} - f_{\mu}^{s,t}\right\| \to 0$ *as* $n \to \infty$,
(3) $\sup_{x\in\mathfrak{X}} \sup_{(s,t)\in\mathbb{T}} M_x\left(\sum_{k:\, s\le t_k<t}\left(F_{n,k}^{\mu}\left(X_n\left(t_k\right)\right) - F_{n,k}^{\nu}\left(X_n\left(t_k\right)\right)\right)^2\right) \le C\delta_n(\mu,\nu)$,

(4) $\sup_{n\geq 1}\int_0^\infty H_n(\epsilon)\,d\epsilon < +\infty$.
(5) $\sup_{\mu,\nu\in\mathfrak{M}\colon \rho_n(\mu,\nu)<\Delta}\sup_{(s,t)\in\mathbb{T}}\left\|f_\mu^{s,t}-f_\nu^{s,t}\right\|\to 0$ *as* $\Delta\to 0$,
(6) *random polygonal lines converge in* $C(\mathbb{T})$, *that is,*

$$\psi_{n,\mu}\overset{\mathrm{w}}{\to}\phi_\mu \quad \text{as } n\to\infty$$

for every fixed μ*; moreover we assume that the joint distributions weakly converge for all finite sets of parameters* μ.

Then the generalized process $\psi_{n,\cdot}$ *with the set of parameters* $\mathfrak{M}$ *and with values in* $C(\mathbb{T})$ *weakly converges in the space* $C(\mathfrak{M},C(\mathbb{T}))$ *to a generalized process* $\phi_\cdot$.

Remark 1. We assume that the generalized processes $\phi_{n,\mu}$ and ϕ_μ are continuous with respect to the parameter μ for all n.

Proof. We recall that a family of probability measures $\mathfrak{P}$ is called relatively compact if, for every sequence

$$\{P_n, n\geq 1\}\subset\mathfrak{P},$$

one can choose a weakly convergent subsequence. A family of measures $\mathfrak{P}$ is called dense if, for every $\epsilon>0$, there exists a compact set K such that

$$\inf_{P\in\mathfrak{P}}P(K)>1-\epsilon.$$

By a well-known result (see Section 5 of Chapter 1 in [5]), the weak convergence in infinite dimensional spaces follows from the convergence of finite dimensional distributions and from relative compactness of such a sequence of measures. Prokhorov's theorem [5] states that a family of measures in a complete separable metric space is relatively compact if it is dense.

Thus the proof of the main result of this section reduces to the proof of the density of the family of measures generated by the generalized process $\{\phi_{n,\mu},\mu\in\mathfrak{M}\}$.

Fix n and consider two arbitrary parameters μ and ν. By $\xi_n\overset{\mathrm{def}}{=}\phi_{n,\mu}$ and $\zeta_n\overset{\mathrm{def}}{=}\phi_{n,\nu}$ we denote the corresponding functionals. Let σ and τ be two Markov stopping times with respect to the filtration $\{\mathcal{F}_t\colon t\geq 0\}$ such that $0<\sigma\leq\tau<T$ and $\sigma,\tau\in n\mathbb{Z}$ almost surely. Now we estimate

$$M_x\left(|\xi_n^{\sigma,\tau}-\zeta_n^{\sigma,\tau}|\,/\mathcal{F}_\sigma\right).$$

First we apply the Cauchy inequality

$$M_x\left(|\xi_n^{\sigma,\tau}-\zeta_n^{\sigma,\tau}|\,/\mathcal{F}_\sigma\right)\leq M_x\left(\left(\xi_n^{\sigma,\tau}-\zeta_n^{\sigma,\tau}\right)^2/\mathcal{F}_\sigma\right)^{\frac{1}{2}} \tag{3.2}$$

and then obtain L_2 estimates similarly to the preceding section. Such estimates are obtained in the following result.

Lemma 3. *The following inequality holds:*

$$\sup_{x\in\mathfrak{X}}M_x\left(|\xi_n^{\sigma,\tau}-\zeta_n^{\sigma,\tau}|/\mathcal{F}_\sigma\right)\leq\left(8C\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|+2C\delta_n(\mu,\nu)\right)^{\frac{1}{2}}\leq C\rho_n(\mu,\nu).$$

As in the preceding section, we put $N_T = |\lambda_n \cap \{0,T\}|$. We assume that $t_{N_T+1} = T$. The square of the right hand side can be rewritten in the following form:

$$\begin{aligned}
&\mathsf{E}\left(\sum_{i=0}^{N_T}\sum_{k=0}^{N_T} \mathbf{I}_{\sigma\le t_i<\tau}\mathbf{I}_{\sigma\le t_k<\tau}\left(\xi_n^{t_i,t_{i+1}} - \zeta_n^{t_i,t_{i+1}}\right)\left(\xi_n^{t_k,t_{k+1}} - \zeta_n^{t_k,t_{k+1}}\right) \,\middle|\, \mathcal{F}_\sigma\right)\\
&\quad = 2\,\mathsf{E}\left(\sum_{i=0}^{N_T} \mathbf{I}_{\sigma\le t_i<\tau}\left(\xi_n^{t_i,t_{i+1}} - \zeta_n^{t_i,t_{i+1}}\right)\left(\xi_n^{t_{i+1},\tau} - \zeta_n^{t_{i+1},\tau}\right) \,\middle|\, \mathcal{F}_\sigma\right)\\
&\qquad + \mathsf{E}\left(\sum_{i=0}^{N_T} \mathbf{I}_{\sigma\le t_i<\tau}\left(\xi_n^{t_i,t_{i+1}} - \zeta_n^{t_i,t_{i+1}}\right)^2 \,\middle|\, \mathcal{F}_\sigma\right)\\
&\quad \le 2\,\mathsf{E}\left(\sum_{i=0}^{N_T} \mathbf{I}_{\sigma\le t_i<\tau}\xi_n^{t_i,t_{i+1}}\left(\xi_n^{t_{i+1},\tau} - \zeta_n^{t_{i+1},\tau}\right) \,\middle|\, \mathcal{F}_\sigma\right)\\
&\qquad + 2\,\mathsf{E}\left(\sum_{i:t_i\le\tau} \mathbf{I}_{\sigma\le t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \,\middle|\, \mathcal{F}_\sigma\right) + C\delta_n(\mu,\nu).
\end{aligned}$$

Here we used the following inequality:

$$\begin{aligned}
&\mathsf{E}\left(\mathbf{I}_{\sigma=t_i}\sum_{k=i}^{N_T-1}\left(\xi_n^{t_{n,k},t_{n,k+1}} - \zeta_n^{t_{n,k},t_{n,k+1}}\right)^2 \,\middle|\, \mathcal{F}_\sigma\right)\\
&\quad = \mathsf{E}\left(\mathbf{I}_{\sigma=t_i}\,\mathsf{E}\left(\sum_{k=i}^{N_T-1}\left(\xi_n^{t_{n,k},t_{n,k+1}} - \zeta_n^{t_{n,k},t_{n,k+1}}\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right) \,\middle|\, \mathcal{F}_\sigma\right) \le C\delta_n(\mu,\nu).
\end{aligned}$$

Next we prove that

$$\begin{aligned}
&\mathsf{E}\left(\mathbf{I}_{\sigma\le t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \,\middle|\, \mathcal{F}_\sigma\right)\\
&\quad = \mathsf{E}\left(\mathsf{E}\left(\mathbf{I}_{\sigma\le t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \,\middle|\, \mathcal{F}_{t_{i+1}}\right) \,\middle|\, \mathcal{F}_\sigma\right).
\end{aligned}\tag{3.3}$$

Indeed, for all $m > 0$ and $B \in \mathcal{F}_\sigma$, we have

$$\begin{aligned}
&\mathsf{E}\left(\mathbf{I}_{\sigma\le t_m}\mathbf{I}_B \times \mathsf{E}\left(\mathbf{I}_{\sigma\le t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \mid \mathcal{F}_{t_{i+1}}\right)\right)\\
&\quad = \mathsf{E}\left(\mathbf{I}_{\sigma\le t_m}\mathbf{I}_B \times \mathsf{E}\left(\mathbf{I}_{\sigma\le t_i}\mathbf{I}_{t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \mid \mathcal{F}_{t_{i+1}}\right)\right)\\
&\quad = \mathsf{E}\left(\mathbf{I}_{\sigma\le t_m}\mathbf{I}_{\sigma\le t_i}\mathbf{I}_B \times \mathsf{E}\left(\mathbf{I}_{t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \mid \mathcal{F}_{t_{i+1}}\right)\right)\\
&\quad = \mathsf{E}\left(\mathbf{I}_{\sigma\le t_i\wedge t_m}\mathbf{I}_B \times \mathsf{E}\left(\mathbf{I}_{t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \mid \mathcal{F}_{t_{i+1}}\right)\right)\\
&\quad = \mathsf{E}\left(M\left(\mathbf{I}_{\sigma\le t_i\wedge t_m}\mathbf{I}_B \times \mathbf{I}_{t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right) \mid \mathcal{F}_{t_{i+1}}\right)\right)\\
&\quad = \mathsf{E}\left(\mathbf{I}_{\sigma\le t_i\wedge t_m}\mathbf{I}_B \times \mathbf{I}_{t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right)\right)\\
&\quad = \mathsf{E}\left(\mathbf{I}_{\sigma\le t_m}\mathbf{I}_B \times \mathbf{I}_{\sigma\le t_i<\tau}\zeta_n^{t_i,t_{i+1}}\left(\zeta_n^{t_{i+1},\tau} - \xi_n^{t_{i+1},\tau}\right)\right),
\end{aligned}$$

where $a \wedge b = \min(a,b)$. We used here the property that

$$\{\sigma \le t_i\} \in \mathcal{F}_{t_{i+1}}, \qquad \{\sigma \le t_i \wedge t_m\} \cap B \in \mathcal{F}_{t_i\wedge t_m} \subset \mathcal{F}_{t_{i+1}}.$$

Summing up the above equalities with respect to m, we prove (3.3) (this means that the balance equation holds in the definition of the conditional expectation; note that the integrability of the variables under consideration follows from the boundedness of the functionals ξ_n and ζ_n for all n).

Now we turn to the initial estimates:

$$\begin{aligned}
&\mathsf{E}\left(\sum_{i=0}^{N_T} \mathbb{1}_{\sigma\le t_i<\tau}\xi_n^{t_i,t_{i+1}}\left(\xi_n^{t_{i+1},\tau}-\zeta_n^{t_{i+1},\tau}\right)\,\middle|\,\mathcal{F}_\sigma\right)\\
&\qquad= \mathsf{E}\left(\sum_{i=0}^{N_T} \mathsf{E}\left(\mathbb{1}_{\sigma\le t_i<\tau}\xi_n^{t_i,t_{i+1}}\left(\xi_n^{t_{i+1},\tau}-\zeta_n^{t_{i+1},\tau}\right)\mid\mathcal{F}_{t_{i+1}}\right)\,\middle|\,\mathcal{F}_\sigma\right)\\
&\qquad= \mathsf{E}\left(\sum_{i=0}^{N_T} \mathbb{1}_{\sigma\le t_i<\tau}\xi_n^{t_i,t_{i+1}}\,\mathsf{E}\left(\xi_n^{t_{i+1},\tau}-\zeta_n^{t_{i+1},\tau}\mid\mathcal{F}_{t_{i+1}}\right)\,\middle|\,\mathcal{F}_\sigma\right).
\end{aligned}\tag{3.4}$$

Consider the variable $\mathsf{E}\big(\xi_n^{t_{i+1},\tau}-\zeta_n^{t_{i+1},\tau}\mid\mathcal{F}_{t_{i+1}}\big)$ in more detail:

$$\begin{aligned}
&\left|\mathsf{E}\left(\xi_n^{t_{i+1},\tau}-\zeta_n^{t_{i+1},\tau}\mid\mathcal{F}_{t_{i+1}}\right)\right|\\
&\quad\le \left|\mathsf{E}\left(\xi_n^{t_{i+1},T}-\zeta_n^{t_{i+1},T}\mid\mathcal{F}_{t_{i+1}}\right)\right| + \mathbb{1}_{\tau>t_{i+1}}\left|\mathsf{E}\left(\xi_n^{\tau,T}-\zeta_n^{\tau,T}\mid\mathcal{F}_{t_{i+1}}\right)\right|\\
&\quad\le \sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\| + \left|\mathsf{E}\left(\sum_{k=i+1}^{N_T}\mathbb{1}_{\tau=t_k}\left(\xi_n^{\tau,T}-\zeta_n^{\tau,T}\right)\,\middle|\,\mathcal{F}_{t_{i+1}}\right)\right|\\
&\quad\le \sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\| + \left|\mathsf{E}\left(\sum_{k=i+1}^{N_T}\mathbb{1}_{\tau=t_k}\left(\xi_n^{t_k,T}-\zeta_n^{t_k,T}\right)\,\middle|\,\mathcal{F}_{t_{i+1}}\right)\right|\\
&\quad\le \sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\| + \left|\mathsf{E}\left(\sum_{k=i+1}^{N_T}\mathbb{1}_{\tau=t_k}\,\mathsf{E}\left(\xi_n^{t_k,T}-\zeta_n^{t_k,T}\mid\mathcal{F}_{t_k}\right)\,\middle|\,\mathcal{F}_{t_{i+1}}\right)\right|\\
&\quad\le \sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\| + \left|\mathsf{E}\left(\sum_{k=i+1}^{N_T}\mathbb{1}_{\tau=t_k}\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|\,\middle|\,\mathcal{F}_{t_{i+1}}\right)\right|\\
&\quad\le 2\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|.
\end{aligned}$$

This implies an upper bound for (3.3):

$$\begin{aligned}
&2\,\mathsf{E}\left(\sum_{i=0}^{N_T}\mathbb{1}_{\sigma\le t_i<\tau}\xi_n^{t_i,t_{i+1}}\sup_{0<s<t<T}\left\|f_n^{s,t}-g_n^{s,t}\right\|\,\middle|\,\mathcal{F}_\sigma\right)\\
&\qquad\le 2\left\|f_n^{s,t}-g_n^{s,t}\right\|\mathsf{E}\left(\xi_n^{\sigma,\tau}\mid\mathcal{F}_\sigma\right).
\end{aligned}$$

Therefore the right hand side of (3.2) is bounded from above as follows:

$$\begin{aligned}
&C\delta_n(\mu,\nu)+\left(4\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|\right)\mathsf{E}\left(\xi_n^{\sigma,\tau}+\zeta_n^{\sigma,\tau}\mid\mathcal{F}_\sigma\right)\\
&\quad\le C\delta_n(\mu,\nu)+\left(4\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|\right)\mathsf{E}\left(\xi_n^{\sigma,\sigma+T}+\zeta_n^{\sigma,\sigma+T}\mid\mathcal{F}_\sigma\right)\\
&\quad\le C\delta_n(\mu,\nu)+\left(4\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|\right)\left(f_n^{\sigma,\sigma+T}\left(X_n(\sigma)\right)+g_n^{\sigma,\sigma+T}\left(X_n(\sigma)\right)\right)\\
&\quad\le C\delta_n(\mu,\nu)+\left(4\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|\right)\left(\sup_{0<s<t<2T}\left\|f_n^{s,t}\right\|+\sup_{0<s<t<2T}\left\|g_n^{s,t}\right\|\right)\\
&\quad\le C\delta_n(\mu,\nu)+4C\sup_{(s,t)\in\mathbb{T}}\left\|f_n^{s,t}-g_n^{s,t}\right\|\le C\rho_n^2(\mu,\nu).
\end{aligned}$$

Hence

$$\mathsf{E}\left(\left|\xi_n^{\sigma,\tau}-\zeta_n^{\sigma,\tau}\right|\,/\,\mathcal{F}_\sigma\right)\le C\rho_n(\mu,\nu).$$

Now we apply Theorem 109 in Section 3, Chapter VI of [6] and obtain

$$\mathsf{E}\exp\left(\lambda \sup_{\substack{sn,tn\in\mathbb{Z}\\ 0<s<t<T}} \left|\xi_n^{s,t}-\zeta_n^{s,t}\right|\right) \le \frac{1}{1-4\lambda C\rho_n(\mu,\nu)},$$

where

$$\lambda \in \left(0, \frac{1}{4\rho_n(\mu,\nu)}\right).$$

Choosing $\lambda_n \stackrel{\text{def}}{=} 1/(8\rho_n(\mu,\nu))$, we prove the following bound:

$$\mathsf{E}\exp\left(\lambda_n \sup_{\substack{sn,tn\in\mathbb{Z}\\ (s,t)\in\mathbb{T}}} \left|\xi_n^{s,t}-\zeta_n^{s,t}\right|\right) \le \frac{1}{1-4\lambda_n\rho_n(\mu,\nu)} = 2.$$

This relation together with the Chebyshev inequality allows one to get an estimate for the tails of the distribution of the difference of two local times:

$$\begin{aligned}\mathsf{P}\left\{\sup_{\substack{sn,tn\in\mathbb{Z}\\ (s,t)\in\mathbb{T}}} \left|\xi_n^{s,t}-\zeta_n^{s,t}\right| > H\right\} &= \mathsf{P}\left\{\lambda_n \sup_{\substack{sn,tn\in\mathbb{Z}\\ (s,t)\in\mathbb{T}}} \left|\xi_n^{s,t}-\zeta_n^{s,t}\right| > \lambda_n H\right\}\\ &\le \mathsf{E}\exp\left(\lambda_n \sup_{\substack{sn,tn\in\mathbb{Z}\\ (s,t)\in\mathbb{T}}} \left|\xi_n^{s,t}-\zeta_n^{s,t}\right|\right)\cdot\exp\left(-\lambda_n H\right)\\ &\le 2\exp\left(-\lambda_n H\right) = 2\exp\left(-H\frac{1}{8\rho_n(\mu,\nu)}\right).\end{aligned}$$

Now we turn to the initial notation, that is, to $\phi_{n,\mu}=\xi_m$ and $\phi_{n,\nu}=\zeta_n$. Note that $\psi_{n,\mu}$ are piecewise linear, whence

$$\|\psi_{n,\mu}-\psi_{n,\nu}\|_{C(\mathbb{T})} = \sup_{(s,t)\in\mathbb{T}}\left|\psi_{n,\mu}^{s,t}-\psi_{n,\nu}^{s,t}\right| = \sup_{\substack{sn,tn\in\mathbb{Z}\\ (s,t)\in\mathbb{T}}}\left|\phi_{n,\mu}^{s,t}-\phi_{n,\nu}^{s,t}\right|.$$

This gives the following estimate for the distribution of the distance between random polygonal lines:

$$\mathsf{P}\left\{\|\psi_{n,\mu}-\psi_{n,\nu}\|_{C(\mathbb{T})} > H\right\} \le 2\exp\left(-\frac{H}{8\rho_n(\mu,\nu)}\right).$$

Next we apply Lemma 2.1 in Section 4.2, Chapter 4 of [8]. Note that a generalization of this result is straightforward for the case where the $X_\lambda(t)$ assume values in $C(\mathbb{T})$ if the metric in this space is uniform.

In the case under consideration, μ and ν are substituted for s and t defining the functionals ξ_n and ζ_n in Lemma 2.1 of [8]. As Λ and the premetric ρ, we use the set of natural numbers $\mathbb{N}$ and ρ_n, respectively.

Further we check the assumptions of Lemma 2.1 of [8]:

(1) for all $A>0$ and $n\in\mathbb{N}$,

$$\mathsf{P}\left\{\|\psi_{n,\mu}-\psi_{n,\nu}\|_{C(\mathbb{T})} > A\right\} \le C\exp\left(-\frac{A}{8\rho_n(\mu,\nu)}\right);$$

(2) this condition holds, since

$$\sup_{n\ge1}\rho_n(\mu,\nu) \le \sqrt{\delta(\mu,\nu)+2\sup_{n,\varkappa}\left\|f_{n,\varkappa}^{0,T}\right\|} < +\infty;$$

(3) the condition

$$\int_{(0,+\infty)} H_{\rho_n}(\epsilon)\, d\epsilon < +\infty$$

follows explicitly from condition (4) of Theorem 2.

Now we conclude that, for all $\epsilon > 0$,

$$\overline{\lim_{\Delta\to 0}} \sup_{n\in\mathbb{N}} \mathsf{P}\left\{ \sup_{\substack{\mu,\nu\in\mathfrak{S}\\ \rho_n(\mu,\nu)<\Delta}} \|\psi_{n,\mu} - \psi_{n,\nu}\| > \epsilon \right\} = 0,$$

where $\mathfrak{S}$ is an arbitrary countable subset of $\mathfrak{M}$.

The rest of the proof is standard and coincides with that of the proof of the weak convergence in metric spaces. Note that the property of the equicontinuity of the family of functionals with respect to the parameter μ proved above implies that the family of distributions of processes $\phi_{n,\cdot}$ is relatively compact. Since the finite dimensional distributions of functionals converge, this completes the proof of the theorem. $\square$

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