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CONVERGENCE OF SERIES OF ELEMENTS OF MULTIDIMENSIONAL GAUSSIAN MARKOV SEQUENCES

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Abstract. Necessary and sufficient conditions are found for the almost sure convergence of random series of elements of a multidimensional Gaussian Markov sequence.

1. Introduction

Asymptotic properties of Gaussian Markov sequences are studied in many papers. For example, the papers [1, 3, 4, 6] study the conditions for the *almost sure* convergence and almost sure boundedness of one-dimensional and multidimensional Gaussian Markov sequences. Necessary and sufficient conditions for the almost sure convergence of a series of Gaussian Markov sequences are found in [7, 8]. A criterion for the almost sure convergence of a random series of Gaussian Markov sequences of random variables is obtained in the paper [10].

The current paper is a continuation of [10]. We study the necessary and sufficient conditions for the almost sure convergence of a random series whose terms are elements of a multidimensional Gaussian Markov sequence.

We consider a finite-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 1$, equipped with the scalar product (X, Y) and norm $\|X\| = \sqrt{(X, X)}$, $X, Y \in \mathbb{R}^d$. For $d \times d$ matrices $A = (a_{ij})_{i,j=1}^d$, we introduce the matrix norm

$$|||A||| = \Big(\sum_{i,j=1}^{d} a_{i,j}^2 \Big)^{1/2}.$$

In the space $\mathbb{R}^d$, consider a multidimensional centered Gaussian Markov sequence $(X_k) = (X_k, k \geq 1)$. In other words, we consider a sequence defined by the recurrence relations

$$X_1 = D_1 \Gamma_1, \qquad X_k = C_k X_{k-1} + D_k \Gamma_k, \quad k \geq 2, \tag{1}$$

where (Γ_k) is a sequence of jointly independent standard Gaussian random vectors in $\mathbb{R}^d$ and where (C_k) and (D_k) are nonrandom sequences of real $d \times d$ matrices. We assume throughout this paper that the matrices C_k, $k \geq 1$, are nonsingular.

Given a sequence (X_k), consider the random series

$$\sum_{k=1}^{\infty} X_k. \tag{2}$$

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Our aim is to find necessary and sufficient conditions for the almost sure convergence of this series.

The results obtained in this paper can be used to determine the necessary and sufficient conditions for the almost sure convergence of series whose terms are elements of a Gaussian m-Markov sequence of random variables. Below we study the case of $m = 2$.

2. Auxiliary results

This section contains some auxiliary results needed in the proofs of the main results of the paper.

Let $c(\mathbb{R}^d)$ be the space of all convergent sequences whose elements belong to $\mathbb{R}^d$, let (Z_n) be a sequence of independent symmetric random vectors of the space $\mathbb{R}^d$, let $\Xi_n = \sum_{k=1}^n Z_k$, $n \geq 1$, and let $(\mathcal{A}_n)$ be a sequence of linear continuous operators acting from $\mathbb{R}^d$ to $\mathbb{R}^d$. By $\mathfrak{R}^\infty$, we denote the class of all monotonic sequences of natural numbers approaching infinity.

Below is the criterion for the almost sure convergence of sums of independent symmetric random vectors with operator normalizations in the space $\mathbb{R}^d$ (see [3]–[6]).

Proposition 2.1. *The relation*

$$(\mathcal{A}_n \Xi_n) \in c(\mathbb{R}^d) \quad \textit{almost surely}$$

is equivalent to the set of the following three conditions:

A) *for all* $k \geq 1$, $(\mathcal{A}_n Z_k) \in c(\mathbb{R}^d)$ *almost surely*;

B) *the series* $\sum_{k=1}^\infty \lim\limits_{n\to\infty} (\mathcal{A}_n Z_k)$ *converges almost surely in* $\mathbb{R}^d$;

C) *for all sequences* (m_j) *of the class* $\mathfrak{R}^\infty$,

$$\|\mathcal{A}_{m_{j+1}}(\Xi_{m_{j+1}} - \Xi_{m_j})\| \underset{j\to\infty}{\longrightarrow} 0 \quad \textit{almost surely.}$$

The following result is used below to check condition C) of Proposition 2.1 for Gaussian random vectors (see, for example, [4]).

Proposition 2.2. *Let* (Γ_k) *be a sequence of centered Gaussian random vectors in the space* $\mathbb{R}^d$, $d \geq 1$. *If*

$$\sum_{k=1}^\infty \exp\left\{-\frac{\varepsilon}{\mathsf{E}\,\|\Gamma_k\|^2}\right\} < \infty \tag{3}$$

for all $\varepsilon > 0$, *then*

$$\lim_{k\to\infty} \|\Gamma_k\| = 0 \qquad \textit{almost surely.} \tag{4}$$

If (Γ_k) *is a sequence of independent random vectors, then relations* (4) *and* (3) *are equivalent.*

3. A criterion for the almost sure convergence of a series whose terms are elements of a multidimensional Gaussian Markov sequence

First we introduce some notation. Let

$$Q(n,k) = \begin{cases} D_k + \sum_{l=1}^{n-k} \left(\prod_{j=k+l}^{k+1} C_j\right) D_k, & 1 \leq k \leq n-1, \\ D_k, & k = n, \\ \mathcal{O}, & k > n, \end{cases}$$

for $n \geq 1$, where $\mathcal{O}$ is the zero $d \times d$ matrix. For $k \geq 1$, consider the following matrix series:

$$\sum_{l=1}^{\infty}\Big(\prod_{j=k+l}^{k+1} C_j\Big)D_k. \tag{5}$$

Put

$$Q(\infty,k) = D_k + \sum_{l=1}^{\infty}\Big(\prod_{j=k+l}^{k+1} C_j\Big)D_k$$

provided the corresponding series converges in the matrix norm.

The following criterion for the almost sure convergence of series (2) holds for a multidimensional Gaussian Markov sequence defined by (1).

Theorem 3.1. *Let matrices C_k, $k \geq 1$, be nonsingular. The random series* (2) *converges almost surely if and only if the following three conditions hold:*

1) *for all $k \geq 1$, series* (5) *converges in the matrix norm*;

2)

$$\sum_{k=1}^{\infty} |||Q(\infty,k)|||^2 < \infty; \tag{6}$$

3) *for all sequences (m_j) of the class $\mathfrak{R}^\infty$ and for all $\varepsilon > 0$,*

$$\sum_{j=1}^{\infty} \exp\left\{-\frac{\varepsilon}{\sum_{k=m_j+1}^{m_{j+1}} |||Q(m_{j+1},k)|||^2}\right\} < \infty. \tag{7}$$

Proof. Consider the sequence (S_n) of partial sums of series (2):

$$S_n = \sum_{k=1}^{n} X_k, \qquad n \geq 1.$$

The method we use to prove Theorem 3.1 is to pass from the sequence (S_n) to the sequence of centered independent Gaussian vectors in the space $\mathbb{R}^{2d}$. A similar method is developed in the monographs [4, 6].

Since $X_n = S_n - S_{n-1}$, $n \geq 1$, relation (1) implies that the sequence (S_n) satisfies the following recurrence relation of the second order:

$$S_{-1} = S_0 = \mathbf{0}, \qquad S_n = (I + C_n)S_{n-1} - C_n S_{n-2} + D_n\Gamma_n, \quad n \geq 1, \tag{8}$$

where I is the unit $d \times d$ matrix and where $\mathbf{0}$ is the zero vector in the space $\mathbb{R}^d$.

Now we pass from relation (8) to a recurrence relation of the first order in the space $\mathbb{R}^{2d}$. This relation can be written as follows:

$$\widetilde{S}_1 = \Theta_1, \qquad \widetilde{S}_n = B_n\widetilde{S}_{n-1} + \Theta_n, \quad n \geq 2, \tag{9}$$

where

$$\widetilde{S}_n = \begin{pmatrix} S_n \\ S_{n-1} \end{pmatrix}, \qquad B_n = \begin{pmatrix} I + C_n & -C_n \\ I & \mathcal{O} \end{pmatrix}, \qquad \Theta_n = \begin{pmatrix} D_n\Gamma_n \\ \mathbf{0} \end{pmatrix}, \qquad n \geq 1.$$

Thus the random series (2) converges almost surely if and only if the limit

$$\lim_{n\to\infty} \widetilde{S}_n$$

exists almost surely.

The recurrence relation (9) implies that

$$\widetilde{S}_n = \left(\prod_{j=n}^{2} B_j\right)\Theta_1 + \left(\prod_{j=n}^{3} B_j\right)\Theta_2 + \cdots + B_n\Theta_{n-1} + \Theta_n, \qquad n \geq 1, \tag{10}$$

where $\prod_{j=n}^{k} B_j = B_n B_{n-1} \dots B_k$, $k \leq n$. Applying the method of mathematical induction we see that

$$\prod_{j=n}^{k} B_j = \begin{pmatrix} I + \sum_{l=1}^{n-k+1}\left(\prod_{j=k+l-1}^{k} C_j\right) & -\sum_{l=1}^{n-k+1}\left(\prod_{j=k+l-1}^{k} C_j\right) \\ I + \sum_{l=1}^{n-k}\left(\prod_{j=k+l-1}^{k} C_j\right) & -\sum_{l=1}^{n-k}\left(\prod_{j=k+l-1}^{k} C_j\right) \end{pmatrix}, \qquad 2 \leq k \leq n.$$

Now we prove the necessity part of the theorem. In other words, we show that the almost sure convergence of random series (2) implies assumptions 1)–3) of Theorem 3.1.

The proof of assumptions 1) and 2) of Theorem 3.1 is the same as the proof of Theorem 3.1 of the paper [8] with the only difference that the scalar coefficients should be exchanged for the matrix coefficients. Thus we omit this part of the proof.

To prove that assumption 3) of Theorem 3.1 holds, we follow the *contraction principle* in the space of convergent sequences (see [2, 4, 6]). Fix an arbitrary sequence (m_j) belonging to the class $\mathfrak{R}^\infty$.

Consider a set of random vectors $(Y_{n,k}; n, k \geq 1)$, where

$$Y_{n,k} = \begin{cases} \left(\prod_{j=n}^{k+1} B_j\right)\Theta_k, & 1 \leq k \leq n-1, \\ \Theta_k, & k = n, \\ \overrightarrow{0}, & k > n, \end{cases}$$

and where $\overrightarrow{0}$ is the zero vector in the space $\mathbb{R}^{2d}$. This set of vectors possesses the following properties:

a) for all $n \geq 1$, the series $\sum_{k=1}^{\infty} Y_{n,k}$ converges almost surely in the space $\mathbb{R}^{2d}$;
b) the sequences $W_k = (Y_{n,k}, n \geq 1)$, $k \geq 1$, are independent and symmetric as elements of the space of sequences.

Moreover,

$$\widetilde{S}_n = \sum_{k=1}^{\infty} Y_{n,k}, \qquad n \geq 1.$$

Along with the set $(Y_{n,k}; n, k \geq 1)$, consider the contraction matrix $(\lambda_{n,k}; n, k \geq 1)$, where

$$\lambda_{n,k} = \begin{cases} 1, & n = m_{j+1},\ m_j < k \leq m_{j+1},\ j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since the random series (2) converges almost surely, the limit

$$\lim_{n\to\infty} \widetilde{S}_n$$

exists almost surely. This implies that the sequence of vectors $(\sum_{k=1}^{\infty} Y_{n,k})$ converges almost surely in the space $\mathbb{R}^{2d}$. Moreover,

$$\|\lambda_{n,k} Y_{n,k}\| \underset{n\to\infty}{\longrightarrow} 0, \qquad k \geq 1, \quad \text{almost surely.}$$

According to the contraction principle, this implies that

$$\left\|\sum_{k=1}^{\infty} \lambda_{n,k} Y_{n,k}\right\| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{almost surely}$$

(see Corollary 2.7.1 in [4]). The latter relation is rewritten as follows:

$$\left\| \sum_{k=m_j+1}^{m_{j+1}} \left(\prod_{i=m_{j+1}}^{k+1} B_i \right) \Theta_k \right\| \underset{j\to\infty}{\longrightarrow} 0 \qquad \text{almost surely,}$$

which is equivalent to

$$\left\| \begin{pmatrix} \sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k)\Gamma_k \\ \sum_{k=m_j+1}^{m_{j+1}-1} Q(m_{j+1}-1,k)\Gamma_k \end{pmatrix} \right\| \underset{j\to\infty}{\longrightarrow} 0 \qquad \text{almost surely,}$$

whence

$$\lim_{j\to\infty} \left\| \sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k)\Gamma_k \right\| = 0 \qquad \text{almost surely.} \tag{11}$$

Since (Γ_k) is a sequence of jointly independent standard Gaussian vectors,

$$\sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k)\Gamma_k$$

is a centered Gaussian random vector for all $j \geq 1$. Moreover, the random vectors

$$\sum_{k=m_{j_1}+1}^{m_{j_1+1}} Q(m_{j_1+1},k)\Gamma_k \quad \text{and} \quad \sum_{k=m_{j_2}+1}^{m_{j_2+1}} Q(m_{j_2+1},k)\Gamma_k$$

are mutually independent for all $j_1 \neq j_2$. Thus (11) and Proposition 2.2 imply that

$$\sum_{j=1}^{\infty} \exp\left\{ -\frac{\varepsilon}{\sum_{k=m_j+1}^{m_{j+1}} |||Q(m_{j+1},k)|||^2} \right\} < \infty$$

for all $\varepsilon > 0$. This means that condition 3) of Theorem 3.1 holds and the necessity part of Theorem 3.1 is proved.

Now we prove that assumptions 1)–3) are sufficient for the almost sure convergence of series (2).

Since all the matrices C_n, $n \geq 1$, are nonsingular, the matrices B_n, $n \geq 1$, in representation (10) also are nonsingular. Thus $\det B_n = \det C_n \neq 0$, $n \geq 1$. Therefore one can pass from relation (10) to the following relation:

$$\widetilde{S}_n = \left(\prod_{j=n}^{2} B_j \right) \left(\Theta_1 + B_2^{-1}\Theta_2 + \left(B_2^{-1}B_3^{-1}\right)\Theta_3 + \cdots + \left(B_2^{-1}B_3^{-1}\ldots B_n^{-1}\right)\Theta_n \right),$$
$$n \geq 1,$$

where B_k^{-1} is the inverse matrix to B_k, $k \geq 1$. Now we represent $(\widetilde{S}_n)$ as a sequence of sums of independent random vectors with operator normalizations (see [4, 5, 6]), namely

$$\widetilde{S}_n = \mathcal{A}_n \sum_{k=1}^{n} V_k = \mathcal{A}_n \Xi_n, \qquad n \geq 1,$$

where

$$\Xi_n = \sum_{k=1}^{n} V_k, \qquad n \geq 1,$$

and

$$V_1 = \Theta_1, \qquad V_k = \left(\prod_{j=2}^{k} B_j^{-1} \right) \Theta_k, \qquad k \geq 2.$$

In the case under consideration

$$\mathcal{A}_1 = I, \qquad \mathcal{A}_n = \prod_{j=n}^{2} B_j = \begin{pmatrix} I + \sum_{l=1}^{n-1}\left(\prod_{j=l+1}^{2} C_j\right) & -\sum_{l=1}^{n-1}\left(\prod_{j=l+1}^{2} C_j\right) \\ I + \sum_{l=1}^{n-2}\left(\prod_{j=l+1}^{2} C_j\right) & -\sum_{l=1}^{n-2}\left(\prod_{j=l+1}^{2} C_j\right) \end{pmatrix}$$

for all $n \geq 2$.

Note that (V_n) is a sequence of independent centered Gaussian random vectors in the space $\mathbb{R}^{2d}$. Hence $(\widetilde{S}_n)$ is represented as a sequence $(\mathcal{A}_n \Xi_n)$ of sums of independent symmetric random vectors with operator normalizations in the space $\mathbb{R}^{2d}$. This implies that the random series (2) converges almost surely if $(\mathcal{A}_n \Xi_n) \in c(\mathbb{R}^{2d})$ almost surely. Now we apply Proposition 2.1 to the sequence $(\mathcal{A}_n \Xi_n)$.

Let the series (5) converge in the matrix norm for all $k \geq 1$. This implies that the limit

$$\lim_{n\to\infty} Q(n,k) = Q(\infty,k)$$

exists for all $k \geq 1$. Taking into account the equalities

$$\mathcal{A}_n V_k = B_n B_{n-1} \dots B_{k+1} \Theta_k = \begin{pmatrix} Q(n,k)\Gamma_k \\ Q(n-1,k)\Gamma_k \end{pmatrix}, \qquad k, n \geq 1,$$

we prove that, for all $k \geq 1$,

$$(\mathcal{A}_n V_k) \in c(\mathbb{R}^{2d}) \quad \text{almost surely},$$

which means that assumption A) of Proposition 2.1 holds. Moreover,

$$\lim_{n\to\infty} \mathcal{A}_n V_k = \lim_{n\to\infty} \begin{pmatrix} Q(n,k)\Gamma_k \\ Q(n-1,k)\Gamma_k \end{pmatrix} = \begin{pmatrix} Q(\infty,k)\Gamma_k \\ Q(\infty,k)\Gamma_k \end{pmatrix}, \qquad k \geq 1.$$

Since

$$\sum_{k=1}^{\infty} (\lim_{n\to\infty} \mathcal{A}_n V_k) = \sum_{k=1}^{\infty} \begin{pmatrix} Q(\infty,k)\Gamma_k \\ Q(\infty,k)\Gamma_k \end{pmatrix},$$

the series

$$\sum_{k=1}^{\infty} \left(\lim_{n\to\infty} \mathcal{A}_n V_k\right)$$

converges almost surely in the space $\mathbb{R}^{2d}$ by condition (6). This means that assumption B) of Proposition 2.1 also holds.

Consider an arbitrary sequence (m_j) belonging to the class $\mathfrak{R}^\infty$. Then

$$\begin{aligned} \mathcal{A}_{m_{j+1}}(\Xi_{m_{j+1}} - \Xi_{m_j}) &= \left(\prod_{j=m_{j+1}}^{m_j+2} B_j\right)\Theta_{m_j+1} + \dots + B_{m_{j+1}}\Theta_{m_{j+1}-1} + \Theta_{m_{j+1}} \\ &= \begin{pmatrix} \sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k)\Gamma_k \\ \sum_{k=m_j+1}^{m_{j+1}-1} Q(m_{j+1}-1,k)\Gamma_k \end{pmatrix}, \qquad j \geq 1. \end{aligned} \tag{12}$$

Let condition (7) hold. Then equality (12) and Proposition 2.2 imply that

$$\lim_{j\to\infty} \left\| \sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k)\Gamma_k \right\| = 0 \quad \text{almost surely}.$$

This together with (12) yields

$$\lim_{j\to\infty} \|\mathcal{A}_{m_{j+1}}(\Xi_{m_{j+1}} - \Xi_{m_j})\| = 0 \quad \text{almost surely}$$

and thus assumption B) of Proposition 2.1 holds too.

Since assumptions A)–C) of Proposition 2.1 hold, $(\mathcal{A}_n\Xi_n) \in c(\mathbb{R}^{2d})$ almost surely, that is, the random series (2) converges almost surely. The proof of Theorem 3.1 is completed. □

Remark 3.1. We assume that the matrices C_k, $k \geq 1$, are nonsingular to make the proof of Theorem 3.1 simpler. The case of the general matrices C_k, $k \geq 1$, can be treated by using the perturbation method (see [3, 4, 6]). This approach is demonstrated in the paper [10] for the case of dimension $d = 1$.

4. A criterion for the almost sure convergence of a series whose terms are elements of a Gaussian 2-Markov sequence of random variables

In this section, we study necessary and sufficient conditions for the almost sure convergence of a random series whose terms are elements of a centered *Gaussian* 2-*Markov sequence*. Namely, we consider a sequence of random variables (ξ_k) defined by the recurrence relation of the second order:

$$\xi_{-1} = \xi_0 = 0, \qquad \xi_k = a_k\xi_{k-1} + b_k\xi_{k-2} + \beta_k\gamma_k, \quad k \geq 1, \tag{13}$$

where (β_k) is a nonnegative nonrandom sequence, (γ_k) is a standard Gaussian sequence, and where (a_k) and (b_k) are nonrandom sequences. In what follows we assume that $b_k \neq 0$, $k \geq 1$.

Given a sequence (ξ_k), consider the following random series:

$$\sum_{k=1}^{\infty} \xi_k. \tag{14}$$

Our aim is to determine necessary and sufficient conditions for the almost sure convergence of this series.

Consider a nonrandom real sequence $\left(u_n^{(k+1)}, n \geq k-1\right)$ defined by the following recurrence relations:

$$u_{k-1}^{(k+1)} = 0, \qquad u_k^{(k+1)} = 1, \qquad u_n^{(k+1)} = a_n u_{n-1}^{(k+1)} + b_n u_{n-2}^{(k+1)}, \qquad n \geq k+1,$$

for $k \geq 1$.

Put

$$U_k = \sum_{l=0}^{\infty} \beta_k u_{k+l}^{(k+1)}$$

for $k \geq 1$ provided the series

$$\sum_{l=0}^{\infty} \beta_k u_{k+l}^{(k+1)} \tag{15}$$

converges.

The criterion for the almost sure convergence of series (14) is given in the following assertion.

Theorem 4.1. *Let* (ξ_k) *be a centered Gaussian* 2-*Markov sequence. The random series* (14) *converges almost surely if and only if the following three conditions hold:*

1) *for all* $k \geq 1$, *the nonrandom series* (15) *converges*;

2)

$$\sum_{k=1}^{\infty} U_k^2 < \infty; \tag{16}$$

3) *for all sequences* (m_j) *belonging to the class* $\mathfrak{R}^\infty$ *and for all* $\varepsilon > 0$,

$$\sum_{j=1}^{\infty} \exp\left\{ -\frac{\varepsilon}{\sum_{k=m_j+1}^{m_{j+1}} \left(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \right)^2 \beta_k^2} \right\} < \infty.$$

Proof. The method we use to prove Theorem 4.1 is to pass from the recurrence relation of the second order (13) to the corresponding recurrence relation of the first order in the space $\mathbb{R}^2$. A similar approach is used in the paper [3] (see monographs [4, 6] for a generalization of this method).

Put

$$X_k = \begin{pmatrix} \xi_k \\ \xi_{k-1} \end{pmatrix}, \qquad \Gamma_k = \begin{pmatrix} \gamma_k \\ 0 \end{pmatrix}, \qquad D_k = \begin{pmatrix} \beta_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad C_k = \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix}, \qquad k \geq 1.$$

Note that C_k is a Frobenius matrix for all $k \geq 1$. Since $b_k \neq 0$, $k \geq 1$, the matrices C_k, $k \geq 1$, are nonsingular.

Now we pass from relation (13) to the recurrence relation of the first order in the space $\mathbb{R}^2$, namely

$$X_1 = D_1 \Gamma_1, \quad X_k = C_k X_{k-1} + D_k \Gamma_k, \quad k \geq 2.$$

Then (X_k) is a centered Gaussian Markov sequence in the space $\mathbb{R}^2$. Thus the random series (14) converges almost surely if and only if the series $\sum_{k=1}^{\infty} X_k$ converges almost surely in the space $\mathbb{R}^2$. We are going to apply Theorem 3.1 to the series $\sum_{k=1}^{\infty} X_k$.

Below we show that assumptions 1)–3) of Theorem 4.1 hold if and only if so do assumptions 1)–3) of Theorem 3.1.

In the case under consideration,

$$Q(n,k) = D_k + \sum_{l=1}^{n-k} \left(\prod_{j=k+l}^{k+1} C_j \right) D_k = \begin{pmatrix} \left(\sum_{l=0}^{n-k} u_{k+l}^{(k+1)} \right) \beta_k & 0 \\ \left(\sum_{l=1}^{n-k} u_{k+l-1}^{(k+1)} \right) \beta_k & 0 \end{pmatrix}$$

for $1 \leq k \leq n-1$ and

$$Q(\infty, k) = \begin{pmatrix} U_k & 0 \\ U_k & 0 \end{pmatrix}, \qquad k \geq 1.$$

Thus the series $\sum_{l=0}^{\infty} u_{k+l}^{(k+1)}$ converges for $k \geq 1$ if and only if the matrix series

$$\sum_{l=1}^{n-k} \prod_{j=k+l}^{k+1} C_j$$

converges in the matrix norm for all $k \geq 1$.

Since (γ_k) is a standard Gaussian sequence and

$$|||Q(\infty, k)|||^2 = 2U_k^2,$$

condition (6) holds if and only if relation (16) is satisfied.

Finally, fix $\varepsilon > 0$ and an arbitrary sequence (m_j) belonging to the class $\mathfrak{R}^\infty$. Then condition

$$\lim_{j \to \infty} \left\| \sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1}, k) \Gamma_k \right\| = 0 \qquad \text{almost surely}$$

holds if and only if

$$\lim_{j \to \infty} \left| \sum_{k=m_j+1}^{m_{j+1}} \left(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \right) \beta_k \gamma_k \right| = 0 \quad \text{almost surely.}$$

Since (γ_k) is a standard Gaussian sequence, Proposition 2.2 implies that the latter relation is equivalent to the condition that

$$\sum_{j=1}^{\infty} \exp\left\{ -\frac{\varepsilon}{\sum_{k=m_j+1}^{m_{j+1}} \left(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \right)^2 \beta_k^2} \right\} < \infty$$

for all $\varepsilon > 0$. Theorem 4.1 is proved. □

Remark 4.1. Following the method of the proof of Theorem 4.1, one can determine necessary and sufficient conditions for the almost sure convergence of the random series whose terms are members of a centered Gaussian m-Markov sequence for $m \geq 1$. In other words, the terms of this series are random variables (ξ_k) defined by the following recurrence relations:

$$\xi_{1-m} = \cdots = \xi_{-1} = \xi_0 = 0, \qquad \xi_k = b_{k_1}\xi_{k-1} + b_{k_2}\xi_{k-2} + \cdots + b_{k_m}\xi_{k-m} + \beta_k\gamma_k, \qquad k \geq 1,$$

where (β_k) is a nonnegative nonrandom sequence, $(b_{k_j}; 1 \leq j \leq m, k \geq 1)$ is a set of nonrandom real numbers, and (γ_k) is a standard Gaussian sequence.

Further we consider some corollaries.

Corollary 4.1. *Let (ξ_k) be a Gaussian 2-Markov sequence such that $a_k \geq 0$, $b_k > 0$, $k \geq 1$. The random series* (14) *converges almost surely if and only if the following two conditions hold:*

1) *for all $k \geq 1$, the nonrandom series* (15) *converges*;
2) *relation* (16) *holds.*

Proof. First we note that conditions 1) and 2) of Corollary 4.1 coincide with conditions 1) and 2) of Theorem 4.1, respectively. Thus we need to show that condition 3) of Theorem 4.1 holds. Fix $\varepsilon > 0$ and an arbitrary sequence (m_j) belonging to the class $\mathfrak{R}^\infty$. Since $a_k \geq 0$ and $b_k > 0$, $k \geq 1$, we have

$$\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \leq \sum_{l=0}^{\infty} u_{k+l}^{(k+1)}, \qquad k \geq 1,$$

for all $j \geq 1$. Thus

$$\begin{aligned} \exp\left\{ -\frac{\varepsilon}{\sum_{i=m_j+1}^{m_{j+1}} \left(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \right)^2 \beta_k^2} \right\} &\leq \frac{1}{\varepsilon} \left(\sum_{i=m_j+1}^{m_{j+1}} \left(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \right)^2 \beta_k^2 \right) \\ &\leq \frac{1}{\varepsilon} \left(\sum_{i=m_j+1}^{m_{j+1}} U_k^2 \right) \end{aligned}$$

for all $j \geq 1$. The convergence of the series

$$\sum_{j=1}^{\infty} \left(\sum_{i=m_j+1}^{m_{j+1}} U_k^2 \right)$$

follows from condition (16). This means that condition 3) of Theorem 4.1 holds. Corollary 4.1 is proved. □

The following assertion contains the criterion for the almost sure convergence of the series whose terms are the elements of a Gaussian 2-Markov sequence with constant coefficients.

Corollary 4.2. *Let a sequence* (ξ_k) *be defined by the following recurrence relations:*

$$\xi_{-1} = \xi_0 = 0, \qquad \xi_k = a\xi_{k-1} + b\xi_{k-2} + \beta_k\gamma_k, \quad k \geq 1,$$

where a *and* b *are some constants and* (β_k) *is a nonrandom nonnegative real sequence containing nonzero members.*

Then the random series (14) *converges almost surely if and only if the following two conditions hold:*

1) *the coefficients* a *and* b *are such that* $-1 < b < 1 - |a|$;

2)

$$\sum_{k=1}^{\infty} \beta_k^2 < \infty. \tag{17}$$

Proof. Consider the following Frobenius matrix:

$$C = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

Without loss of generality we assume that $b \neq 0$, that is, the matrix C is nonsingular. Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 - a\lambda - b = 0. \tag{18}$$

In the general case, the roots can be complex numbers. The spectral radius of the matrix C is denoted by r, that is,

$$r = \max\{|\lambda_1|, |\lambda_2|\}.$$

Let μ be the maximal multiplicity of the roots λ_k, $k = 1, 2$.

We also put

$$X_k = \begin{pmatrix} \xi_k \\ \xi_{k-1} \end{pmatrix}, \qquad \Gamma_k = \begin{pmatrix} \gamma_k \\ 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_k = \beta_k M, \qquad k \geq 1.$$

Then one can rewrite the recurrence relation of the second order in the space $\mathbb{R}$ in the form of a recurrence relation of the first order in the space $\mathbb{R}^2$. More precisely,

$$X_1 = D_1\Gamma_1, \qquad X_k = CX_{k-1} + D_k\Gamma_k, \qquad k \geq 2.$$

It is clear that (X_k) is a centered Gaussian Markov sequence in the space $\mathbb{R}^2$. Thus the random series (14) converges almost surely if and only if the random series $\sum_{k=1}^{\infty} X_k$ converges almost surely in the space $\mathbb{R}^2$. Thus Theorem 3.1 applies to the series $\sum_{k=1}^{\infty} X_k$.

Consider the following matrix series:

$$\sum_{l=0}^{\infty} C^l D_k.$$

Note that

$$\left|\left|\left|C^l D_k\right|\right|\right| = \left|\left|\left|C^l \beta_k M\right|\right|\right| = \beta_k \left|\left|\left|C^l M\right|\right|\right|, \qquad l \geq 1,$$

for all $k \geq 1$. Since C is a Frobenius matrix, the estimates obtained in the paper [9] (also see Lemma 7.7.3 in [6]) imply that

$$c_1 \cdot r^l \cdot l^{\mu-1} \leq \left|\left|\left|C^l M\right|\right|\right| \leq c_2 \cdot r^l \cdot l^{\mu-1} \tag{19}$$

for all $l \geq 1$, where c_1 and c_2 are some constants such that $c_2 > c_1 > 0$. Then we conclude that

$$c_1 \cdot r^l \cdot l^{\mu-1} \cdot \beta_k \leq \left|\left|\left|C^l D_k\right|\right|\right| \leq c_2 \cdot r^l \cdot l^{\mu-1} \cdot \beta_k, \qquad l \geq 1,$$

if the corresponding β_k is nonzero. For this k, the matrix series $\sum_{l=0}^{\infty} C^l D_k$ converges in the matrix norm if and only if

$$\sum_{l=0}^{\infty} r^l \cdot l^{\mu-1} < \infty.$$

Since $\mu = 1$ or $\mu = 2$, the latter relation holds if and only if $r < 1$. Solving the characteristic equation (18) we prove that $r < 1$ if and only if the coefficients a and b are such that

$$-1 < b < 1 - |a|.$$

Therefore condition 1) of Theorem 3.1 holds if and only if so does condition 1) of Corollary 4.2.

Further,

$$|||Q(\infty, k)|||^2 = \left|\left|\left|\left(\sum_{l=0}^{\infty} C^l\right)\beta_k M\right|\right|\right|^2 = \beta_k^2 \left|\left|\left|\left(\sum_{l=0}^{\infty} C^l\right) M\right|\right|\right|^2, \qquad k \geq 1,$$

whence we conclude that condition (6) holds if and only if condition (17) is satisfied.

Finally, fix $\varepsilon > 0$ and an arbitrary sequence (m_j) belonging to the class $\mathfrak{R}^{\infty}$. According to bounds (19),

$$\begin{aligned}
\exp\left\{-\frac{\varepsilon}{\sum_{k=m_j+1}^{m_{j+1}} |||Q(m_{j+1}, k)|||^2}\right\} &\leq \frac{1}{\varepsilon} \sum_{k=m_j+1}^{m_{j+1}} |||Q(m_{j+1}, k)|||^2 \\
&= \frac{1}{\varepsilon} \sum_{k=m_j+1}^{m_{j+1}} \left|\left|\left|\left(\sum_{l=0}^{m_{j+1}-k} C^l\right)\beta_k M\right|\right|\right|^2 \\
&\leq \frac{1}{\varepsilon} \sum_{k=m_j+1}^{m_{j+1}} \beta_k^2 \left(\sum_{l=0}^{m_{j+1}-k} |||C^l M|||\right)^2 \\
&\leq \frac{c_2^2}{\varepsilon} \sum_{k=m_j+1}^{m_{j+1}} \beta_k^2 \left(\sum_{l=0}^{m_{j+1}-k} r^l \cdot l^{\mu-1}\right)^2 \\
&\leq \frac{c_2^2}{\varepsilon} \left(\sum_{l=0}^{\infty} r^l \cdot l^{\mu-1}\right)^2 \cdot \sum_{k=m_j+1}^{m_{j+1}} \beta_k^2.
\end{aligned}$$

This means that condition (7) follows from (17).

Corollary 4.2 is proved. □

An approach similar to that demonstrated in Corollary 4.2 can be used to find the necessary and sufficient conditions for the almost sure convergence of random series (14) whose terms are elements of a Gaussian m-Markov sequence of random variables with constant coefficients.

5. Concluding remarks

The criterion for the almost sure convergence of a random series whose terms are elements of a multidimensional Gaussian Markov sequence is obtained in the paper. This criterion is used to find the necessary and sufficient conditions for the almost sure convergence of partial sums of a Gaussian 2-Markov sequence of random variables.

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