

MINIMAL MARTINGALE MEASURE ON A FINITE PROBABILITY SPACE

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ABSTRACT. Necessary and sufficient conditions are established for the existence of a minimal martingale measure for a discrete time financial market on a finite probability space.

INTRODUCTION

The definition of the minimal martingale measure is introduced in the papers [1] and [2]. The minimal martingale measure is used as an auxiliary technical tool in the problem of the local minimization of the risk in [1] and [2]. The definition of the minimal martingale measure used in this paper is given in [3].

Minimal martingale measures are used in models with continuous time, too. Some of the properties of minimal martingale measures are considered in [4]–[7]; applications in hedging are described in [1, 8] and those in pricing the contingent claims are given in [9, 10].

In the current paper, we consider the question of the existence and evaluation of the minimal martingale measure for a financial market defined on a finite probability space. We develop a procedure to evaluate a minimal martingale measure. Examples of evaluation of minimal martingale measures are also given.

The paper is organized as follows. Section 1 introduces some necessary notation. Section 2 considers a one-period model of the financial market. Necessary and sufficient conditions for the existence of a minimal martingale measure are given in Theorem 2.1. Section 3 considers a multiperiod model of the financial market. We propose a procedure that determines whether or not a minimal martingale measure exists and allows one to evaluate it in the case where such a measure exists. Section 4 contains some examples of the evaluation of minimal martingale measures by using the procedure developed in Section 3.

1. DEFINITIONS

Consider a model of the financial market with discrete time and $d + 1$ assets. We assume that one of these assets is risk-free, while the others d assets are risky. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\emptyset, X\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_T = \mathcal{F}$ be a filtration of σ -algebras in this space. The asset prices at moments $t = 0, \dots, T$ are given by

$$S_t = (S_t^0, S_t^1, \dots, S_t^d),$$

where $S_t^0 > 0$ is nonrandom and the S_t^k are \mathcal{F}_t -measurable functions.

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The symbol \mathbb{T} denotes the set $\{0, \dots, T\}$.

Definition 1.1. The processes

$$X_t^i = \frac{S_t^i}{S_t^0}, \quad t \in \mathbb{T}, \quad i = 1, \dots, d,$$

are called the discounted price processes.

Denote by $X^k = \{X_t^k, t \in \mathbb{T}\}$ the price process for the k -th asset.

Definition 1.2. A stochastic process $(U_t)_{t \in \mathbb{T}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called adapted with respect to a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_T$ if the random variable U_t is \mathcal{F}_t -measurable for all $t = 0, \dots, T$.

Definition 1.3. A measure $\tilde{\mathbb{P}}$ is called a martingale measure if it is equivalent to \mathbb{P} and if all price processes X^k , $k = 1, \dots, N$, are martingales with respect to $\tilde{\mathbb{P}}$.

Definition 1.4. Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_T$ be a filtration of σ -algebras defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Two processes $U = (U_t)_{t \in \mathbb{T}}$ and $V = (V_t)_{t \in \mathbb{T}}$ are called strongly orthogonal if they are adapted with respect to this filtration and if

$$\text{cov}((U_{t+1} - U_t)(V_{t+1} - V_t) \mid \mathcal{F}_t) = 0 \pmod{\mathbb{P}}, \quad t = 0, \dots, T-1.$$

We write $U \perp V$ if two processes U and V are strongly orthogonal.

Note that if one of the adapted processes $(U_t)_{t \in \mathbb{T}}$ or $(V_t)_{t \in \mathbb{T}}$ in Definition 1.4 is a martingale, then

$$\text{cov}((U_{t+1} - U_t)(V_{t+1} - V_t) \mid \mathcal{F}_t) = \mathbb{E}((U_{t+1} - U_t)(V_{t+1} - V_t) \mid \mathcal{F}_t).$$

We also say that two processes are \mathbb{P} -strongly orthogonal to indicate the measure used to evaluate expectations in Definition 1.4.

Definition 1.5. A measure \mathbb{Q} on (Ω, \mathcal{F}) is called a minimal martingale measure if

- (1) \mathbb{Q} is a martingale probability measure that is equivalent to \mathbb{P} ;
- (2) $\mathbb{E}(d\mathbb{Q}/d\mathbb{P})^2 < \infty$;
- (3) each square integrable \mathbb{P} -martingale that is strongly orthogonal to X^i , $i = 1, \dots, d$, is a \mathbb{Q} -martingale.

In what follows we consider a finite probability space

$$\Omega = \{\omega_1, \dots, \omega_N\}, \quad \mathcal{F} = 2^\Omega,$$

where $N \geq 1$ is a fixed number. We assume that $\mathbb{P}(\{\omega_i\}) > 0$, $i = 1, \dots, N$.

The aim of this paper is to find necessary and sufficient conditions for the existence of a minimal martingale measure in the case where a financial market is defined on a finite probability space.

Since the probability space is assumed to be finite, all probabilities as well as random variables defined on it can naturally be identified with vectors of the Euclidean space \mathbb{R}^N . Let (a, b) be the standard scalar product of two vectors in \mathbb{R}^N , namely

$$(a, b) = \sum_{k=1}^N a_k b_k,$$

where $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N) \in \mathbb{R}^N$.

If $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N) \in \mathbb{R}^N$, then ab denotes the vector

$$ab = (a_1 b_1, \dots, a_N b_N).$$

2. ONE-PERIOD MODEL OF THE FINANCIAL MARKET

First we consider a one-period model of the financial market, that is, the case of $T = 1$. Let $Y^i = X_1^i - X_0^i$, $i = 1, \dots, d$, denote the gains. Put $p_i = \mathbb{P}(\{\omega_i\})$, $q_i = \mathbb{Q}(\{\omega_i\})$, and $y_{ji} = Y^j(\omega_i)$, $i = 1, \dots, N$, $j = 1, \dots, d$.

Lemma 2.1. *A minimal martingale measure \mathbb{Q} is a linear combination of \mathbb{P} and $\mathbb{P}Y^1, \dots, \mathbb{P}Y^d$.*

Proof. Let $M = (M_0, M_1)$ be a \mathbb{P} -martingale that is orthogonal to the price processes $X^i = (X_0^i, X_1^i)$, $i = 1, \dots, d$. Put $m_k = M_1(\omega_k)$, $k = 1, \dots, N$, $m = M_0$.

The property that M is a \mathbb{P} -martingale can be written as follows:

$$\mathbb{E}_{\mathbb{P}}(M_1 - M_0) = \sum_{k=1}^N p_k(m_k - m) = (\mathbb{P}, M_1 - M_0) = 0.$$

Thus $\mathbb{P} \perp (M_1 - M_0)$.

Since M is strongly orthogonal to (X_0^i, X_1^i) , $i = 1, \dots, d$, we deduce that

$$\mathbb{E}_{\mathbb{P}}(M_1 - M_0)(X_1^i - X_0^i) = \mathbb{E}_{\mathbb{P}}(M_1 - M_0)Y^i = \sum_{k=1}^N p_k y_{ik}(m_k - m) = 0,$$

that is, $\mathbb{P}Y^i \perp (M_1 - M_0)$. Since M is a \mathbb{Q} -martingale, we have $\mathbb{Q} \perp (M_1 - M_0)$.

If $M_1 \in (\langle \mathbb{P}, \mathbb{P}Y^1, \dots, \mathbb{P}Y^d \rangle)^\perp$ is an arbitrary vector, then the stochastic process $M = (0, M_1)$ is a \mathbb{P} -martingale, whence

$$\mathbb{Q} \in (\langle \mathbb{P}, \mathbb{P}Y^1, \dots, \mathbb{P}Y^d \rangle)^{\perp\perp} = \langle \mathbb{P}, \mathbb{P}Y^1, \dots, \mathbb{P}Y^d \rangle. \quad \square$$

According to Lemma 2.1, there are $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that

$$(1) \quad \mathbb{Q} = \sum_{k=1}^d \alpha_k \mathbb{P}Y^k - \beta \mathbb{P}.$$

If the family of vectors Y^i , $i = 1, \dots, d$, is linearly dependent, then we consider its maximal linearly independent subfamily. In what follows we assume that the family of vectors Y^i , $i = 1, \dots, d$, is linearly independent.

Since \mathbb{Q} is a martingale measure, $(\mathbb{Q}, Y^i) = 0$, $i = 1, \dots, d$. This property can be written in the form of a system of d equations and $d + 1$ unknowns, namely

$$(2) \quad \begin{aligned} \alpha_1(\mathbb{P}Y^1, Y^1) + \dots + \alpha_d(\mathbb{P}Y^d, Y^1) &= \beta(\mathbb{P}, Y^1), \\ &\dots \\ \alpha_1(\mathbb{P}Y^1, Y^d) + \dots + \alpha_d(\mathbb{P}Y^d, Y^d) &= \beta(\mathbb{P}, Y^d). \end{aligned}$$

Now we introduce a scalar product $[U, V]$ in \mathbb{R}^N by

$$[U, V] = (\mathbb{P}U, V),$$

where $U = (u_1, \dots, u_N)$ and $V = (v_1, \dots, v_N)$. It is not complicated to check that $[U, V]$ is a scalar product, indeed.

Then the left hand side of the system (2) is the Gram matrix with respect to the scalar product $[\cdot, \cdot]$. Since the family of vectors Y^1, \dots, Y^d is linearly independent, the matrix of system (2) is nonsingular. Therefore, for each fixed β , there exists a unique solution of system (2). All the solutions of system (2) are proportional. Since $\mathbb{Q} = (q_1, \dots, q_N)$ is a probability measure, $\sum_{k=1}^N q_k = 1$. Hence only one of the proportional solutions determines a probability measure.

The following result is known in the general theory of minimal martingale measures (see [11]).

Proposition 1. *There exists at most one minimal martingale measure in a one-period model of the financial market.*

Proof. Assume that there are two minimal martingale measures. Then the system of equations (2) has at least two solutions for an arbitrary maximal linearly independent subfamily of the system of vectors Y^i , $i = 1, \dots, d$. This is impossible and Proposition 1 is proved by contradiction. \square

Now we are ready to provide necessary and sufficient conditions for the existence of a minimal martingale measure.

Since $(PY^i, Y^j) = \mathbb{E}_P Y_i Y_j$ and $(P, Y^i) = \mathbb{E}_P Y_i$, $i, j = 1, \dots, d$, one can rewrite system (2) as follows:

$$(3) \quad \begin{aligned} \alpha_1 \mathbb{E}_P (Y_1)^2 + \dots + \alpha_d \mathbb{E}_P Y_d Y_1 &= \beta \mathbb{E}_P Y_1, \\ &\dots \\ \alpha_1 \mathbb{E}_P Y_d Y_1 + \dots + \alpha_d \mathbb{E}_P (Y_d)^2 &= \beta \mathbb{E}_P Y^d. \end{aligned}$$

A minimal martingale measure exists if and only if system (3) has a solution

$$(\alpha_1, \dots, \alpha_n, \beta)$$

such that $\mathbf{Q} = \sum_{k=1}^d \alpha_k PY^k - \beta P$ is a probability vector with nonzero coordinates. Put

$$(4) \quad \Delta = \det \begin{pmatrix} \mathbb{E}_P (Y^1)^2 & \dots & \mathbb{E}_P Y^1 Y^d \\ \dots & \dots & \dots \\ \mathbb{E}_P Y^d Y^1 & \dots & \mathbb{E}_P (Y^d)^2 \end{pmatrix},$$

$$(5) \quad \Delta_i = \det \begin{pmatrix} \mathbb{E}_P (Y^1)^2 & \dots & \mathbb{E}_P Y^1 & \dots & \mathbb{E}_P Y^1 Y^d \\ \dots & \dots & \dots & \dots & \dots \\ \mathbb{E}_P Y^d Y^1 & \dots & \mathbb{E}_P Y^d & \dots & \mathbb{E}_P (Y^d)^2 \end{pmatrix},$$

where the numbers $\mathbb{E}_P Y^j$, $j = 1, \dots, d$, form the i -th column in Δ_i .

According to Cramer's rule,

$$\alpha_i = \beta \frac{\Delta_i}{\Delta}, \quad i = 1, \dots, d.$$

Thus

$$(6) \quad \mathbf{Q} = \beta \sum_{k=1}^d \frac{\Delta_k}{\Delta} PY^k - \beta P = -\beta \left(P - \sum_{k=1}^d \frac{\Delta_k}{\Delta} PY^k \right).$$

Note that if a vector \mathbf{Q} corresponds to a probability measure, then $q_i > 0$, $i = 1, \dots, N$, and thus the following result holds.

Theorem 2.1. *A minimal martingale measure exists if and only if all the numbers*

$$(7) \quad r_i = 1 - \sum_{k=1}^d \frac{\Delta_k}{\Delta} y_{ki}, \quad i = 1, \dots, d,$$

are of the same sign.

Lemma 2.2. *Let $d = 1$. A minimal martingale measure exists if and only if*

$$y_{1i} \mathbb{E}_P Y^1 < \mathbb{E}_P (Y^1)^2, \quad i = 1, \dots, N.$$

Proof. Since $d = 1$, we have

$$\Delta = (\mathbb{P}Y^1, Y^1) = \mathbb{E}_{\mathbb{P}}(Y^1)^2, \quad \Delta_1 = (\mathbb{P}, Y^1) = \mathbb{E}_{\mathbb{P}} Y^1.$$

Theorem 2.1 implies that a minimal martingale measure exists if and only if all the numbers $r_i = \mathbb{E}_{\mathbb{P}}(Y^1)^2 - \mathbb{E}_{\mathbb{P}} Y^1 y_{1i}$, $i = 1, \dots, N$, are of the same sign. Since

$$\sum_{i=1}^N p_i a_i = \mathbb{E}_{\mathbb{P}}(Y^1)^2 - (\mathbb{E}_{\mathbb{P}} Y^1)^2 > 0,$$

we conclude that all the a_i cannot be negative. Therefore a minimal martingale measure exists if and only if all numbers r_i are positive. This completes the proof of the lemma. \square

3. MULTIPERIOD MODEL OF THE FINANCIAL MARKET

Now we treat the case of $T > 1$.

Let \mathbb{Q} be a probability measure on the space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_T = \mathcal{F}$. Assume that \mathbb{Q} is equivalent to the measure \mathbb{P} . Our aim is to find a condition imposed on \mathbb{Q} ensuring that \mathbb{Q} is a minimal martingale measure.

Let R be an arbitrary probability measure on (Ω, \mathcal{F}) that is equivalent to the measure \mathbb{P} . For all $t \in \mathbb{T}$, denote by R_t the restriction of the measure R to the σ -algebra \mathcal{F}_t . Then $R_0(\emptyset) = 0$, $R_0(\Omega) = 1$, and the measure R_T coincides with the measure R .

Clearly,

$$(8) \quad \mathbb{E}_{R_t}(\xi | \mathcal{F}_s) = \mathbb{E}_R(\xi | \mathcal{F}_s), \quad 0 \leq s \leq t \leq T,$$

for an \mathcal{F} -measurable random variable ξ .

We denote by FM_t the model of the financial market with t periods defined on a probability space equipped with a filtration $(\Omega, \mathcal{F}_0 \subset \dots \subset \mathcal{F}_t, \mathbb{P}_t)$ and with price processes (X_0^k, \dots, X_t^k) , $k = 1, \dots, d$.

Let $t = 0, \dots, T - 1$. Given an arbitrary atom A of the σ -algebra \mathcal{F}_t we define the σ -algebra $\mathcal{F}_{t+1}(A) = \{B \in \mathcal{F}_{t+1} \mid B \subset A\}$. Put

$$R_{t+1}^A(B) = \frac{R(B)}{R(A)}$$

for an arbitrary event $B \in \mathcal{F}_{t+1}(A)$.

Lemma 3.1. *Let $t = 0, \dots, T - 1$ and let A_1, \dots, A_k be all the atoms of the σ -algebra \mathcal{F}_t . Then*

$$R_{t+1}(B) = \sum_{i=1}^k R_t(A_i) R_{t+1}^{A_i}(A_i \cap B)$$

for $B \in \mathcal{F}_{t+1}$.

Proof. The proof of the lemma follows from the definition of the probability measures R_t and R_{t+1}^A . \square

Lemma 3.2. *The following two statements hold.*

- 1) \mathbb{Q} is a martingale measure if and only if \mathbb{Q}_t is a martingale measure for the model FM_t of the financial market with t periods for all $t = 1, \dots, T$.
- 2) \mathbb{Q} is a minimal martingale measure if and only if \mathbb{Q}_t a minimal martingale measure for the model FM_t of the financial market with t periods for all $t = 1, \dots, T$.

Proof of statement 1). Sufficiency follows from the equality $Q_T = Q$.

Necessity. Let Q be a martingale measure for the model FM_T of the financial market with T periods. For $t = 1, \dots, T$, consider the measure Q_t for the model FM_t of the financial market with t periods. We check that Q_t is a martingale measure for FM_t . Let $Y^k = (X_0^k, \dots, X_t^k)$, $k = 1, \dots, d$, be the prices up to the moment t . By the definition of the measure Q_t , it is equivalent to the measure P_t , since Q is equivalent to P . Then equality (8) implies that Y^k is a Q_t -martingale. \square

Proof of statement 2). Sufficiency follows from the equality $Q_T = Q$.

Necessity. Let Q be a minimal martingale measure for the model FM_T of the financial market with T periods. For $t = 1, \dots, T$, consider the measure Q_t for the model FM_t of the financial market with t periods. Statement 1) implies that Q_t is a martingale measure.

Now we check that Q_t is a minimal martingale measure for the model FM_t . Let $M = (M_0, \dots, M_t)$ be a P_t -martingale defined on the probability space equipped with a filtration $(\Omega, \mathcal{F}_0 \subset \dots \subset \mathcal{F}_t, P_t)$. Assume that M is strongly P_t -orthogonal to Y^k , $k = 1, \dots, d$. We extend the stochastic process M to the set $\{t+1, \dots, T\}$ by putting $M_{t+1} = \dots = M_T = M_t$. Then the stochastic process $N = (M_0, \dots, M_T)$ is a P -martingale.

Next we prove that the process N is strongly P -orthogonal to the price processes X^k , $k = 1, \dots, d$. Let $k = 1, \dots, d$. Then

$$\begin{aligned} & \mathbb{E}_P((M_{s+1} - M_s)(X_{s+1}^k - X_s^k) \mid \mathcal{F}_s) \\ &= \begin{cases} \mathbb{E}_{P_t}((M_{s+1} - M_s)(X_{s+1}^k - X_s^k) \mid \mathcal{F}_s) = 0, & \text{for } s < t, \text{ since } M \perp Y^k, k = 1, \dots, d; \\ 0, & \text{for } s \geq t, \text{ since } M_s = M_{s+1} \text{ by definition.} \end{cases} \end{aligned}$$

Thus N is a P -martingale that is strongly P -orthogonal to the price processes X_1, \dots, X_d . Since Q is a minimal martingale measure, N is a Q -martingale. Lemma 3.3 implies that M is a Q_t -martingale.

Therefore Q_t is a minimal martingale measure. \square

Therefore the probability measure Q_{t+1} , $t = 0, \dots, T-1$, is uniquely determined by the measure Q_t and the family of conditional probability measures

$$\{Q_{t+1}^A \mid A \text{ is an atom of the } \sigma\text{-algebra } \mathcal{F}_t\}.$$

Given an arbitrary atom A of the σ -algebra \mathcal{F}_t , consider the probability space

$$PS_{t+1}^A = (A, \mathcal{F}_{t+1}(A), P_{t+1}^A).$$

Let $Y_t^{iA} = (X_{t+1}^i - X_t^i)|_A$, $i = 1, \dots, d$, be the gain processes.

For every atom of the σ -algebra \mathcal{F}_t , we have a one-period model of the financial market defined on the probability space $(A, \mathcal{F}_t(A), P_A)$ for which $Y_t^{1A}, \dots, Y_t^{dA}$ are the gain vectors. We denote these one-period models of the financial market by FM_t^A .

The indicator function of an arbitrary set $A \in \mathcal{F}$ is denoted by χ_A .

Lemma 3.3. *Let $t = 0, \dots, T-1$. Then Q_{t+1} is a martingale measure for the model FM_{t+1} of the financial market with $t+1$ periods if and only if*

- (i) Q_t is a martingale measure for the model FM_t of the financial market with t periods, and
- (ii) Q_{t+1}^A is a martingale measure for the one-period model FM_t^A of the financial market, where A is an arbitrary atom of the σ -algebra \mathcal{F}_t .

Proof. Since \mathbb{P} and \mathbb{Q} are equivalent probability measures, \mathbb{P}_t and \mathbb{Q}_t are equivalent probability measures for all $t \in \mathbb{T}$.

Necessity. Let \mathbb{Q}_{t+1} be a martingale measure. Then Lemma 3.2 implies that \mathbb{Q}_t is a martingale measure for the model FM_{t+1} . Let A be an atom of the σ -algebra \mathcal{F}_t . First we prove that \mathbb{Q}_{t+1}^A is a martingale measure. For all $k = 1, \dots, d$, we have

$$\mathbb{E}_{\mathbb{Q}_{t+1}^A} X_{t+1}^k = \frac{\mathbb{E}_{\mathbb{Q}_{t+1}}(X_{t+1}^k \chi_A)}{\mathbb{Q}_t(A)} = \frac{\mathbb{E}_{\mathbb{Q}_{t+1}}(X_t^k \chi_A)}{\mathbb{Q}_t(A)} = \frac{\mathbb{E}_{\mathbb{Q}_t}(X_t^k \chi_A)}{\mathbb{Q}_t(A)} = X_t^k |_A.$$

The second equality above holds in view of the property that \mathbb{Q}_{t+1} is a martingale measure, while the last equality holds, since the random variable X_t^k is constant on every atom of the σ -algebra \mathcal{F}_t .

Sufficiency. Let the probability measure \mathbb{Q}_t be a martingale measure for the model FM_t and let \mathbb{Q}_t be martingale measures for all models FM_t^A , where A denotes an arbitrary atom of the σ -algebra \mathcal{F}_t . Since \mathbb{Q}_t is a martingale measure for the model FM_t , equality (8) implies that

$$\mathbb{E}_{\mathbb{Q}_{t+1}}(X_{s+1}^k | \mathcal{F}_s) = X_s^k$$

for all $k = 0, \dots, d$ and $s = 0, \dots, t-1$.

It remains to show that $\mathbb{E}_{\mathbb{Q}_{t+1}}(X_{t+1} | \mathcal{F}_t) = X_t$. Given an arbitrary atom A of the σ -algebra \mathcal{F}_t , we have

$$\mathbb{E}_{\mathbb{Q}_{t+1}}(X_{t+1}^k - X_t^k) \chi_A = \mathbb{Q}_t(A) \mathbb{E}_{\mathbb{Q}_{t+1}^A} Y_t^{kA} = 0,$$

where the last equality holds, since \mathbb{Q}_t^A is a martingale measure for the model FM_t^A . \square

Lemma 3.4. *Let $t = 0, \dots, T-1$. Then \mathbb{Q}_{t+1} is a minimal martingale measure for the model FM_{t+1} with $t+1$ periods if and only if*

- (i) \mathbb{Q}_t is a minimal martingale measure for the model FM_t with t periods, and
- (ii) \mathbb{Q}_{t+1}^A is a minimal martingale measure for the one-period model FM_t^A for all atoms A of the σ -algebra \mathcal{F}_t .

Proof. Necessity. Let \mathbb{Q}_{t+1} be a minimal martingale measure for the model FM_{t+1} . Lemma 3.3 implies that \mathbb{Q}_t is a martingale measure for the model FM_t and that \mathbb{Q}_t^A is a martingale measure for the model FM_t^A for all atoms of the σ -algebra \mathcal{F}_t .

Let $M = (M_0, \dots, M_t)$ be an arbitrary \mathbb{P}_t -martingale that is strongly \mathbb{P}_t -orthogonal to the price processes (X_0^k, \dots, X_t^k) , $k = 1, \dots, d$. We extend the process M by putting $M_{t+1} = M_t$. The resulting process $\widetilde{M} = (M_0, \dots, M_{t+1})$ is a \mathbb{P}_{t+1} -martingale. Note that the process \widetilde{M} is strongly \mathbb{P}_{t+1} -orthogonal to $(X_0^k, \dots, X_{t+1}^k)$, $k = 1, \dots, d$. Since \mathbb{Q}_{t+1} is a minimal martingale measure, we derive that \widetilde{M} is a \mathbb{Q}_{t+1} -martingale. Then equality (8) implies that M is a \mathbb{Q}_t -martingale. Therefore \mathbb{Q}_t is a minimal martingale measure for the model FM_t .

Let $M = (M_0, M_1)$ be an arbitrary \mathbb{P}_{t+1}^A -martingale for the model FM_{t+1}^A . Assume that M is orthogonal to the price processes and introduce another stochastic process $N = (N_0, \dots, N_{t+1})$, where $N_0 = \dots = N_t = 0$ and $N_{t+1} = M_1 - M_0$. Then N is a \mathbb{P}_{t+1} -martingale that is orthogonal to the price processes, whence we conclude that it is a \mathbb{Q}_{t+1} -martingale. Thus M is a \mathbb{Q}_{t+1}^A -martingale for the model FM_{t+1}^A . Therefore \mathbb{Q}_{t+1}^A is a minimal martingale measure for the model FM_{t+1}^A .

Sufficiency. Let \mathbb{Q}_t be a minimal martingale measure for the model FM_t and let \mathbb{Q}_{t+1}^A be a minimal martingale measure for the model FM_t^A and for all atoms A_1, \dots, A_k of the σ -algebra \mathcal{F}_t , $k > 0$. We check that \mathbb{Q}_{t+1} is a minimal martingale measure for the model FM_{t+1} . It follows from Lemma 3.3 that \mathbb{Q}_{t+1} is a martingale measure.

Let $M = (M_0, \dots, M_{t+1})$ be an arbitrary \mathbb{P}_{t+1} -martingale that is strongly \mathbb{P}_{t+1} -orthogonal to the price processes $(X_0^k, \dots, X_{t+1}^k)$, $k = 1, \dots, d$. Let $N = (M_0, \dots, M_t)$. The stochastic process N is a \mathbb{P}_t -martingale that is strongly \mathbb{P}_t -orthogonal to the price processes (X_0^k, \dots, X_t^k) , $k = 1, \dots, d$. Since \mathbb{Q}_t is a minimal orthogonal measure for the model FM_t , the process N is a \mathbb{Q}_t -martingale. Using equality (8), we obtain

$$(9) \quad 0 = \mathbb{E}_{\mathbb{Q}_t}(M_{s+1} - M_s \mid \mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}_{t+1}}(M_{s+1} - M_s \mid \mathcal{F}_s)$$

for all $s = 0, \dots, t - 1$.

Let A be an arbitrary atom of the σ -algebra \mathcal{F}_t . The process $N_A = (M_t|_A, M_{t+1}|_A)$ is a \mathbb{P}_t^A -martingale for the model FM_t^A that is strongly \mathbb{P}_t^A -orthogonal to the price processes. Since \mathbb{P}_t^A is minimal for the model FM_t^A , we conclude that N_A is a \mathbb{Q}_t^A -martingale for \mathbb{P}_t^A . Then

$$(10) \quad \mathbb{E}_{\mathbb{Q}_{t+1}}(M_{t+1} - M_t)\chi_A = \frac{\mathbb{E}_{\mathbb{Q}_{t+1}^A}(M_{t+1} - M_t)}{\mathbb{Q}_t(A)} = 0.$$

Equalities (9) and (10) imply that M is a \mathbb{Q}_{t+1} -martingale. Thus \mathbb{Q}_{t+1} is a minimal martingale measure for the model FM_{t+1} . \square

Theorem 3.1. 1. *There exists at most one minimal martingale measure.*

2. *A minimal martingale measure exists if and only if, for all $t = 0, \dots, T - 1$ and for an arbitrary atom A of the σ -algebra \mathcal{F}_t , there exists a minimal martingale measure for the one-period model FM_t^A .*

Proof. Since there exists at most one minimal martingale measure for the one-period models FM_t^A for all $t = 0, \dots, T - 1$ and an arbitrary atom of the σ -algebra \mathcal{F}_t , the proof of statement 1 follows.

The proof of statement 2 follows from Lemmas 3.1–3.4. \square

Corollary 3.1 (A procedure to construct a minimal martingale measure). *The procedure consists of the following five steps.*

Step 1 Set $t = 0$ and define the initial measure \mathbb{Q}_0 by putting

$$\mathbb{Q}_0(\emptyset) = 0 \quad \text{and} \quad \mathbb{Q}_0(\Omega) = 1.$$

Step 2 The measure \mathbb{Q}_t is defined. For an arbitrary atom A of the σ -algebra \mathcal{F}_t , consider the one-period model FM_t^A . Using equalities (4), (5), and (7) evaluate the numbers r_1, \dots, r_d . If all these numbers are of the same sign, then the minimal martingale measure exists for the model FM_t^A and is given by

$$\mathbb{Q}_{t+1}^A(\{\omega_i\}) = \frac{r_i}{\sum_{k=1}^d r_k}, \quad i = 1, \dots, N.$$

Otherwise, if not all numbers r_1, \dots, r_d have the same sign, then there is no minimal martingale measure for the model FM_t^A .

Step 3 If a minimal martingale measure does not exist in the one-period model FM_t^A for at least one atom A of the σ -algebra \mathcal{F}_t , then the minimal martingale measure does not exist and the procedure stops.

Step 4 Applying Lemma 3.1 we evaluate the measure \mathbb{Q}_{t+1} as follows. Let $t = 0, \dots, T - 1$ and let A_1, \dots, A_k be all the atoms of the σ -algebra \mathcal{F}_t . For $B \in \mathcal{F}_{t+1}$, we put

$$\mathbb{Q}_{t+1}(B) := \sum_{i=1}^k \mathbb{Q}_t(A_i) \mathbb{Q}_{t+1}^A(A_i \cap B).$$

Step 5 Increase t by one. If $t = T$, then the minimal martingale measure $\mathbb{Q} = \mathbb{Q}_T$ is found and the procedure stops; otherwise go to Step 2.

4. EXAMPLES OF THE EVALUATION OF A MINIMAL MARTINGALE MEASURE

4.1. **A one-period model of the financial market with two risky assets.** Let $N > 0$, $\Omega = (\omega_1, \dots, \omega_N)$, $P(\omega_i) = p_i$, $i = 1, \dots, N$, and

$$Y^i(\omega_j) = y_{ij}, \quad i = 1, 2, \quad j = 1, \dots, N.$$

By Q we denote the minimal martingale measure to be evaluated,

$$Q(\omega_i) = q_i, \quad i = 1, \dots, N.$$

The numbers r_1, \dots, r_N are evaluated according to Theorem 2.1. Let

$$\Delta = \begin{vmatrix} E(Y^1)^2 & E Y^1 Y^2 \\ E Y^1 Y^2 & E(Y^2)^2 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} E Y^1 & E Y^1 Y^2 \\ E Y^2 & E(Y^2)^2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} E(Y^1)^2 & E Y^1 \\ E Y^1 Y^2 & E Y^2 \end{vmatrix}.$$

Then we have

$$(11) \quad r_i = \Delta - \Delta_1 y_{1i} - \Delta_2 y_{2i}, \quad i = 1, \dots, N.$$

If all numbers r_i are of the same sign, then the minimal martingale measure exists and the q_i are proportional to the r_i (see Theorem 2.1). Otherwise the minimal martingale measure does not exist.

Below are two numerical examples of the evaluation of a minimal martingale measure.

Example 4.1. Let $N = 3$ and let

$$P(\omega_1) = \frac{1}{6}, \quad P(\omega_2) = \frac{1}{3}, \quad P(\omega_3) = \frac{1}{2},$$

$$Y^1(\omega_1) = Y^1(\omega_2) = 1, \quad Y^1(\omega_3) = -1,$$

$$Y^2(\omega_1) = 1, \quad Y^2(\omega_2) = -2, \quad Y^2(\omega_3) = 1.$$

Using equalities (11) we get

$$r_1 = r_2 = r_3 = 1.$$

Since the numbers r_1 , r_2 , and r_3 are of the same sign, the minimal martingale measure exists and its values are proportional to r_1 , r_2 , and r_3 . Therefore

$$q_1 = q_2 = q_3 = \frac{1}{3}.$$

Example 4.2. Let $N = 3$ and let

$$P(\omega_1) = \frac{1}{6}, \quad P(\omega_2) = \frac{1}{3}, \quad P(\omega_3) = \frac{1}{2},$$

$$Y^1(\omega_1) = Y^1(\omega_2) = 1, \quad Y^1(\omega_3) = -1,$$

$$Y^2(\omega_1) = -3, \quad Y^2(\omega_2) = -2, \quad Y^2(\omega_3) = 1.$$

Using equalities (11) we get

$$r_1 = -\frac{1}{3}, \quad r_2 = \frac{1}{3}, \quad r_3 = \frac{1}{9}.$$

Since not all numbers r_1 , r_2 , r_3 have the same sign, the minimal martingale measure does not exist.

4.2. A two-period model of the financial market with two risky assets. Let $\Omega = \{\omega_{ij} \mid i, j = 1, 2, 3\}$, $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{2}$,

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_1 = \sigma a(\{A_i \mid i = 1, 2, 3\}), \quad \mathcal{F}_2 = 2^\Omega,$$

where $A_i = \{\omega_{ij} \mid j = 1, 2, 3\}$, $i = 1, 2, 3$, and $\mathbb{P}(\omega_{ij}) = p_i p_j$, $i, j = 1, 2, 3$.

Let $a_1 = a_2 = 1$, $a_3 = -1$ and $b_1 = 1$, $b_2 = -2$, $b_3 = 1$. Put

$$X_1^1(\omega_{ij}) = 1 + a_i, \quad X_2^1(\omega_{ij}) = 1 + a_i + a_j, \quad i, j = 1, 2, 3,$$

and

$$X_1^2(\omega_{ij}) = 1 + b_i, \quad X_2^2(\omega_{ij}) = 1 + b_i + b_j, \quad i, j = 1, 2, 3.$$

Following the procedure of Corollary 3.1, we first find the measure \mathbb{Q}_1 . In fact, the measure \mathbb{Q}_1 is found in Example 4.1. Thus

$$\mathbb{Q}_1(A_1) = \mathbb{Q}_1(A_2) = \mathbb{Q}_1(A_3) = \frac{1}{3}.$$

Then we evaluate the measure \mathbb{Q}_2 . For this, we need to evaluate the measures $\mathbb{Q}_2^{A_1}$, $\mathbb{Q}_2^{A_2}$, and $\mathbb{Q}_2^{A_3}$. The problems of the evaluation of the measures $\mathbb{Q}_2^{A_1}$, $\mathbb{Q}_2^{A_2}$, and $\mathbb{Q}_2^{A_3}$ coincide with the problem solved in Example 4.1. Thus

$$\mathbb{Q}^{A_i}(\omega_{ij}) = \frac{1}{3}, \quad i, j = 1, 2, 3.$$

Then we find from Lemma 3.1 that

$$\mathbb{Q}_2(\omega_{ij}) = \frac{1}{27}, \quad i, j = 1, 2, 3.$$

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