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ON THE DISTRIBUTION OF STORAGE PROCESSES FROM THE CLASS $V(\varphi,\psi)$

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Abstract. Estimates for the distribution of a storage process

$$Q(t)=\sup_{s\le t}\bigl(X(t)-X(s)-(f(t)-f(s))\bigr)$$

are obtained in the paper, where $(X(t),t\in T)$ is a stochastic process belonging to the class $V(\varphi,\psi)$ and where the service output rate $f(t)$ is a continuous function. In particular, the results hold if $(X(t),t\in T)$ is a Gaussian process. Several examples of applications of the results obtained in the paper are given for sub-Gaussian stationary stochastic processes.

1. Introduction

A considerable number of papers are devoted nowadays to studies of the behavior of the data traffic in computer networks (see, for example, [1, 2, 4, 8, 10]). A special place in this topic is occupied by the problem of estimating the reliability when transferring the data. Several stochastic models of the data traffic have been studied over recent years. Among those models are stochastic processes belonging to some classes describing a specific traffic of the data, namely Markov stochastic processes, processes with long as well as with short memory, automodel processes, Gaussian processes, especially fractional Brownian motion, autoregressive integrated moving average processes (ARIMA), etc.

The current paper deals with the distribution of a storage process generated by a stochastic process $\{X(t),t\in T\}$ belonging to a rather general class $V(\varphi,\psi)$ containing, for example, Gaussian stochastic processes.

Let T be a set of parameters.

The process under consideration is defined as follows.

Definition 1.1. A process $Q(t)=\{Q(t),t\in T\}$ given by

$$Q(t)=\sup_{s\le t}\bigl(X(t)-X(s)-(f(t)-f(s))\bigr),\qquad s,t\in T,$$

is called a *storage process* generated by the input $X(t)=\{X(t),t\in T\}$ and a service output rate $f(t)$.

Assume that a work arrives to a server. If more work arrives than can be processed, the surplus waits in the queue buffer of length $x\ge 0$. A work that arrives after the buffer overflows is lost. An important problem of queuing theory is to estimate the overflow probability, namely

$$\mathsf{P}\{Q(t)>x\}.$$

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The same problem is stated in terms of the ruin probability for the corresponding risk process in risk theory.

We consider the case where the input process $X(t)$ belongs to the class $V(\varphi,\psi)$. Recall that $V(\varphi,\psi)$ contains the class of φ-sub-Gaussian stochastic processes. More detail and results concerning the processes of $V(\varphi,\psi)$ can be found in [3, 13], and other sources are included in the list of references at the end of the paper.

The paper is organized as follows. Section 2 contains necessary definitions and results concerning φ-sub-Gaussian random variables and stochastic processes. Section 3 is devoted to the main results proved with the help of the metric entropy method. An application of the results obtained in Section 2 to stationary sub-Gaussian stochastic processes is given in Section 3.

2. Necessary definitions and results

Let $\{\Omega,\mathcal{B},P\}$ be a standard probability space and let T be a space of parameters.

2.1. Orlicz N-functions, Young–Fenchel transform.

Definition 2.1 ([3])**.** A function $U=\{U(x),x\in\mathbb{R}\}$ is called an *Orlicz N-function* if U is continuous, even, and such that

1) $U(0)=0$,
2) $U(x)$ increases for $x>0$,
3) $U(x)/x\to 0$ as $x\to 0$ and $U(x)/x\to\infty$ as $x\to\infty$.

Condition Q. We say that condition **Q** holds for an Orlicz N-function φ if

$$\liminf_{x\to 0}\frac{\varphi(x)}{x^2}=c>0. \tag{1}$$

Definition 2.2. We say that an N-function φ_1 is subordinated to an N-function φ_2 and denote $\varphi_1\prec\varphi_2$ if there are some constants $c>0$ and $x_0>0$ such that $\varphi_1(x)<\varphi_2(cx)$ for $x>x_0$.

We say that two N-functions φ_1 and φ_2 are equivalent if $\varphi_1\prec\varphi_2$ and $\varphi_2\prec\varphi_1$.

Definition 2.3. Let $\varphi=\{\varphi(x),x\in\mathbb{R}\}$ be an N-function. Then

$$\varphi^*(x)=\sup_{y\in\mathbb{R}}(xy-\varphi(y))$$

is called the Young–Fenchel transform of the function φ.

2.2. φ-sub-Gaussian random variables and stochastic processes.

Definition 2.4 ([3])**.** Let φ be an N-function satisfying condition **Q**. We say that a random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if

1) $\mathsf{E}\,\xi=0$,
2) $\mathsf{E}\exp\{\lambda\xi\}$ is finite for all $\lambda\in\mathbb{R}$, and
3) there exists a constant $a>0$ such that

$$\mathsf{E}\exp\{\lambda\xi\}\le\exp\{\varphi(\lambda a)\} \tag{2}$$

for all $\lambda\in\mathbb{R}$.

Theorem 2.1 ([3])**.** *The space* $\mathrm{Sub}_\varphi(\Omega)$ *is a Banach space with respect to the norm*

$$\tau_\varphi(\xi)=\sup_{\lambda>0}\frac{\varphi^{(-1)}(\log\mathsf{E}\exp\{\lambda\xi\})}{\lambda}, \tag{3}$$

where $\varphi^{(-1)}$ *is the inverse function to* φ*. In addition,*

$$\mathsf{E}\exp\{\lambda\xi\} \le \exp\{\varphi(\lambda\tau_\varphi(\xi))\} \tag{4}$$

for all $\lambda \in \mathbb{R}$*.*

Lemma 2.1 ([13])**.** *Let a random variable* $\xi \in \mathrm{Sub}_\varphi(\Omega)$ *be such that* $\tau_\varphi(\xi) > 0$ *and let* $\varepsilon > 0$*. Then*

$$\mathsf{P}\{\xi > \varepsilon\} \le \exp\biggl\{-\varphi^*\biggl(\frac{\varepsilon}{\tau_\varphi(\xi)}\biggr)\biggr\},$$
$$\mathsf{P}\{\xi < -\varepsilon\} \le \exp\biggl\{-\varphi^*\biggl(\frac{\varepsilon}{\tau_\varphi(\xi)}\biggr)\biggr\},$$
$$\mathsf{P}\{|\xi| > \varepsilon\} \le 2\exp\biggl\{-\varphi^*\biggl(\frac{\varepsilon}{\tau_\varphi(\xi)}\biggr)\biggr\}.$$

Lemma 2.2 ([13])**.** *Let a random variable* ξ *belong to the space* $\mathrm{Sub}_\varphi(\Omega)$*. Then*

$$\bigl|\mathsf{E}\,\xi^k\bigr| \le \mathsf{E}\,|\xi|^k \le 2(\tau_\varphi(\xi))^k\frac{e^k}{(\varphi^{(-1)}(k))^k}k! \tag{5}$$

for $k = 1, 2, \dots$ *.*

Example 2.1 ([3])*.* Let ξ be a centered Gaussian random variable, that is, $\xi \in N(0,\sigma^2)$. Then ξ belongs to the space $\mathrm{Sub}_{x^2/2}(\Omega)$ and $\tau(\xi) = (\mathsf{E}\,\xi^2)^{1/2}$.

Definition 2.5. A stochastic process $X = \{X(t), t \in T\}$ is called *φ-sub-Gaussian* if the random variables $X(t)$, $t \in T$, are φ-sub-Gaussian. We write $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ in this case.

If $\varphi(x) = x^2/2$, then the processes $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ are called *sub-Gaussian*.

Example 2.2. Every centered Gaussian stochastic process is a sub-Gaussian process.

2.3. Strictly φ-sub-Gaussian random variables and stochastic processes.

Definition 2.6 ([6])**.** A family Δ of random variables belonging to the space $\mathrm{Sub}_\varphi(\Omega)$ is called *strictly φ-sub-Gaussian* if there exists a constant $C_\Delta > 0$ such that

$$\tau_\varphi\biggl(\sum_{i\in I}\lambda_i\xi_i\biggr) \le C_\Delta\left(\mathsf{E}\biggl(\sum_{i\in I}\lambda_i\xi_i\biggr)^2\right)^{1/2} \tag{6}$$

for all finite sets I, all $\xi_i \in \Delta$, $i \in I$, and all $\lambda_i \in \mathbb{R}^1$.

The constant C_Δ is called a *defining constant* for the family Δ.

Theorem 2.2 ([6])**.** *Let* Δ *be a strictly* φ*-sub-Gaussian family. Then the linear closure of the family* Δ *in the space* $\mathrm{Sub}_\varphi(\Omega)$ (*or in* $L_2(\Omega)$) *also is a strictly* φ*-sub-Gaussian family with the same defining constant.*

The linear closure of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables forms the space of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables. This space is denoted by $\mathrm{SSub}_\varphi(\Omega)$.

Definition 2.7. A stochastic process $X = \{X(t), t \in T\}$ is called *strictly φ-sub-Gaussian* if the random variables $\{X(t), t \in T\}$ form a strictly φ-sub-Gaussian family.

The defining constant of this family is called the *defining constant* of the process X and is denoted by C_X.

2.4. **Stochastic processes belonging to the class** $V(\varphi,\psi)$.

Definition 2.8 ([13]). Let $\varphi \prec \psi$ be two Orlicz N-functions. We say that a stochastic process $X=\{X(t), t\in T\}$ belongs to the class $V(\varphi,\psi)$ if the random variable $X(t)$ belongs to the space $\mathrm{Sub}_\psi(\Omega)$ for all $t\in T$ and if the increments $X(t)-X(s)$ belong to the space $\mathrm{Sub}_\varphi(\Omega)$ for all $s,t\in T$.

Example 2.3 ([3]). Every sub-Gaussian stochastic process belongs to the class $V(\varphi,\varphi)$, where $\varphi = x^2/2$.

Example 2.4 ([13]). Let

$$X(t)=\xi_0+\sum_{k=1}^{\infty}\xi_k f_k(t),$$

where the random variable ξ_0 belongs to $\mathrm{Sub}_\psi(\Omega)$, $\{\xi_k, k=1,2,\dots\}\in \mathrm{Sub}_\varphi(\Omega)$, and where φ is an Orlicz N-function such that $\varphi(\sqrt{x})$ is even and

$$\sum_{k=1}^{\infty}\tau_\varphi(\xi_k)|f_k(t)|<\infty.$$

Then the stochastic process $X(t)$ belongs to the class $V(\varphi,\psi)$.

3. Main results

Let (T,ρ) be a pseudo-metric (metric) separable space equipped with a pseudo-metric (metric) ρ. Recall that a pseudo-metric satisfies all the conditions for a metric except the following one: if $\rho(t,s)=0$, then $t=s$. The latter means that the set $\{(t,s)\colon \rho(t,s)=0\}$ is possibly wider than the "diagonal" $\{(t,s)\colon t=s\}$ (see [3]).

Consider a separable stochastic process $X=\{X(t),\ t\in T\}$ belonging to the class $V(\varphi,\psi)$. Assume that there exists a continuous increasing function $\sigma=\{\sigma(h), h>0\}$ such that $\sigma(h)\to 0$ as $h\to 0$ and

$$\sup_{\rho(t,s)\le h}\tau_\varphi(X(t)-X(s))\le\sigma(h). \tag{7}$$

Note that the function

$$\sigma(h)=\sup_{\rho(t,s)\le h}\tau_\varphi(X(t)-X(s))$$

can be used on the right hand side of (7) if the process $X(t)$ is continuous in the norm $\tau_\varphi(\cdot)$.

Let B be a compact set, $B\subset T$. In what follows we use the following notation:

1) $\gamma(u)=\tau_\psi(X(u))$;
2) $B_t=\{u\in B\colon u\le t\}$;
3) $\beta>0$ is a number such that $\beta\le\sigma\left(\inf_{s\in B}\sup_{t\in B}\rho(t,s)\right)$;
4) $\zeta_\varphi(v)=\frac{v}{\varphi^{(-1)}(v)}$;
5) $\mathsf{N}_B(u)=\mathsf{N}_{(B,\rho)}(u)$ is the metric capacity of the space (B,ρ), that is, the minimum number of closed balls of radius u covering the space (B,ρ);
6) $\mathsf{H}_B(u)=\mathsf{H}_{(B,\rho)}(u)=\ln \mathsf{N}_B(u)$ is the metric entropy of the space (B,ρ).

Let a separable stochastic process $X=\{X(t),\ t\in B\}$ belong to the class $V(\varphi,\psi)$ and be defined on a compact set B. Assume that $\gamma(u)<\infty$ and that $f=\{f(t), t\in B\}$ is a continuous function. The following result contains conditions for the boundedness of the stochastic process

$$Q(t)=\sup_{s\le t}\bigl(X(t)-X(s)-(f(t)-f(s))\bigr),\qquad s,t\in B. \tag{8}$$

Lemma 3.1 also provides the bounds for the exponential moments of this process.

Lemma 3.1. *Let $f=\{f(t), t\in B\}$ be a continuous function such that*

$$|f(u)-f(w)|\le \delta(\rho(u,w)),$$

where $\delta=\{\delta(s), s>0\}$ is a nonnegative increasing function. Let $\{q_k, k=1,2,\dots\}$ be a sequence such that $q_k>1$, $k\ge 1$, and

$$\sum_{k=1}^{\infty}\frac{1}{q_k}\le 1.$$

Then

$$\begin{aligned}
\mathsf{E}\exp\{\lambda Q(t)\} &\le W(\lambda,t,p)\left(\prod_{k=2}^{\infty}\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)^{1/q_k}\right)\\
&\quad\times\left(\prod_{k=2}^{\infty}\exp\left\{\frac{1}{q_k}\varphi\left(\lambda q_k\beta p^{k-1}\right)+\lambda\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right\}\right)
\end{aligned}\tag{9}$$

for all $\lambda>0$ and $p\in(0,1)$, where

$$W(\lambda,t,p)=\min\left\{X_{\lambda,t,p};Y_{\lambda,t,p}\right\}\tag{10}$$

and

$$X_{\lambda,t,p}=\left(\sum_{j=1}^{\mathsf{N}_{B_t}\left(\sigma^{(-1)}(\beta p)\right)}\exp\left\{\varphi\left(q_1\lambda\sigma\left(2j\sigma^{(-1)}(\beta p)\right)\right)+q_1\lambda\delta\left(2j\sigma^{(-1)}(\beta p)\right)\right\}\right)^{1/q_1},$$

$$\begin{aligned}
Y_{\lambda,t,p}&=\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)^{\frac{1}{q_1}}\\
&\quad\times\inf_{w>1}\exp\left\{\frac{\psi(wq_1\lambda\gamma(t))}{wq_1}\right.\\
&\qquad\left.+\max_{u\in B_t}\left\{\frac{(w-1)\psi\left(\frac{w}{w-1}q_1\lambda\gamma(u)\right)}{wq_1}+\lambda f(u)\right\}-\lambda f(t)\right\}.
\end{aligned}$$

Remark 3.1. Clearly, inequality (9) makes sense if the denominators on the right hand side are finite.

Proof. Denote by V_{ε_k} the set of centers of closed balls of radius

$$\varepsilon_k=\sigma^{(-1)}\left(\beta p^k\right),\qquad p\in(0,1),\qquad k=0,1,2,\dots,$$

that form the minimal covering of the space (B,ρ). The number of sets in the set V_{ε_k} is equal to $\mathsf{N}_B(\varepsilon_k)$.

Note that $X(t)$ and thus $X_f(t)=X(t)-f(t)$ are separable processes.

Lemma 2.1 and assumption (7) imply that

$$\begin{aligned}
\mathsf{P}\{|X(t)-X(s)|>\varepsilon\}&\le 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(X(t)-X(s))}\right)\right\}\\
&\le 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\sigma(\rho(t,s))}\right)\right\}
\end{aligned}$$

for all $\varepsilon>0$. Hence the process X and thus X_f, too, are continuous in probability. If a separable stochastic process defined on (B,ρ) is continuous in probability, then every finite and everywhere dense set with respect to ρ is the set of separability for this process. Thus $V = \bigcup_{k=1}^{\infty} V_{\varepsilon_k}$ is a set of ρ-separability of the process X_f. Moreover, with probability one,

$$Q(t) = \sup_{s\le t, s\in B} (X_f(t) - X_f(s)) = \sup_{s\le t, s\in V} (X_f(t) - X_f(s)). \tag{11}$$

Consider the mapping $\alpha_n = \{\alpha_n(s), n = 0, 1, \ldots\}$ of the set V to V_{ε_n} such that $\alpha_n(s)$ is a point of the set V_{ε_n} for which $\rho(s, \alpha_n(s)) < \varepsilon_n$. If $s \in V_{\varepsilon_n}$, then $\alpha_n(s) = s$. If there are several points in the set V_{ε_n} such that $\rho(s, \alpha_n(s)) < \varepsilon_n$, then we choose an arbitrary point among them and denote it by $\alpha_n(s)$.

Chebyshev's inequality, Lemma 2.1, and assumption (7) imply that

$$\begin{aligned} \mathsf{P}\left\{|X(s) - X(\alpha_n(s))| > p^{n/2}\right\} &\le \frac{\mathsf{E}\big(X(s) - X(\alpha_n(s))\big)^2}{p^n} \\ &\le \frac{c^2\tau_\varphi^2\big(X(s) - X(\alpha_n(s))\big)}{p^n} \le \frac{c^2\sigma^2(\varepsilon_n)}{p^n} = c^2\beta^2 p^n, \end{aligned}$$

where

$$c = \frac{2e}{\varphi^{(-1)}(2)}.$$

The latter inequality means that

$$\sum_{n=1}^{\infty} \mathsf{P}\left\{|X(s) - X(\alpha_n(s))| > p^{n/2}\right\} < \infty.$$

Now the Borel–Cantelli lemma implies that $X(s) - X(\alpha_n(s)) \to 0$ as $n \to \infty$ with probability one. Since the function f is continuous,

$$X_f(s) - X_f(\alpha_n(s)) \to 0 \quad \text{as } n \to \infty \tag{12}$$

with probability one. The set V is countable; hence $X(s) - X(\alpha_n(s)) \to 0$ with probability one as $n \to \infty$ uniformly in s.

Let s be an arbitrary point of the set V. Let, for all $m \ge 1$,

$$s_m = \alpha_m(s), \qquad s_{m-1} = \alpha_{m-1}(s_m), \qquad \ldots, \qquad s_1 = \alpha_1(s_2).$$

Then

$$X_f(s) = X_f(s_1) + \sum_{k=2}^{m}\big(X_f(s_k) - X_f(s_{k-1})\big) + X_f(s) - X_f(\alpha_m(s))$$

for all $m \ge 2$ and similarly

$$-X_f(s) \le -\min_{u\in V_{\varepsilon_1}} X_f(u) - \sum_{k=2}^{m} \min_{u\in V_{\varepsilon_k}} \big(X_f(u) - X_f(\alpha_{k-1}(u))\big) - X_f(s) + X_f(\alpha_m(s)). \tag{13}$$

Relations (8) and (13) imply that

$$\begin{aligned} Q(t) &= \sup_{\substack{s\le t,\\ s\in V}} (X_f(t) - X_f(s)) \\ &\le \liminf_{m\to\infty} \Bigg(X_f(t) - \min_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}} X_f(u) - \sum_{k=2}^{m} \min_{\substack{u\le t,\\ u\in V_{\varepsilon_k}}} \big(X_f(u) - X_f(\alpha_{k-1}(u))\big) \\ &\qquad\qquad - X_f(s) + X_f(\alpha_m(s)) \Bigg) \end{aligned} \tag{14}$$

with probability one.

Now Hölder's inequality, (12), (14), and Fatou's lemma yield

(15)

$$
\begin{aligned}
&\mathsf{E}\exp\{\lambda Q(t)\}\\
&\quad\le \mathsf{E}\liminf_{m\to\infty}\exp\left\{\lambda\left(X_f(t)-\min_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}}X_f(u)-\sum_{k=2}^{m}\min_{\substack{u\le t,\\ u\in V_{\varepsilon_k}}}\big(X_f(u)-X_f(\alpha_{k-1}(u))\big)\right)\right\}\\
&\quad\le \liminf_{m\to\infty}\mathsf{E}\exp\left\{\lambda\left(X_f(t)-\min_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}}X_f(u)-\sum_{k=2}^{m}\min_{\substack{u\le t,\\ u\in V_{\varepsilon_k}}}\big(X_f(u)-X_f(\alpha_{k-1}(u))\big)\right)\right\}\\
&\quad\le \liminf_{m\to\infty}\left(\left(\mathsf{E}\exp\left\{q_1\lambda\left(X_f(t)-\min_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}}X_f(u)\right)\right\}\right)^{1/q_1}\right.\\
&\qquad\qquad\left.\times\prod_{k=2}^{m}\left(\mathsf{E}\exp\left\{q_k\lambda\max_{\substack{u\le t,\\ u\in V_{\varepsilon_k}}}\big(X_f(\alpha_{k-1}(u))-X(u)\big)\right\}\right)^{1/q_k}\right)\\
&\quad\le \left(\mathsf{E}\exp\left\{q_1\lambda\left(X_f(t)-\min_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}}X_f(u)\right)\right\}\right)^{1/q_1}\\
&\quad\quad\times\prod_{k=2}^{\infty}\left(\mathsf{E}\exp\left\{q_k\lambda\max_{\substack{u\le t,\\ u\in V_{\varepsilon_k}}}\big(X_f(\alpha_{k-1}(u))-X(u)\big)\right\}\right)^{1/q_k}\\
&\quad= I_1\cdot\prod_{k=2}^{\infty}I_k
\end{aligned}
$$

for all $\lambda>0$.

Every term on the right hand side of (15) is considered separately. Theorem 2.1 and assumption (7) imply that

$$
\begin{aligned}
\mathsf{E}\exp\big\{q_1\lambda(X(t)-X(u))\big\}&\le\exp\big\{\varphi\big(q_1\lambda\tau_\varphi(X(t)-X(u))\big)\big\}\\
&\le\exp\big\{\varphi\big(q_1\lambda\sigma(\rho(t,u))\big)\big\}.
\end{aligned}
$$

Then the condition $|f(u)-f(w)|\le\delta(\rho(u,w))$ yields

$$
\begin{aligned}
(I_1)^{q_1}&\le\sum_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}}\mathsf{E}\exp\big\{q_1\lambda(X(t)-X(u))\big\}\exp\big\{-q_1\lambda(f(t)-f(u))\big\}\\
&\le\sum_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}}\exp\big\{\varphi\big(q_1\lambda\sigma(\rho(t,u))\big)+q_1\lambda\delta(\rho(t,u))\big\}\\
&\le\sum_{j=1}^{\mathsf{N}_{B_t}(\varepsilon_1)}\exp\big\{\varphi(q_1\lambda\sigma(2j\varepsilon_1))+q_1\lambda\delta(2j\varepsilon_1)\big\}.
\end{aligned}
\tag{16}
$$

On the other hand, the Hölder inequality and (4) imply that, for all $w > 1$,

(17)

$$
\begin{aligned}
(I_1)^{q_1} &\le \sum_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}} \left(\mathsf{E}\exp\{wq_1\lambda X(t)\}\right)^{\frac{1}{w}} \left(\mathsf{E}\exp\left\{-\frac{w}{w-1}q_1\lambda X(u)\right\}\right)^{1-\frac{1}{w}} \\
&\quad \times \exp\Big\{-q_1\lambda(f(t)-f(u))\Big\} \\
&\le \sum_{\substack{u\le t,\\ u\in V_{\varepsilon_1}}} \exp\left\{\frac{\psi(wq_1\lambda\gamma(t))}{w} - q_1\lambda f(t) + \frac{(w-1)\psi\left(\frac{w}{w-1}q_1\lambda\gamma(u)\right)}{w} + q_1\lambda f(u)\right\} \\
&\le \mathsf{N}_{B_t}(\varepsilon_1)\exp\left\{\frac{\psi(wq_1\lambda\gamma(t))}{w} - q_1\lambda f(t)\right\} \\
&\quad \times \exp\left\{\max_{u\in B_t}\left(\frac{(w-1)\psi\left(\frac{wq_1\lambda\gamma(u)}{w-1}\right)}{w} + q_1\lambda f(u)\right)\right\}.
\end{aligned}
$$

Further, Theorem 2.1 and (7) imply that

$$
\mathsf{E}\exp\left\{q_k\lambda\big(X(u)-X(\alpha_{k-1}(u))\big)\right\} \le \exp\{\varphi(q_k\lambda\sigma(\varepsilon_{k-1}))\}.
$$

Then

(18)

$$
\begin{aligned}
(I_k)^{q_k} &\le \mathsf{N}_{B_t}(\varepsilon_k)\max_{\substack{u\le t,\\ u\in V_{\varepsilon_k}}} \mathsf{E}\exp\{q_k\lambda\left(X_f(\alpha_{k-1}(u))-X(u)\right)\} \\
&\quad \times \exp\{q_k\lambda[f(u)-f(\alpha_{k-1}(u))]\} \\
&\le \mathsf{N}_{B_t}(\varepsilon_k)\max_{\substack{u\le t,\\ u\in V_{\varepsilon_k}}} \exp\left\{\varphi(q_k\lambda\sigma(\varepsilon_{k-1})) + q_k\lambda\delta\big(\rho(u,\alpha_{k-1}(u))\big)\right\} \\
&\le \mathsf{N}_{B_t}(\varepsilon_k)\exp\left\{\varphi\left(q_k\lambda\beta p^{k-1}\right) + q_k\lambda\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right\}.
\end{aligned}
$$

Inequalities (15)–(18) complete the proof of Lemma 3.1. □

Theorem 3.1. *Let all the assumptions of Lemma* 3.1 *hold. Then*

$$
\mathsf{P}\left\{Q(t) > x\right\} \le Z(p,t,x) \tag{19}
$$

for all $p \in (0,1)$ *and* $x > 0$, *where*

$$
\begin{aligned}
Z(p,t,x) = \inf_{\lambda>0} W(\lambda,t,p)&\left(\prod_{k=2}^{\infty}\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)^{\frac{1}{q_k}}\right) \\
&\times\exp\left\{\sum_{k=2}^{\infty}\left(\frac{1}{q_k}\varphi\left(\lambda q_k\beta p^{k-1}\right) + \lambda\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right) - \lambda x\right\}
\end{aligned}
$$

and where $W(\lambda,t,p)$ *is defined in equation* (10).

Proof. The proof of Theorem 3.1 follows directly from Lemma 3.1 and Chebyshev's inequality. □

Choosing a specific sequence q_k in Lemma 3.1, we obtain the following result.

Lemma 3.2. *Let* $f = \{f(t), t \in B\}$ *be a continuous function such that*

$$
|f(u) - f(w)| \le \delta(\rho(u,w)),
$$

where $\delta=\{\delta(s), s>0\}$ *is an increasing nonnegative function. Further let*

$$\int_0^{\beta}\zeta_\varphi\left(\mathsf{H}_{B_t}\left(\sigma^{(-1)}(u)\right)\right)du<\infty. \tag{20}$$

Then

$$\begin{aligned}\mathsf{E}\exp\{\lambda Q(t)\}&\le W_1(\lambda,t,p)\\&\times\exp\left\{\varphi\left(\frac{\lambda\beta}{1-p}\right)p+\frac{2\lambda}{p(1-p)}\int_0^{\beta p^2}\zeta_\varphi\left(\mathsf{H}_{B_t}\left(\sigma^{(-1)}(u)\right)\right)du\right\}\\&\times\exp\left\{\lambda\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right\}\end{aligned} \tag{21}$$

for all $\lambda>0$ *and* $p\in(0,1)$*, where*

$$W_1(\lambda,t,p)=\inf_{v\ge\frac{1}{1-p}}\min\left\{X_{v,\lambda,t,p},Y_{v,\lambda,t,p}\right\} \tag{22}$$

and

$$X_{v,\lambda,t,p}=\left(\int_1^{\mathsf{N}_{B_t}\left(\sigma^{(-1)}(\beta p)\right)+1}\exp\left\{\varphi\left(v\lambda\sigma\left(2x\sigma^{(-1)}(\beta p)\right)\right)+v\lambda\delta\left(2x\sigma^{(-1)}(\beta p)\right)\right\}dx\right)^{\frac{1}{v}},$$

$$\begin{aligned}Y_{v,\lambda,t,p}&=\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)^{\frac{1}{v}}\\&\times\inf_{w>1}\exp\left\{\frac{\psi(wv\lambda\gamma(t))}{wv}\right.\\&\qquad\left.\times\max_{u\in B_t}\left\{\frac{(w-1)\psi\left(\frac{wv}{w-1}\lambda\gamma(u)\right)}{wv}+\lambda f(u)\right\}-\lambda f(t)\right\}.\end{aligned}$$

Proof. Inequality (9) implies that

$$\begin{aligned}\mathsf{E}\exp\{\lambda Q(t)\}&\le W(\lambda,t,p)\\&\times\exp\left\{\lambda\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right\}\\&\times\exp\left\{\sum_{k=2}^{\infty}\frac{\mathsf{H}_{B_t}(\varepsilon_k)+\varphi(\lambda q_k\beta p^{k-1})}{q_k}\right\}\end{aligned} \tag{23}$$

for all q_k, $k\ge1$, such that

$$q_k>1,\quad k\ge1,\qquad\sum_{k=1}^{\infty}\frac{1}{q_k}\le1, \tag{24}$$

and for all $\lambda>0$, where $\varepsilon_k=\sigma^{(-1)}(\beta p^k)$.

Let v be a number such that $v\ge(1-p)^{-1}$ and put $q_1=v$. Note that $q_1>1$. Further let, for $k=2,3,\dots$,

$$q_k=\frac{1}{\lambda\beta p^{k-1}}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\beta}{1-p}\right)+\mathsf{H}_{B_t}(\varepsilon_k)\right). \tag{25}$$

Then

$$q_k>\frac{1}{p^{k-1}(1-p)}>1,\qquad k=2,3,\dots.$$

Since

$$\frac{1}{q_k} \le \frac{\lambda\beta p^{k-1}}{\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\beta}{1-p}\right)\right)} = p^{k-1}(1-p), \qquad k \ge 2,$$

we have

$$\sum_{k=1}^{\infty} \frac{1}{q_k} \le \sum_{k=1}^{\infty} p^{k-1}(1-p) = 1.$$

Thus conditions (24) hold.

Now we consider

$$\tilde{Z} = \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k) + \varphi(\lambda q_k \beta p^{k-1})}{q_k}.$$

For the sequence q_k defined by (25), we get

$$\begin{aligned}
\tilde{Z} &= \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k)}{q_k} + \sum_{k=2}^{\infty} \frac{1}{q_k}\varphi\left(\lambda\beta p^{k-1}\frac{\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\beta}{1-p}\right) + H_{B_t}(\varepsilon_k)\right)}{\lambda\beta p^{k-1}}\right) \\
&= \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k)}{q_k} + \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k)}{q_k} + \varphi\left(\frac{\lambda\beta}{1-p}\right)\sum_{k=2}^{\infty}\frac{1}{q_k} \\
&\le 2\sum_{k=2}^{\infty} H_{B_t}(\varepsilon_k)\frac{\lambda\beta p^{k-1}}{\varphi^{(-1)}(H_{B_t}(\varepsilon_k))} + \varphi\left(\frac{\lambda\beta}{1-p}\right)\sum_{k=2}^{\infty} p^{k-1}(1-p) \\
&= \varphi\left(\frac{\lambda\beta}{1-p}\right)p + 2\lambda\sum_{k=2}^{\infty}\zeta_\varphi\left(H_{B_t}\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)\beta p^{k-1}.
\end{aligned} \tag{26}$$

The function $\varphi(x)/x$ increases for $x > 0$. Thus the function $\zeta_\varphi(x) = x/\varphi^{(-1)}(x)$ increases for $x > 0$, too. Then

$$\int_{\beta p^{k+1}}^{\beta p^k} \zeta_\varphi\left(H_{B_t}\left(\sigma^{(-1)}(u)\right)\right)du \ge \zeta_\varphi\left(H_{B_t}\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)\beta p^k(1-p). \tag{27}$$

Relations (26) and (27) imply that

$$\tilde{Z} \le \varphi\left(\frac{\lambda\beta}{1-p}\right)p + \frac{2\lambda}{p(1-p)}\int_0^{\beta p^2} \zeta_\varphi\left(H_{B_t}\left(\sigma^{(-1)}(u)\right)\right)du. \tag{28}$$

Therefore inequality (21) follows from (9) and (28). □

Choosing the sequence q_k defined in Lemma 3.2, we obtain the following result.

Theorem 3.2. *Let a stochastic process $X(t) = \{X(t), t \in B\}$ belong to the class $V(\varphi, \psi)$ and let $\varphi \prec \psi$. Assume that condition (7) holds. Let $f = \{f(t), t \in B\}$ be a continuous function such that*

$$|f(u) - f(w)| \le \delta(\rho(u, w)),$$

where $\delta = \{\delta(s), s > 0\}$ is an increasing nonnegative function. Further let

$$r = \{r(u) \colon u \ge 1\}$$

be a continuous function such that $r(1) = 0$, $r(u) > 0$ for $u > 1$, and the function $s(t) = r(\exp\{t\})$, $t \ge 0$, is convex. If

$$\int_0^{\beta} \frac{r\left(N_{B_t}\left(\sigma^{(-1)}(u)\right)\right)}{\varphi^{(-1)}\left(H_{B_t}\left(\sigma^{(-1)}(u)\right)\right)}du < \infty \tag{29}$$

for the process $Q(t)$ defined by (8), then

$$\mathsf{P}\left\{Q(t)>x\right\}\le Z_r(p,t,x)$$

for all $p\in(0,1)$ and $x>0$, where

$$Z_r(p,t,x)=\inf_{\lambda>0}W_1(\lambda,t,p)\exp\left\{p\varphi\left(\frac{\lambda\beta}{1-p}\right)+\lambda\left(\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)-x\right)\right\}$$
$$\times\left(r^{(-1)}\left(\frac{\lambda}{p(1-p)}\int_0^{\beta p^2}\frac{r\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}(u)\right)\right)}{\varphi^{(-1)}\left(\mathsf{H}_{B_t}\left(\sigma^{(-1)}(u)\right)\right)}\,du\right)\right)^2$$

and where $W_1(\lambda,t,p)$ is defined by (22).

Theorem 3.2 follows from Lemmas 3.1 and 3.2. The proof is similar to that of Theorem 3.2 of the paper [12] or to that of Theorem 3.4 in [7].

Using the sequence $q_k=1/\left((1-p)p^{k-1}\right)$ in inequalities (9) we get the following result. An advantage of this result is that its conditions are easier to check than those of Theorem 3.2.

Theorem 3.3. *Let $f=\{f(t),t\in B\}$ be a continuous function such that*

$$|f(u)-f(w)|\le\delta(\rho(u,w)),$$

where $\delta=\{\delta(s),s>0\}$ is a nonnegative increasing function and let $r_1=\{r_1(u),u\ge1\}$ be a continuous function such that $r_1(u)>0$ for $u>1$ and $s(t)=r_1(\exp\{t\})$, $t\ge0$, is a convex function. If

$$\int_0^{\beta}r_1\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}(u)\right)\right)du<\infty \tag{30}$$

for the process $Q(t)$ defined by (8), then

$$\mathsf{P}\left\{Q(t)>x\right\}\le Z_{r_1}(p,t,x)$$

for all $p\in(0,1)$ and $x>0$, where

(31)
$$Z_{r_1}(p,t,x)=\inf_{\lambda>0}W_2(\lambda,p,t)\exp\left\{p\varphi\left(\frac{\lambda\beta}{1-p}\right)+\lambda\left(\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)-x\right)\right\}$$
$$\times r_1^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p}r_1\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}(u)\right)\right)du\right)$$

and

(32)
$$W_2(\lambda,t,p)=\min\left\{X_{\lambda,t,p};Y_{\lambda,t,p}\right\},$$

$$X_{\lambda,t,p}=\left(\int_1^{\mathsf{N}_{B_t}\left(\sigma^{(-1)}(\beta p)\right)+1}\exp\left\{\varphi\left(\frac{\lambda\sigma\left(2x\sigma^{(-1)}(\beta p)\right)}{1-p}\right)+\frac{\lambda\delta\left(2x\sigma^{(-1)}(\beta p)\right)}{1-p}\right\}dx\right)^{1-p},$$

$$Y_{\lambda,t,p}=\left(\mathsf{N}_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)^{1-p}\inf_{v>1}\exp\left\{\frac{(1-p)\psi\left(\frac{v\lambda\gamma(t)}{1-p}\right)}{v}\right.$$
$$\left.-\lambda f(t)+\max_{u\in B_t}\left(\frac{(v-1)(1-p)\psi\left(\frac{v\lambda\gamma(t)}{(v-1)(1-p)}\right)}{v}+\lambda f(u)\right)\right\}.$$

Theorem 3.2 follows from Lemma 3.1. The proof is similar to that of Theorem 3.1 of [12] or that of Theorem 3.5 of [7].

4. An estimate for the distribution of a stationary φ-sub-Gaussian storage process

Consider a stationary φ-sub-Gaussian stochastic process $X = \{X(t), t \in B\}$ defined on a compact set $B \subset T$, where T is a set of parameters. Recall (see [13]) that a φ-sub-Gaussian process X is called *stationary* if

1) its norm $\tau_\varphi(X(t)) = c_\varphi = \gamma$ is constant for all $t, s \in B$, and if
2) $\tau_\varphi(X(t) - X(s)) = \sigma_\varphi(t - s)$.

Assume that X satisfies conditions (7). Let a continuous function $f = \{f(t), t \in B\}$ be such that $|f(u) - f(w)| \le \delta(\rho(u, w))$, where $\delta = \{\delta(s), s > 0\}$ is an increasing nonnegative function.

Example 4.1. We apply Lemma 3.2 to X. Then we estimate both terms on the right hand side of (22) defining the minimum of the function $W_1(\lambda, t, p)$.

First we consider the case of

$$v = \frac{w-1}{\lambda\gamma w}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\gamma w}{(1-p)(w-1)}\right) + H_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right).$$

Then

$$v \ge \frac{w-1}{\lambda\gamma w}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\gamma w}{(1-p)(w-1)}\right)\right) = \frac{1}{1-p}.$$

In this case,

$$\begin{aligned}
&W_1(\lambda, t, p)\\
&\quad \le \left(N_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)^{\frac{1}{v}}\\
&\quad\quad \times \inf_{w>1} \exp\left\{\frac{\varphi\left(w\frac{w-1}{\lambda\gamma w}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\gamma w}{(1-p)(w-1)}\right) + H_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)\lambda\gamma\right)}{wv}\right.\\
&\quad\quad\quad \left. + \frac{(w-1)\varphi\left(\frac{w}{w-1}\frac{w-1}{\lambda\gamma w}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\gamma w}{(1-p)(w-1)}\right) + H_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)\lambda\gamma\right)}{wv}\right\}\\
&\quad = \left(N_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)^{\frac{1}{v}}\\
&\quad\quad \times \inf_{w>1} \exp\left\{\frac{\varphi\left((w-1)\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\gamma w}{(1-p)(w-1)}\right) + H_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)\right)}{wv}\right.\\
&\quad\quad\quad \left. + \frac{(w-1)\left(\varphi\left(\frac{\lambda\gamma w}{(1-p)(w-1)}\right) + H_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right)}{wv}\right\}.
\end{aligned}$$

Now let

$$v = \frac{1}{\lambda}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda}{1-p}\right) + H_{B_t}\left(\sigma^{(-1)}(\beta p)\right)\right).$$

Then

$$v \ge \frac{1}{\lambda}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda}{1-p}\right)\right) = \frac{1}{1-p},$$

whence

$$\begin{aligned}
&W_1(\lambda,t,p)\\
&\quad\le\left(\int_1^{N_{B_t}\left(\sigma^{(-1)}(\beta p)\right)+1}\exp\left\{\varphi\left(\frac{1}{\lambda}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda}{1-p}\right)+J(t,p)\right)\lambda\sigma(K(x,p))\right)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{1}{\lambda}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda}{1-p}\right)+J(t,p)\right)\lambda\delta(K(x,p))\right\}dx\right)^{\frac{1}{v}}\\
&\quad=\left(\int_1^{N_{B_t}\left(\sigma^{(-1)}(\beta p)\right)+1}\exp\left\{\left(\varphi\left(\frac{\lambda}{1-p}\right)+J(t,p)\sigma(K(x,p))\right)\right.\right.\\
&\qquad\qquad\left.\left.+\varphi^{(-1)}\left(\varphi\left(\frac{\lambda}{1-p}\right)+J(t,p)\right)\delta(K(x,p))\right\}dx\right)^{\frac{1}{v}},
\end{aligned}$$

where

$$J(t,p)=N_{B_t}\left(\sigma^{(-1)}(\beta p)\right),$$
$$K(x,p)=2x\sigma^{(-1)}(\beta p).$$

Example 4.2. Let X be a stationary sub-Gaussian stochastic process defined in the interval $B=[a,b]$, $-\infty<a<b<\infty$, and let $\rho(u,v)=|u-v|$. Then

$$\inf_{u\in B_t}\sup_{v\in B_t}\rho(u,v)=\frac{t-a}{2}$$

and

$$\beta\le\sigma\left(\frac{b-a}{2}\right).$$

Thus

$$N_{B_t}(u)\le\frac{t-a}{2u}+1,$$

whence

$$N_{B_t}(u)\le\frac{t-a}{u}$$

if

$$u\le\frac{t-a}{2}.$$

As in Example 4.1, we apply Lemma 3.2 to the process X. Since $\psi(u)=\varphi(u)=u^2/2$, the expression in (22) simplifies:

(33)

$$\begin{aligned}
W_1(\lambda,t,p)&=\inf_{v\ge\frac{1}{1-p}}\min\left\{\left(\int_1^{M(t,p)}\exp\left\{\frac{v^2\lambda^2\sigma^2(K(x,p))}{2}+v\lambda\delta(K(x,p))\right\}dx\right)^{\frac{1}{v}};\right.\\
&\qquad\left.(M(t,p)-1)^{\frac{1}{v}}\inf_{w>1}\exp\left\{\frac{v\lambda^2\gamma^2w^2}{2(w-1)}\right\}\right\}\\
&\le\min\left\{\left(\int_1^{M(t,p)}\exp\left\{\frac{\lambda^2\sigma^2(K(x,p))}{2(1-p)^2}+\frac{\lambda\delta(K(x,p))}{1-p}\right\}dx\right)^{1-p};\right.\\
&\qquad\left.(M(t,p)-1)^{2\lambda^2\gamma^2}\right\},
\end{aligned}$$

where

$$M(t,p) = \frac{t-a}{\sigma^{(-1)}(\beta p)} + 1$$

and where $K(x,p)$ is defined in Example 4.1. Moreover,

$$\int_0^{\beta p^2} \zeta_\varphi\left(\mathsf{H}_{B_t}\left(\sigma^{(-1)}(u)\right)\right) du = \int_0^{\beta p^2} \left(2\ln\left(\frac{t-a}{\sigma^{(-1)}(u)}\right)\right)^{\frac{1}{2}} du.$$

Therefore we proved the following result.

Theorem 4.1. *Let $X(t)$ be a stationary sub-Gaussian stochastic process defined on a finite interval and let condition (7) hold. Further let $f = \{f(t), t \in B\}$ be a continuous function such that $|f(u) - f(w)| \le \delta(\rho(u,w))$, where $\delta = \{\delta(s), s > 0\}$ is an increasing nonnegative function. If*

$$\int_0^{\beta} \left(\ln\left(\frac{t-a}{\sigma^{(-1)}(u)}\right)\right)^{\frac{1}{2}} du < \infty$$

for all $p \in (0,1)$ and $x > 0$, then

$$\begin{aligned}
\mathsf{P}\{Q(t) > x\} \le {} & \inf_{\lambda>0} W_3(\lambda,t,p) \\
& \times \exp\left\{\frac{\lambda^2\beta^2 p}{2(1-p)^2} + \frac{\lambda\sqrt{2}}{p(1-p)}\int_0^{\beta p^2}\left(\ln\left(\frac{t-a}{\sigma^{(-1)}(u)}\right)\right)^{\frac{1}{2}} du\right\} \\
& \times \exp\left\{\lambda\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right) - \lambda x\right\},
\end{aligned} \tag{34}$$

where $W_3(\lambda,t,p) = \min\{X_{\lambda,t,p}; Y_{\lambda,t,p}\}$ and where

$$X_{\lambda,t,p} = \left(\frac{t-a}{\sigma^{(-1)}(\beta p)}\right)^{2\lambda^2\gamma^2},$$

$$Y_{\lambda,t,p} = \left(\int_1^{\frac{t-a}{\sigma^{(-1)}(\beta p)}+1} \exp\left\{\frac{\lambda^2\sigma^2\left(2x\sigma^{(-1)}(\beta p)\right)}{2(1-p)^2} + \frac{\lambda\delta\left(2x\sigma^{(-1)}(\beta p)\right)}{1-p}\right\} dx\right)^{1-p}.$$

Example 4.3. Let a stochastic process X in Theorem 4.1 be such that $\sigma(u) = u^{1/2}$ and let $f(u) = \delta(u) = cu$, where $c > 0$ is a constant. Then

$$\begin{aligned}
W_3(\lambda,t,p) &= \min\left\{D(t,p); \left(\int_1^{\frac{t-a}{\beta^2p^2}+1} \exp\left\{x\left(\frac{\lambda^2\beta p}{(1-p)^2} + \frac{2\lambda\beta^2p^2}{1-p}\right)\right\}dx\right)^{1-p}\right\} \\
&= \min\left\{D(t,p); \left(\frac{(1-p)^2}{\lambda\beta p(\lambda + 2\beta p(1-p))}\right)^{1-p} \exp\left\{\lambda\beta p\left(\frac{\lambda}{1-p} + 2\beta p\right)\right\}\right. \\
&\qquad\qquad \left. \times\left(\exp\left\{\frac{t-a}{\beta p}\left(\frac{\lambda^2}{1-p} + 2\lambda\right)\right\} - 1\right)\right\},
\end{aligned}$$

where

$$D(t,p) = \left(\frac{t-a}{\beta^2p^2}\right)^{2\lambda^2\gamma^2}.$$

Changing the variables

$$s = \left(\ln\frac{t-a}{u^2}\right)^{\frac{1}{2}},$$

we obtain

$$\begin{aligned}
&\int_0^{\beta p^2}\left(\ln\left(\frac{t-a}{\sigma^{(-1)}(u)}\right)\right)^{\frac12}du\\
&\quad=\int_0^{\beta p^2}\left(\ln\left(\frac{t-a}{u^2}\right)\right)^{\frac12}du\\
&\quad=(t-a)^{\frac12}\int_{\left(\ln\frac{t-a}{\beta p^2}\right)^{\frac12}}^{\infty}s^2\exp\left\{-\frac{s^2}{2}\right\}ds\\
&\quad=(t-a)^{\frac12}\left(-s\exp\left\{-\frac{s^2}{2}\right\}\Big|_{\left(\ln\frac{t-a}{\beta p^2}\right)^{\frac12}}^{\infty}+\int_{\left(\ln\frac{t-a}{\beta p^2}\right)^{\frac12}}^{\infty}\exp\left\{-\frac{s^2}{2}\right\}ds\right)\\
&\quad=(t-a)^{\frac12}\left(\left(\ln\frac{t-a}{\beta p^2}\right)^{\frac12}\exp\left\{-\frac12\ln\frac{t-a}{\beta p^2}\right\}+\frac{\sqrt{\pi}}{2}-\int_0^{\left(\ln\frac{t-a}{\beta p^2}\right)^{\frac12}}\exp\left\{-\frac{s^2}{2}\right\}ds\right)\\
&\quad\le(t-a)^{\frac12}\frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Finally

$$\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)=\sum_{k=1}^{\infty}c\beta^2p^{2(k-1)}=\frac{c\beta^2}{1-p^2}.$$

Therefore bound (34) becomes of the following form in the case of a stationary sub-Gaussian process:

$$\mathsf{P}\left\{Q(t)>x\right\}\le\inf_{\substack{\lambda>0,\\ p\in(0,1)}}\min\left\{X_{\lambda,p,t};Y_{\lambda,p,t}\right\},$$

$$X_{\lambda,p,t}=\left(\frac{t-a}{\beta^2p^2}\right)^{2\lambda^2\gamma^2};$$

$$\begin{aligned}
Y_{\lambda,p,t}=&\left(\frac{(1-p)^2}{\lambda\beta p(\lambda+2\beta p(1-p))}\right)^{1-p}\exp\left\{\lambda\beta p\left(\frac{\lambda}{1-p}+2\beta p\right)\right\}\\
&\times\left(\exp\left\{\frac{t-a}{\beta p}\left(\frac{\lambda^2}{1-p}+2\lambda\right)\right\}-1\right)\\
&\times\exp\left\{\frac{\lambda^2\beta^2p}{2(1-p)^2}+\frac{\lambda\sqrt{\pi}(t-a)^{\frac12}}{\sqrt{2}p(1-p)}+\frac{\lambda c\beta^2}{1-p^2}-\lambda x\right\}.
\end{aligned}$$

Concluding remarks

We studied the properties of stochastic processes X belonging to the class $V(\varphi,\psi)$ and obtained some bounds for the distribution of the storage process

$$Q(t)=\sup_{s\le t}\big(X(t)-X(s)-(f(t)-f(s))\big)$$

generated by X, where a continuous increasing function $f(t)$ is the service output rate. The results of the paper can be applied to a wide class of stochastic processes, in particular to Gaussian processes. As an example, the case of stationary sub-Gaussian processes is considered in the paper.

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Bibliography

1. R. Addie, P. Mannersalo, and I. Norros, *Most probable paths and performance formulae for buffers with Gaussian input traffic*, Eur. Trans. Telecommun. **13 (3)** (2002), 183–196.
2. P. Boulongne, D. Pierre-Loti-Viaud, and V. Piterbarg, *On average losses in the ruin problem with fractional Brownian motion as input*, Extremes **12** (2009), 77–91. MR2480724 (2010f:60115)
3. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, TViMS, Kiev, 1998; English transl., American Mathematical Society, Providence, RI, 2000. MR1743716 (2001g:60089)
4. N. G. Duffield and N. O'Connell, *Large deviations and overflow probabilities for the general single-server queue, with applications*, Math. Proc. Cambridge Phil. Soc. **118** (1995), 363–374. MR1341797 (96f:60039)
5. Yu. V. Kozachenko and E. I. Ostrovskii, *Banach spaces random variables of sub-Gaussian type*, Teor. Veroyatnost. i Mat. Statist. **32** (1985), 42–53; English transl. in Theory Probab. Math. Statist. **32** (1986), 45–56. MR882158 (88e:60009)
6. Yu. V. Kozachenko and Yu. A. Kovalchuk, *Boundary value problems with random initial conditions, and functional series from* $\mathrm{Sub}_\varphi(\Omega)$*, I*, Ukrain. Mat. Zh. **50** (1998), no. 4, 504–515; English transl. in Ukrainian Math. J. **50** (1999), no. 4, 572–585. MR1698149 (2000f:60029)
7. Yu. Kozachenko, O. Vasylyk, and R. Yamnenko, *Upper estimate of overrunning by* $\mathrm{Sub}_\varphi(\Omega)$ *random process the level specified by continuous function*, Random Oper. and Stoch. Equ. **13** (2005), no. 2, 111–128. MR2152102 (2006b:60207)
8. L. Massoulie and A. Simonian, *Large buffer asymptotics for the queue with fractional Brownian input*, J. Appl. Probab. **36** (1999), 894–906. MR1737061 (2000i:60108)
9. I. Norros, *A storage model with self-similar input*, Queueing Syst. **16** (1994), 387–396. MR1278465 (95a:60142)
10. I. Norros, *On the use of fractional Brownian motions in the theory of connectionless networks*, IEEE Journal on selected areas in communications **13** (1995), no. 6, 953–962.
11. R. Yamnenko, *Ruin probability for generalized* φ*-sub-Gaussian fractional Brownian motion*, Theory Stoch. Process. **12 (28)** (2006), no. 1–2, 261–275. MR2316577 (2008g:60109)
12. R. Yamnenko and O. Vasylyk, *Random process from the class* $V(\varphi,\psi)$*: exceeding a curve*, Theory Stoch. Process. **13 (29)** (2007), no. 4, 219–232. MR2482262 (2010a:60125)
13. Yu. V. Kozachenko, O. I. Vasylyk, and R. E. Yamnenko, φ*-sub-Gaussian Stochastic Processes*, Kyiv University, Kyiv, 2008. (Ukrainian)

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