Теорія Ймовір. та Матем. Статист.
Вип. 84, 2011

Theor. Probability and Math. Statist.
No. 84, 2012, Pages 99–106
S 0094-9000(2012)00863-6
Article electronically published on July 31, 2012

LIMIT BEHAVIOR OF THE PRICES OF A BARRIER OPTION IN THE BLACK–SCHOLES MODEL WITH RANDOM DRIFT AND VOLATILITY

UDC 519.21

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Abstract. A generalized Black–Scholes model with random drift and volatility dependent on a parameter is studied in the paper. Sufficient conditions for the convergence of a sequence of prices of a European barrier option are established.

1. Introduction

Conditions for convergence of prices of a barrier option in the classical Black–Scholes model are studied in the paper [1]. In the current paper, we extend these results to the Black–Scholes model with random drift and volatility. Convergence of a sequence of prices of a European barrier call option is studied. One of the conditions needed for such a convergence is convergence of the finite-dimensional distributions of parameters of the model, namely convergence of the drift and volatility. The proof of the main results is based on theorems on weak convergence of stochastic integrals with respect to semimartingales obtained in the papers [2] and [3].

2. The Black–Scholes model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space with filtration $\{\mathcal{F}_t, t \geq 0\}$. Consider a family of financial markets with continuous time in the interval $[0, T]$. The prices of the riskless and risky assets are described by

$$
\begin{aligned}
B_n(t) &= \exp\left\{\int_0^t r_n(s)\,ds\right\}, \\
S_n(t) &= S_n^0 \exp\left\{\int_0^t \mu_n(s)\,ds + \int_0^t \sigma_n(s)\,dW(s)\right\}, \qquad n \in \mathbf{Z}_+,\ t \in [0, T],
\end{aligned} \tag{1}
$$

respectively, where $W = \{W_t, \mathcal{F}_t, t \geq 0\}$ is a Wiener process, S_n^0 a positive constant, and $\mu = \mu_n(t)$, $\sigma = \sigma_n(t)$, and $r_n = r_n(t)$ are adapted stochastic processes such that σ_n is predictable and positive P-almost surely for all $t \in [0, T]$; $r_n(t) \geq 0$; and all integrals in (1) are well defined, that is,

$$
\int_0^t r_n(s)\,ds < \infty, \qquad \int_0^t |\mu_n(s)|\,ds < \infty, \qquad \int_0^t \sigma_n^2(s)\,ds < \infty \tag{2}
$$

P-almost surely and for all $t > 0$.

2010 *Mathematics Subject Classification.* Primary 60G44, 60F05, 60B12.

Key words and phrases. Stochastic integrals, functional limit theorems, weak convergence, semimartingales, barrier options.

Note that the discounted price process for the risky assets is given by

$$X_n(t) = S_n^0 \exp\left\{\int_0^t \hat{\mu}_n(s)\,ds + \int_0^t \sigma_n(s)\,dW(s)\right\}, \qquad n \in \mathbf{Z}_+, \tag{3}$$
$$\hat{\mu}_n(s) = \mu_n(s) - r_n(s), \qquad n \in \mathbf{Z}_+.$$

The limit behavior of prices of certain contingent claims for this model is of interest in view of applications to real financial markets. First we recall the definition of barrier options. A barrier option is either activated or extinguished only if the underlying reaches a predetermined level (the barrier).

The barrier options are path-dependent and come in various flavors and forms, that is, they are either knocked in or knocked out, and each of this form is defined for either American or European type options. The knock-out option becomes activated if the underlying does not reach a barrier. In contrast, the knock-in option becomes activated if the underlying breaks a barrier.

Consider a European call up-and-out option. The payoff at the moment T for each market belonging to the family mentioned above is equal to

$$(S_n(T) - K_n)^+ I\left\{\max_{0\le t\le T} S_n(t) < H_n\right\},$$

where K_n is the strike price and $H_n > S_n^0$ is the barrier.

Consider measure P_n^* with Radon–Nikodym derivative

$$\left.\frac{dP_n^*}{d\,\mathsf{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t \zeta_n(s)\,dW(s) - \frac{1}{2}\int_0^t \zeta_n^2(s)\,ds\right\},$$

where

$$\zeta_n(s) = -\frac{\mu_n(s) - r_n(s)}{\sigma_n(s)} - \frac{1}{2}\sigma_n(s).$$

If

$$\mathsf{E}\exp\left\{\frac{1}{2}\int_0^t \zeta_n^2(s)\,ds\right\} < \infty, \tag{4}$$

then P_n^* is a neutral measure, that is, P_n^* is a martingale measure for the discounted price process for a risky asset. In this case, the price of a contingent claim is equal to the expectation evaluated with respect to this measure.

Therefore, the price of a European type up-and-out option is given by

$$C_n = \mathsf{E}_{P_n^*}\left(\exp\left\{-\int_0^T r_n(t)\,dt\right\}(S_n(T) - K_n)^+ I\left\{\max_{0\le t\le T} S_n(t) < H_n\right\}\right). \tag{5}$$

In what follows the symbol $\Rightarrow$ stands for weak convergence of finite-dimensional distributions, while weak convergence of probability measures in the interval $[0,T]$ is denoted by $\stackrel{C[0,T]}{\longrightarrow}$.

Fix a countable and everywhere dense set $I \subset \mathbf{R}_+$ in the interval $[0,T]$. Let

$$T_I := I \cap [0,T].$$

By L_{T_I}, we denote the class of all sequences

$$\alpha_k = \{0 = t_{0k} < t_{1k} < \dots < t_{k_T k} < T\}$$

of finite partitions of the interval $[0,T]$ such that

1) $\alpha_k \subseteq \alpha_{k+1} \subseteq T_I$,
2) for all $t \in T_I$, there exists $k(t)$ such that $t \in \alpha_k$ for $k > k(t)$.

Put

$$\triangle_{jk}x := x(t_{j\,k}) - x(t_{j-1\,k}), \qquad \triangle_{jk} := t_{j\,k} - t_{j-1\,k},$$
$$\omega_{j\,k}x = \sup_{t_{j-1\,k}\le s<t\le t_{j\,k}} |x(t) - x(s)|,$$
$$k_t = \sup\{j\colon t_{jk} \le t\}.$$

Theorem 1. *Let the price processes for risky assets be defined by equalities* (1). *Assume also that*

1) $S_n^0 \to S_0^0$, $H_n \to H_0$, *and* $K_n \to K_0$ *as* $n \to \infty$;
2) $(r_n(t), \hat\mu_n(t), \sigma_n(t))$, $t \in T_I$, $\Rightarrow$ $(r_0(t), \hat\mu_0(t), \sigma_0(t))$, $t \in T_I$, *as* $n \to \infty$;
3) $\sup_{n\ge 0} \mathsf{E} \sup_{0\le t\le T} |y_n(t)| < \infty$ *for all* $y \in \{\hat\mu, r, \sigma^2, (\hat\mu/\sigma)^2\}$;
4) $\lim_{\delta\to 0} \limsup_{n\to\infty} \mathsf{E} \sup_{0\le t<s\le t+\delta\le T} |z_n(s) - z_n(t)| = 0$ *for* $z = \hat\mu$ *and for* $z = r$;

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathsf{E} \sup_{0\le t<s\le t+\delta\le T} |z_n(s) - z_n(t)|^2 = 0$$

for $z = \sigma$ *and for* $z = \hat\mu/\sigma$;

5) $\mathsf{P}\big(\max_{0\le t\le T} X_0(t) \exp\{\int_0^T r_0(s)\,ds\} = H_0\big) = 0$;
6) *there exists* $\varepsilon > 0$ *such that*

$$\sup_{n\ge 0} \mathsf{E}\left[\exp\left\{(1+\varepsilon)\int_0^T \left(\frac{1}{2}\sigma_n(s) - \frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2 ds\right\}\right] < \infty.$$

Then $C_n \to C_0$ *as* $n \to \infty$.

Proof. The formula for the price of a barrier option can be rewritten in the following form:

$$\begin{aligned}
C_n &= \mathsf{E}_{P_n^*}\left(\left(X_n(T) - K_n \exp\left\{-\int_0^T r_n(t)\,dt\right\}\right)^+ I\left\{M_n(T) < H_n\right\}\right) \\
&= \mathsf{E}\left(\frac{dP_n^*}{d\mathsf{P}}\left(X_n(T) - K_n \exp\left\{-\int_0^T r_n(t)\,dt\right\}\right)^+ I\left\{M_n(T) < H_n\right\}\right) \\
&= \mathsf{E}\left((\alpha_n(T) - \beta_n(T))^+ I\left\{M_n(T) < H_n\right\}\right),
\end{aligned}$$

where

$$M_n(T) = \max_{0\le t\le T} X_n(t) \exp\left\{\int_0^T r_n(t)\,dt\right\},$$

$$\begin{aligned}
\alpha_n(t) = S_n^0 \exp\Bigg\{&\int_0^t \hat\mu_n(s)\,ds + \int_0^t \sigma_n(s)\,dW(s) + \int_0^t \left(-\frac{\hat\mu_n(s)}{\sigma_n(s)} - \frac{1}{2}\sigma_n(s)\right) dW(s) \\
&- \frac{1}{2}\int_0^t \left(\frac{\hat\mu_n(s)}{\sigma_n(s)} + \frac{1}{2}\sigma_n(s)\right)^2 ds\Bigg\} \\
= S_n^0 \exp\Bigg\{&\int_0^t \left(\frac{1}{2}\sigma_n(s) - \frac{\hat\mu_n(s)}{\sigma_n(s)}\right) dW(s) \\
&+ \int_0^t \left(\frac{1}{2}\hat\mu_n(s) - \frac{1}{2}\left(\frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2 - \frac{1}{8}(\sigma_n(s))^2\right) ds\Bigg\},
\end{aligned}$$

$$\beta_n(t) = K_n \exp\left\{\int_0^t \left(-\frac{\hat{\mu}_n(s)}{\sigma_n(s)} - \frac{1}{2}\sigma_n(s)\right) dW(s) - \frac{1}{2}\int_0^t \left(\frac{\hat{\mu}_n(s)}{\sigma_n(s)} + \frac{1}{2}\sigma_n(s)\right)^2 ds\right\}$$
$$\times \exp\left\{-\int_0^t r_n(s)\,ds\right\}$$
$$= K_n \exp\left\{\int_0^t \left(-\frac{1}{2}\hat{\mu}_n(s) - \frac{1}{2}\left(\frac{\hat{\mu}_n(s)}{\sigma_n(s)}\right)^2 - \frac{1}{8}(\sigma_n(s))^2 - r_n(s)\right) ds \right.$$
$$\left. + \int_0^t \left(-\frac{\hat{\mu}_n(s)}{\sigma_n(s)} - \frac{1}{2}\sigma_n(s)\right) dW(s)\right\}.$$

The functionals

$$f_1(x) = \max_{0\le t\le T} x(t)$$

and

$$f_2(x) = (x(T) - K)^+$$

are continuous in the Skorokhod topology. In view of assumptions 2) and 5), weak convergence of probability measures generated by the processes

$$(\alpha_n(\cdot) - \beta_n(\cdot))^+ I\left\{\max_{0\le s\le \cdot} X_n(s)\exp\left\{\int_0^\cdot r_n(s)\,ds\right\} < H_n\right\}$$

is implied by weak convergence of probability measures generated by the family of the following integrals:

$$\int_0^\cdot r_n(s)\,ds, \qquad \int_0^\cdot \hat{\mu}_n(s)\,ds, \qquad \int_0^\cdot \sigma_n(s)\,dW(s), \qquad \int_0^\cdot \frac{\hat{\mu}_n(s)}{\sigma_n(s)}\,dW(s),$$
$$\int_0^\cdot \left(\frac{\hat{\mu}_n(s)}{\sigma_n(s)}\right)^2 ds, \qquad \int_0^\cdot (\sigma_n(s))^2\,ds.$$

This, in turn, follows from the weak convergence of probability measures corresponding to an arbitrary linear combination of the form

$$a_1\int_0^\cdot \hat{\mu}_n(s)\,ds + a_2\int_0^\cdot \sigma_n(s)\,dW(s) + a_3\int_0^\cdot \frac{\hat{\mu}_n(s)}{\sigma_n(s)}\,dW(s) + a_4\int_0^\cdot \left(\frac{\hat{\mu}_n(s)}{\sigma_n(s)}\right)^2 ds$$
$$+ a_5\int_0^\cdot (\sigma_n(s))^2\,ds + a_6\int_0^\cdot r_n(s)\,ds, \qquad a_i \in \mathbf{R},\ 1\le i\le 6. \tag{6}$$

We use Theorem 5 of [2], which provides sufficient conditions for convergence of stochastic integrals in the Skorokhod topology. First we rewrite (6) as an integral with respect to a two-dimensional semimartingale, namely

$$\int_0^t \xi_n(s)\,dX(s) = \int_0^t \xi_n^1(s)\,dX_1(s) + \int_0^t \xi_n^2(s)\,dX_2(s)$$
$$= \int_0^t \left(a_1\hat{\mu}_n(s) + a_4\left(\frac{\hat{\mu}_n(s)}{\sigma_n(s)}\right)^2 + a_5(\sigma_n(s))^2 + a_6 r_n(s)\right) ds \tag{7}$$
$$+ \int_0^t \left(a_2\sigma_n(s) + a_3\frac{\hat{\mu}_n(s)}{\sigma_n(s)}\right) dW(s).$$

It is clear that one needs to check the conditions

$$\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}\left|\xi_n^1(t)\right|\ge C\right\}=0, \tag{8}$$

$$\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}\left|\xi_n^2(t)\right|\ge C\right\}=0, \tag{9}$$

$$\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\omega_{ik}\xi_n^1\triangle_{ik}=0, \tag{10}$$

$$\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\omega_{ik}\xi_n^2\triangle_{ik}=0 \tag{11}$$

in order to apply Theorem 5 of [2], since assumptions 1)–3) of this theorem follow from (8)–(11).

First we prove relation (8):

$$\begin{aligned}
&\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}\left|\xi_n^1(t)\right|\ge C\right\}\\
&\quad=\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}\left|a_1\hat\mu_n(t)+a_4\left(\frac{\hat\mu_n(t)}{\sigma_n(t)}\right)^2+a_5(\sigma_n(t))^2+a_6r_n(t)\right|\ge C\right\}\\
&\quad\le\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}|\hat\mu_n(t)|\ge C\right\}+\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}\left|\frac{\hat\mu_n(t)}{\sigma_n(t)}\right|\ge C\right\}\\
&\qquad+\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}|\sigma_n(t)|\ge C\right\}+\lim_{C\to\infty}\limsup_{n\to\infty}\mathsf{P}\left\{\sup_{0\le t\le T}|r_n(t)|\ge C\right\}\\
&\quad=0.
\end{aligned} \tag{12}$$

The proof of relation (9) is analogous. Now we prove relation (10):

$$\begin{aligned}
&\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\omega_{ik}\xi_n^1\triangle_{ik}\\
&\quad=\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}|\xi_n^1(t)-\xi_n^1(s)|\triangle_{ik}\\
&\quad\le|a_1|\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\omega_{ik}\hat\mu_n\triangle_{ik}\\
&\qquad+|a_4|\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}\left|\left(\frac{\hat\mu_n(t)}{\sigma_n(t)}\right)^2-\left(\frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2\right|\triangle_{ik}\\
&\qquad+|a_5|\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}\left|(\sigma_n(t))^2-(\sigma_n(s))^2\right|\triangle_{ik}\\
&\qquad+|a_6|\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\omega_{ik}r_n\triangle_{ik}.
\end{aligned}$$

The first and fourth terms on the right hand side of the latter relation are equal to zero by assumption 4). Now we are going to prove that the second and third terms in the same sum are also vanishing.

Indeed, assumptions 3) and 4) imply that

$$\begin{aligned}
&\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}\left|\left(\frac{\hat\mu_n(t)}{\sigma_n(t)}\right)^2-\left(\frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2\right|\triangle_{ik}\\
&\le \lim_{k\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{k_T+1}\sqrt{\mathsf{E}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}\left[\left(\frac{\hat\mu_n(t)}{\sigma_n(t)}-\frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2\right]}\\
&\qquad\times\sqrt{\mathsf{E}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}\left[\left(\frac{\hat\mu_n(t)}{\sigma_n(t)}+\frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2\right]}\cdot\triangle_{ik}\\
&\le C\sqrt{\sup_{n\ge0}\mathsf{E}\sup_{0\le t\le T}\left(\frac{\hat\mu_n(t)}{\sigma_n(t)}\right)^2}\\
&\qquad\times\sqrt{\lim_{\delta\to0}\limsup_{n\to\infty}\mathsf{E}\sup_{0\le s<t\le s+\delta<T}\left(\frac{\hat\mu_n(t)}{\sigma_n(t)}-\frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2}\\
&=0.
\end{aligned}\tag{13}$$

The third term is considered analogously.

Therefore equality (10) is proved. Now we turn to the proof of equality (11). Using condition 4), we get

$$\begin{aligned}
\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\omega_{ik}\xi_n^2\triangle_{ik}&=\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}\left|\xi_n^2(t)-\xi_n^2(s)\right|\triangle_{ik}\\
\le |a_2|\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\omega_{ik}\sigma_n\triangle_{ik}&\\
+|a_3|\lim_{k\to\infty}\limsup_{n\to\infty}\mathsf{E}\sum_{i=1}^{k_T+1}\sup_{t_{i-1\,k}\le s<t\le t_{i\,k}}\left|\frac{\hat\mu_n(t)}{\sigma_n(t)}-\frac{\hat\mu_n(s)}{\sigma_n(s)}\right|\triangle_{ik}&=0.
\end{aligned}$$

Hence (8)–(11) imply that

$$\int_0^{\cdot}\xi_n(s)\,dX(s)\overset{D[0,T]}{\longrightarrow}\int_0^{\cdot}\xi_0(s)\,dX(s)$$

by Theorem 5 in [2].

Taking into account equality (7) and definition of $\xi_n(s)$ and $X(s)$ we prove the weak convergence of the probability measures:

$$\begin{aligned}
&\left\{\int_0^{\cdot}r_n(s)\,ds,\int_0^{\cdot}\hat\mu_n(s)\,ds,\int_0^{\cdot}\sigma_n(s)\,dW(s),\int_0^{\cdot}\frac{\hat\mu_n(s)}{\sigma_n(s)}\,dW(s),\right.\\
&\qquad\qquad\left.\int_0^{\cdot}\left(\frac{\hat\mu_n(s)}{\sigma_n(s)}\right)^2ds,\int_0^{\cdot}(\sigma_n(s))^2ds\right\}\\
&\overset{D[0,T]}{\longrightarrow}\left\{\int_0^{\cdot}r_0(s)\,ds,\int_0^{\cdot}\hat\mu_0(s)\,ds,\int_0^{\cdot}\sigma_0(s)\,dW(s),\int_0^{\cdot}\frac{\hat\mu_0(s)}{\sigma_0(s)}\,dW(s),\right.\\
&\qquad\qquad\left.\int_0^{\cdot}\left(\frac{\hat\mu_0(s)}{\sigma_0(s)}\right)^2ds,\int_0^{\cdot}(\sigma_0(s))^2\,ds\right\}.
\end{aligned}\tag{14}$$

Equality (14) and assumptions 2) and 5) imply that

$$(15)\qquad \begin{aligned}&(\alpha_n(\cdot)-\beta_n(\cdot))^+ I\left\{\max_{0\le t\le\cdot} X_n(t)\exp\left\{\int_0^{\cdot} r_n(t)\,dt\right\}<H_n\right\}\\ &\stackrel{C[0,T]}{\longrightarrow}(\alpha_0(\cdot)-\beta_0(\cdot))^+ I\left\{\max_{0\le t\le\cdot} X_0(t)\exp\left\{\int_0^{\cdot} r_0(t)\,dt\right\}<H_0\right\}\end{aligned}$$

by Theorem 5.2 in [4].

Next we prove that the sequence in (15) is uniformly integrable. Put

$$Y_n(t)=\left(\frac{1}{2}\sigma_n(t)-\frac{\hat\mu_n(t)}{\sigma_n(t)}\right),\qquad 0\le t\le T,\ n\in\mathbf{Z}_+.$$

We have

$$\begin{aligned}&\mathsf{E}\left[(\alpha_n(T)-\beta_n(T))^+ I\left\{\max_{0\le t\le T} X_n(t)\exp\left\{\int_0^T r_n(t)\,dt\right\}<H_n\right\}\right]^{1+\varepsilon}\\ &\le \mathsf{E}\left[S_n^0\exp\left\{\int_0^T Y_n(s)\,dW(s)-\frac{1}{2}\int_0^T Y_n^2(s)\,ds\right\}\right]^{1+\varepsilon}\\ &\le C\,\mathsf{E}\left[\exp\left\{(1+\varepsilon)\int_0^T Y_n(s)\,dW(s)-\frac{1+\varepsilon}{2}\int_0^T Y_n^2(s)\,ds\right\}\right]\\ &= C\,\mathsf{E}\Bigg[\exp\left\{\left((1+\varepsilon)\int_0^T Y_n(s)\,dW(s)-\frac{(1+\varepsilon)^2}{2(1-\varepsilon)}\int_0^T Y_n^2(s)\,ds\right)\right\}\\ &\qquad\qquad\times\exp\left\{\left(\frac{(1+\varepsilon)^2}{2(1-\varepsilon)}-\frac{1+\varepsilon}{2}\right)\int_0^T Y_n^2(s)\,ds\right\}\Bigg]\end{aligned}$$

for some $\varepsilon>0$ and $C>0$. The latter expectation is uniformly bounded. This property can be checked by applying Hölder's inequality with $p=1/(1-\varepsilon)$ and $q=1/\varepsilon$. FInally, Theorem 5.4 of [4] completes the proof of the theorem. □

It is worth mentioning that one can assume only the convergence in probability of r_n, since the integral $\int_0^t r_n(s)\,ds$ is separated from other terms in the definition of the option price.

This gives us a modification of the preceding theorem.

Theorem 2. *Suppose the price processes for risky assets are given by equalities* (1). *Further, let assumptions* 1), 5), *and* 6) *of Theorem* 1 *hold. If*

2′) $(\hat\mu_n(t),\sigma_n(t)),\ t\in T_I\Rightarrow(\hat\mu_0(t),\sigma_0(t)),\ t\in T_I$, *as* $n\to\infty$;
3′) $\sup_{n\ge0}\mathsf{E}\sup_{0\le t\le T}|y_n(t)|<\infty$ *for all* $y\in\{\hat\mu,\sigma^2,(\hat\mu/\sigma)^2\}$;
4′) $\lim_{\delta\to0}\limsup_{n\to\infty}\mathsf{E}\sup_{0\le t<s\le t+\delta\le T}|z_n(s)-z_n(t)|=0$ *for* $z=\hat\mu$;

$$\lim_{\delta\to0}\limsup_{n\to\infty}\mathsf{E}\sup_{0\le t<s\le t+\delta\le T}|z_n(s)-z_n(t)|^2=0$$

for $z=\sigma$ *and for* $z=\hat\mu/\sigma$;
7) $\int_0^T|r_n(s)-r(s)|\,ds\to0$ *in probability,*

then $C_n\to C_0$ *as* $n\to\infty$.

3. Concluding remarks

Convergence of prices of a European barrier up-and-out option is proved for a sequence of generalized Black–Scholes markets with random drift and volatility. The main assumption is weak convergence of the parameters of the financial market. Some analogs of the above results can be obtained for other types of convergence (this, for example, is demonstrated in Theorem 2 for convergence in probability).

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Received 10/JAN/2011

Translated by N. SEMENOV