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STOCHASTIC ANALYSIS WITH THE GAMMA MEASURE—MOVING A DENSE SET

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Abstract. The Gamma measure corresponds to a measure on a marked configuration space with an infinite measure on the marks. We construct Dirichlet forms for the movement of marks and positions. These include the movement of the support, which is a dense set in $\mathbb{R}^d$, $d \in \mathbb{N}$. The key ingredient is a recently discovered integration by parts formula for the directional derivative w.r.t. the positions. We briefly introduce the geometry and then concentrate on the construction of the Dirichlet forms.

1. Introduction

A Gamma measure can be considered as a measure on a "marked configuration space with an infinite measure on the marks". To my knowledge only Dirichlet forms moving the marks and positions were considered for marked configuration spaces with finite measure on the marks.

In this article we concentrate on the Dirichlet forms which we can construct for a Gamma measure. It is supported by the cone of positive, locally finite discrete measures, which are of the form

$$\eta = \sum_{x \in \tau(\eta)} s_x \delta_x.$$

Here, $\tau(\eta) \subset \mathbb{R}^d$, $d \in \mathbb{N}$ fixed, denotes the support of η, which is typically dense in $\mathbb{R}$. Moreover, $s_x \in \mathbb{R}_+ := (0, \infty)$ and δ_x is the Dirac measure at $x \in \mathbb{R}^d$. We refer to the points $x \in \tau(\eta)$ of the support of each finite discrete measure η as positions and to the s_x as marks.

In [4] a proper geometry on the cone of positive finite discrete measures is introduced and the existence of an integration by parts rule is shown. Here, we only briefly introduce the directional derivatives, gradients and tangent spaces involved and state the integration by parts formula. Regarding the Dirichlet forms we are more detailed. For further applications we refer the reader to [4].

Projecting the associated stochastic process to the support, we get a stochastic process moving a dense set: If we consider a compact subset of the support, we encounter typically

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infinitely many points lying dense in this set. For the other mentioned Dirichlet forms, the number of points in each compact set of the support is a finite number.

The difference to e.g. [1] is that we obtain some Dirichlet forms without using a quasi-invariance formula. Although in [4] we also obtain an integration by parts formula for the directional derivative w.r.t. the marks via the quasi-invariance, this is not possible for the one w.r.t. the movement of the positions. Nevertheless, we have deduced an integration by parts formula for this directional derivative.

We first calculate for each gradient its adjoint and then produce the corresponding Dirichlet forms which correspond to stochastic processes on the cone.

2. Gamma measure

Let $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m)$ denote the measure space of $\mathbb{R}^d$, $d \in \mathbb{N}$ fixed, with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and the Lebesgue measure m. Let $\mathbb{R}_+$ be equipped with $\mathcal{B}(\mathbb{R}_+)$ being the trace Borel σ-algebra $\mathcal{B}(\mathbb{R}) \cap \mathbb{R}_+$.

The cone of positive finite real discrete measures is defined as

$$K := \Big\{ \eta = \sum z_i \delta_{x_i} \;\Big|\; z_i \in \mathbb{R}_+, x_i \in \mathbb{R}^d,\ \forall i, j \in \mathbb{N}\ x_i \neq x_j \text{ whenever } i \neq j \text{ and } \forall \Lambda \in B_c\big(\mathbb{R}^d\big) \colon \eta(\Lambda) < \infty \Big\}, \tag{2.1}$$

where $B_c(\mathbb{R}^d)$ denotes the collection of measurable sets lying in a compact set in $\mathbb{R}^d$.

Definition 2.1. A *Levy measure* λ is a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ satisfying

$$\lambda((1,\infty)) < \infty, \qquad \int_0^1 z \, d\lambda(z) < \infty$$

and $\lambda((0,\infty)) = \infty$.

Definition 2.2 (cf. [7, Definition 2.1])**.** A *Levy process* on the space (X, m) with *Levy measure* λ on $\mathbb{R}_+$ is a Poisson process on K, whose law P_λ has Laplace transform

$$\mathbb{E}_{P_\lambda}\left[\exp\left(-\langle a, \cdot\rangle\right)\right] = \exp\left(- \int_{\mathbb{R}_+ \times \mathbb{R}^d} \left(1 - e^{-a(x)z}\right) d\lambda(z)\, dm(x) \right),$$

where $\langle a, \eta\rangle := \int_{\mathbb{R}^d} a(x)\, d\eta(x)$ and $a \colon \mathbb{R}^d \to \mathbb{R}$ is a compactly supported, bounded, non-negative Borel function.

Definition 2.3 (Gamma measure)**.** A *Gamma process* with shape parameter $\theta > 0$ is the Levy process defined by the Levy measure

$$d\lambda_\theta(t) = \theta t^{-1} e^{-t}\, dt \quad \forall t > 0.$$

Its law $\mathcal{G}_\theta$, given by the Laplace transform

$$\mathbb{E}_{\mathcal{G}_\theta}\left[\exp\left(-\langle a, \cdot\rangle\right)\right] = \exp\left(-\theta \int_{\mathbb{R}^d} \log(1 + a(x))\, dm(x) \right), \tag{2.2}$$

is called *Gamma measure* where $a \colon \mathbb{R}^d \to \mathbb{R}$ is such that $\log(1 + a(\cdot)) \in L^1(\mathbb{R}^d, m)$.

In [8] the Gamma measures are discussed in the context of representation theory of groups. A constructive approach is presented in [7, Definition 2.2], where $\mathbb{R}^d$ is replaced by $[0, 1]$.

Quasi-invariance of $\mathcal{G}_\theta$. Fix $\theta > 0$ and set

$$\mathcal{M} := \left\{ f \in C_b(\mathbb{R}^d) \;\middle|\; f \text{ is supported by a compact set} \right\}. \tag{2.3}$$

Definition 2.4 (cf. [7, Chapter 3])**.** For each $h \in \mathcal{M}$ we define the multiplicator

$$\begin{array}{rccl} M_h\colon & K & \to & K \\ & \eta & \mapsto & e^h\eta := \sum_{x\in\tau(\eta)} e^{h(x)} s_x \delta_x. \end{array}$$

That is, $(M_h\eta)(x) = e^{h(x)}\eta(x)$.

This multiplicator changes the weights of the discrete measure $\eta \in K$ at a point $x \in \tau(\eta)$ depending on that point; i.e., in our interpretation the mark of a particle is changed.

Theorem 2.5 (see [4] and cf. [7, Theorem 3.1])**.** *For each $h \in \mathcal{M}$, the Gamma measure $\mathcal{G}_\theta$ is quasi-invariant under M_h, and the corresponding density is given by*

$$\frac{d(M_h\mathcal{G}_\theta)}{d\mathcal{G}_\theta}(\eta) = \exp\left(-\theta \int_{\mathbb{R}^d} h(x)\, m(dx)\right) \exp\left(-\int_{\mathbb{R}^d} \left(e^{-h(x)} - 1\right) d\eta(x)\right).$$

3. Differential geometry

After introducing a gradient w.r.t. the movement of the marks and one w.r.t. changing the positions, we merge them to obtain one acting on both components. The results of this section are explained in more detail in [4, 3].

3.1. **Gradient w.r.t. the motion of marks.** If no confusion is possible without any further remarks, we denote by F that cylindrical function $F \in \mathcal{F}C_b^1(K, \mathcal{M})$ given by

$$F(\eta) = g_F(\langle \rho_1, \eta\rangle, \ldots, \langle \rho_N, \eta\rangle), \tag{3.1}$$

where $g_F \in C_b^1(\mathbb{R}^N)$, $\eta \in K$, and for $i = 1, \ldots, N$, $N \in \mathbb{N}$ and $\rho_i \in \mathcal{M}$.

We note that the transformation M_{th}, $t \in \mathbb{R}$ and $h \in \mathcal{M}$, changes only the marks of the discrete measure η. Thus the related directional derivative is a property of the marks.

Definition 3.1. The *directional derivative* of a function $F : K \to \mathbb{R}$ in direction $h \in \mathcal{M}$ is defined as

$$\nabla^K_{\mathbb{R}_+,h} F(\eta) := \frac{d}{dt} F\left(M_{th}(\eta)\right)\Big|_{t=0}$$

whenever the expression on the right-hand side exists.

We set the tangent space to K at $\eta \in K$ as

$$T_{\mathbb{R}_+,\eta} := L^2\left(\mathbb{R}^d, \eta\right).$$

Definition 3.2. We define the gradient $\nabla^K_{\mathbb{R}_+}$ of a function $F\colon K \to \mathbb{R}$ by

$$\begin{array}{rl} \nabla^K_{\mathbb{R}_+} F\colon & K \to T.K \\ & \eta \to \left(\nabla^K_{\mathbb{R}_+} F\right)(\eta) \quad \in T_\eta K, \end{array}$$

whenever the directional derivative of F in each direction $h \in \mathcal{M}$ exists and

$$\nabla^K_{\mathbb{R}_+,h} F(\eta) = \left\langle \nabla^K_{\mathbb{R}_+} F(\eta), h \right\rangle_{T^K_{\mathbb{R}_+,\eta}} =: \left\langle \nabla^K_{\mathbb{R}_+} F, h \right\rangle(\eta) \quad \text{for all } h \in \mathcal{M}. \tag{3.2}$$

Lemma 3.3. *For each $F \in \mathcal{F}C_b^1(K, \mathcal{M})$ the gradient is well-defined (and independent of the representation of F):*

$$\left(\nabla^K_{\mathbb{R}_+} F\right)(\eta) := \sum_{i=1}^N \partial_i g_F(\langle \rho_1, \eta\rangle, \dots, \langle \rho_N, \eta\rangle)\rho_i,$$

where $\partial_i g_F$ denotes the partial derivative w.r.t. the i-th component of g_F.

3.2. **Gradient w.r.t. the change of the positions.** Let $V_0(\mathbb{R}^d)$ denote the set of all C^∞-vector fields on $\mathbb{R}^d$ with compact support. For any $x \in \mathbb{R}^d$, $v \in V_0(\mathbb{R}^d)$ the curve $\mathbb{R} \ni t \mapsto \phi_t^v(x) \in \mathbb{R}^d$ is defined as the solution to the following Cauchy problem:

$$\begin{cases} \frac{d}{dt}\phi_t^v(x) = v(\phi_t^v(x)), \\ \phi_0^v(x) = x. \end{cases}$$

We fix $v \in V_0(\mathbb{R}^d)$. Having the group ϕ_t^v, $t \in \mathbb{R}$, we can consider for any $\eta \in K$ the curve $\mathbb{R} \ni t \mapsto \phi_t^v(\eta) \in K$.

Definition 3.4. For a function $F\colon K \to \mathbb{R}$ we define the *directional derivative* along the vector field $v \in V_0(\mathbb{R}^d)$ as

$$\left(\nabla^K_{\mathbb{R}^d, v} F\right)(\eta) := \frac{d}{dt} F(\phi_t^{v\star}\eta)\Big|_{t=0},$$

provided the right-hand side exists. Here, $\star$ means that we take the image measure.

Definition 3.5. We define the tangent space $T^K_{\mathbb{R}^d,\eta}$ to the cone K at the positive discrete measure $\eta \in K$ to be the Hilbert space of measurable η-square integrable sections (measurable vector fields) $V_\eta\colon \mathbb{R}^d \to \mathbb{R}^d$ with the scalar product

$$\langle V_\eta^1, V_\eta^2\rangle_{T^K_{\mathbb{R}^d,\eta}} := \int_{\mathbb{R}^d} \langle V_\eta^1(x), V_\eta^2(x)\rangle_{\mathbb{R}^d}\, \eta(dx),$$

where $V_\eta^1, V_\eta^2 \in T^K_{\mathbb{R}^d,\eta}$.

Definition 3.6. Let the function $F\colon K \to \mathbb{R}$ be such that for all $v \in V_0(\mathbb{R}^d)$ the directional derivative $\nabla^K_{\mathbb{R}^d,v} F$ exists. The intrinsic gradient of such a function $F\colon K \to \mathbb{R}$ is defined as the mapping $K \ni \eta \mapsto (\nabla^K_{\mathbb{R}^d} F)(\eta) \in T_\eta K$ such that for any $v \in V_0(\mathbb{R}^d)$

$$\left(\nabla^K_{\mathbb{R}^d,v} F\right)(\eta) = \left\langle \left(\nabla^K_{\mathbb{R}^d} F\right)(\eta), v\right\rangle_{T^K_{\mathbb{R}^d,\eta}}. \tag{3.3}$$

The intrinsic gradient $\nabla^K_{\mathbb{R}^d}$ is defined for all those functions for which the above holds.

3.3. **Gradient on the cone.** After having defined the gradient w.r.t. the motion of the marks and the one w.r.t. the change of positions, we glue the pieces together.

Definition 3.7. Let $h \in \mathcal{M}$ and $v \in V_0(\mathbb{R}^d)$. Then the directional derivative of a function $F\colon K \to \mathbb{R}$ at the point $\eta \in K$ is defined to be

$$\left(\nabla^K_{h,v} F\right)(\eta) := \left(\nabla^K_{\mathbb{R}_+,h} F\right)(\eta) + \left(\nabla^K_{\mathbb{R}^d,v} F\right)(\eta)$$

and the gradient as

$$\nabla^K := \left(\nabla^K_{\mathbb{R}_+}, \nabla^K_{\mathbb{R}^d}\right)$$

whenever the objects exist.

Furthermore, we set the tangent space of K at $\eta \in K$ to be

$$T_\eta K := T^K_{\mathbb{R}_+,\eta} \oplus T^K_{\mathbb{R}^d,\eta}. \tag{3.4}$$

Lemma 3.8. *For each $F \in \mathcal{F}C_b^1(K, C_0^1(\mathbb{R}^d))$ the gradient ∇^K exists.*

4. Dirichlet forms

Our aim is to define some Dirichlet forms on the cone K of positive discrete measures over $\mathbb{R}^d$. In the framework of [5] the case of a finite measure on the marks is treated. Translating our task to that framework requires an infinite measure on the marks. Although we lack a quasi-invariance formula on the space of points, we obtain a Dirichlet form on the cone K. The Dirichlet form approach (cf. e.g. [2, 6]) is used to obtain the Dirichlet forms; i.e., we define a bilinear form whose closure is a Dirichlet form.

4.1. **Integration by parts formula.**

Theorem 4.1 (see [4]). *For each $h \in \mathcal{M}$, $v \in V_0(\mathbb{R}^d)$ and $\eta \in K$ we define the following logarithmic derivative:*

$$\begin{aligned}\left\langle \beta^{\mathcal{G}_\theta}(\eta), (h,v)\right\rangle_{T_\eta K} &:= \left\langle \beta^{\mathcal{G}_\theta}_{ext}(\eta), h\right\rangle_{T^K_{\mathbb{R}_+,\eta}} + \left\langle \beta^{\mathcal{G}_\theta}_{int}(\eta), v\right\rangle_{T^K_{\mathbb{R}^d,\eta}} \\ &= -\theta\langle h, m\rangle + \langle h, \eta\rangle + \int_{\mathbb{R}^d} \operatorname{div}^{\mathbb{R}^d} v(x)\, \eta(dx).\end{aligned}$$

We obtain for all F, $G \in \mathcal{F}C_b^\infty(K, C_0^\infty(\mathbb{R}^d))$, all $h \in \mathcal{M}$ and all $v \in V_0(\mathbb{R}^d)$ an integration by parts formula, i.e.,

$$\begin{aligned}\int_K \nabla^K_{h,v} F(\eta) G(\eta)\, \mathcal{G}_\theta(d\eta) = &- \int_K F(\eta) \nabla^K_{h,v} G(\eta)\, \mathcal{G}_\theta(d\eta) \\ &- \int_K F(\eta) G(\eta) \left\langle \beta^{\mathcal{G}_\theta}(\eta), (h,v)\right\rangle_{T_\eta K} \mathcal{G}_\theta(d\eta).\end{aligned}$$

Definition 4.2. A function $V\colon K \to \mathbb{R}$ is called a differentiable cylindrical vector field iff it is of the form

$$V(\eta) := \left(\sum_{i=1}^N g_i(\eta)\phi_i, \sum_{i=1}^N h_i(\eta) v_i\right)$$

where for $i = 1, \dots, N$,

$$g_i, h_i \in \mathcal{F}C_b^\infty\left(K, C_0^\infty(\mathbb{R}^d)\right), \qquad \phi_i \in \mathcal{M}$$

and $v_j \in V_0(\mathbb{R}^d)$. By V^K_{cyl} we denote the set of such functions. Moreover,

$$V_{\mathbb{R}_+} := \sum_{i=1}^N g_i(\eta)\phi_i \quad \text{and} \quad V_{\mathbb{R}^d} := \sum_{i=1}^N h_i(\eta) v_i.$$

The gradient of a function $F \in \mathcal{F}C_b^\infty(K, C_0^\infty(\mathbb{R}^d))$ is exactly of that form, i.e.,

$$\nabla^K F \in V^K_{\text{cyl}}.$$

Before we derive a formula for the adjoint of such vector fields, we have to check that the following integrals are finite. This ensures that the adjoint of the gradient can be well-defined.

Lemma 4.3. *Let $V_1, V_2 \in V^K_{cyl}$. Then*

$$\int_K \langle V_1(\eta), V_2(\eta)\rangle_{T_\eta K}\, \mathcal{G}_\theta(d\eta) < \infty. \tag{4.1}$$

Proof. The differentiable cylindrical vector fields are bounded and finitely supported. Thus the integral is finite because the moments of the Gamma measure $\mathcal{G}_\theta$ exist and are finite: We know that there exist $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $C, \tilde{C} > 0$ such that

$$\begin{aligned}\int_K \langle V_1(\eta), V_2(\eta)\rangle_{T_\eta K}\, \mathcal{G}_\theta(d\eta) &= \int_{K(\Lambda)} \langle V_1(\eta), V_2(\eta)\rangle_{T_\eta K}\, \mathcal{G}_{\theta,\Lambda}(d\eta)\\ &\le C \sum_{i=1}^N \sum_{j=1}^{\tilde{N}} \int_{K(\Lambda)} \int_\Lambda \phi_i(x)\phi_j(x) + \langle v_i, v_j\rangle_{\mathbb{R}^d}\, \eta(dx)\, \mathcal{G}_{\theta,\Lambda}(d\eta)\\ &\le \tilde{C} \int_{K(\Lambda)} \langle \mathbb{1}_\Lambda, \eta\rangle\, \mathcal{G}_{\theta,\Lambda}(d\eta) < \infty. \qquad \square\end{aligned}$$

Theorem 4.4. *Fix*

$$V(\eta) := \left(\sum_{i=1}^N g_i(\eta)\phi_i, \sum_{i=1}^N h_i(\eta) v_i\right) \in V^K_{cyl}.$$

Then we have for all $F \in \mathcal{F}C_b^\infty(K, C_0^\infty(\mathbb{R}^d))$ *that*

$$\begin{aligned}\int_K \left\langle \nabla^K F(\eta), V(\eta)\right\rangle_{T_\eta K}\, \mathcal{G}_\theta(d\eta) &= \int_K F(\eta) \left(\left(\nabla^K\right)^{\star,\mathcal{G}_\theta} V\right)(\eta)\, \mathcal{G}_\theta(d\eta)\\ &= -\int_K F(\eta) \sum_{i=1}^N \left\langle \nabla^K_{\mathbb{R}_+} g_i, \phi_i\right\rangle_{T^K_{\mathbb{R}_+,\eta}} + \left\langle \nabla^K_{\mathbb{R}^d} h_i, v_i\right\rangle_{T^K_{\mathbb{R}^d,\eta}} \mathcal{G}_\theta(d\eta)\\ &\quad + \int_K F(\eta) \left(\left\langle \beta^{\mathcal{G}_\theta}_{ext}(\eta), V_{\mathbb{R}_+}(\eta)\right\rangle_{T^K_{\mathbb{R}_+,\eta}} + \left\langle \beta^{\mathcal{G}_\theta}_{int}(\eta), V_{\mathbb{R}^d}(\eta)\right\rangle_{T^K_{\mathbb{R}^d,\eta}}\right) \mathcal{G}_\theta(d\eta).\end{aligned} \tag{4.2}$$

Proof. This follows by the definition of the tangent space (cf. (3.4)) and using Theorem 4.1 twice (once for $h = 0$ and once for v being the identity). The finiteness of the involved integrals follows by Lemma 4.3.

In detail, we see that

$$\begin{aligned}&\int_K \left\langle \left(\nabla^K F\right)(\eta), V(\eta)\right\rangle_{T_\eta K}\, \mathcal{G}_\theta(d\eta)\\ &\quad= \int_K \left\langle \left(\nabla^K_{\mathbb{R}_+} F\right)(\eta), V_{\mathbb{R}_+}(\eta)\right\rangle_{T^K_{\mathbb{R}_+,\eta}} \mathcal{G}_\theta(d\eta) + \int_K \left\langle \left(\nabla^K_{\mathbb{R}^d} F\right)(\eta), V_{\mathbb{R}^d}(\eta)\right\rangle_{T^K_{\mathbb{R}^d,\eta}} \mathcal{G}_\theta(d\eta)\\ &\quad= \int_K F(\eta) \left(-\sum_{i=1}^N \left\langle \nabla^K_{\mathbb{R}_+} g_i, \phi_i\right\rangle_{T^K_{\mathbb{R}_+,\eta}} + \left\langle \beta^{\mathcal{G}_\theta}_{\text{ext}}(\eta), V_{\mathbb{R}_+}(\eta)\right\rangle_{T^K_{\mathbb{R}_+,\eta}}\right) \mathcal{G}_\theta(d\eta)\\ &\qquad + \int_K F(\eta) \left(-\sum_{i=1}^N \left\langle \nabla^K_{\mathbb{R}^d} h_i, v_i\right\rangle_{T^K_{\mathbb{R}^d,\eta}} + \left\langle \beta^{\mathcal{G}_\theta}_{\text{int}}(\eta), V_{\mathbb{R}^d}(\eta)\right\rangle_{T^K_{\mathbb{R}^d,\eta}}\right) \mathcal{G}_\theta(d\eta)\\ &\quad= \int_K F(\eta) \left(\left(\nabla^K\right)^{\star,\mathcal{G}_\theta} V\right)(\eta) \mathcal{G}_\theta \phi(d\eta),\end{aligned}$$

where we used the definition of the adjoint in the last line. $\square$

4.2. **Extrinsic Dirichlet form.** We define for $F, G \in \mathcal{F}C_b^1(K, \mathcal{M})$,

$$\mathcal{E}^{\mathcal{G}_\theta}_{\text{ext}}(F, G) := \int_K \left\langle \nabla^K_{\mathbb{R}_+} F(\eta), \nabla^K_{\mathbb{R}_+} G(\eta)\right\rangle_{T^K_{\mathbb{R}_+,\eta}} \mathcal{G}_\theta(d\eta).$$

It is well-defined by Lemma 4.3.

Proposition 4.5. *We have for $F, G \in \mathcal{F}C_b^2(K, \mathcal{M})$ that*

$$\mathcal{E}_{ext}^{\mathcal{G}_\theta}(F, G) = \int_K \left(L_{ext}^{\mathcal{G}_\theta} F\right)(\eta) G(\eta)\, \mathcal{G}_\theta(d\eta).$$

Here,

$$\left(L_{ext}^{\mathcal{G}_\theta} F\right)(\eta) := -\sum_{l,k=1}^{N} g_{l,k} \langle \rho_k, \rho_l \rangle_{T^K_{\mathbb{R}_+,\eta}} + \beta_{ext}^{\mathcal{G}_\theta}\left(\nabla^K_{\mathbb{R}_+} F(\eta), \eta\right)$$

where for $1 \le l, k \le N$,

$$g_{l,k}(\eta) := \partial_l \partial_k g(\langle \rho_1, \eta \rangle, \dots, \langle \rho_N, \eta \rangle) \quad \forall \eta \in K.$$

Proof. This follows by Theorem 4.4. Namely, for an arbitrary cylindrical function

$$F(\eta) = g_F(\langle \rho_1, \eta \rangle, \dots, \langle \rho_N, \eta \rangle) \in \mathcal{F}C_b^2(K, \mathcal{M})$$

we choose for $i = 1, \dots, N$, $\phi_i = \rho_i$ and

$$g_i(\eta) := \partial_i g_F(\langle \rho_1, \eta \rangle, \dots, \langle \rho_N, \eta \rangle). \tag{4.3}$$

Moreover, the generator is independent of the representation of F (cf. [3, Section 6.2]). □

Proposition 4.6. *$(\mathcal{E}_{ext}^{\mathcal{G}_\theta}, \mathcal{F}C_b^2(K, \mathcal{M}))$ is a closable, symmetric positive definite bilinear form.*

Proof. Since

$$\mathcal{F}C_b^\infty(K, \mathcal{M})$$

is a dense linear subspace of $L^2(K, \mathcal{G}_\theta)$ (cf. [3, Subsection 6.2.1]), the bilinear form is densely defined. Obviously $\mathcal{E}_{\mathcal{G}_\theta}$ is a symmetric bilinear form. We prove the positive definiteness: For $F \in \mathcal{F}C_b^2(K, \mathcal{M})$ it holds that (using (4.3))

$$\begin{aligned} \mathcal{E}_{\text{ext}}^{\mathcal{G}_\theta}(F, F) &= \int_K \int_{\mathbb{R}^d} \sum_{k=1}^{N} \sum_{l=1}^{N} g_k(\eta) g_l(\eta) \rho_k(x) \rho_l(x)\, \eta(dx)\, \mathcal{G}_\theta(d\eta) \\ &= \int_K \int_{\mathbb{R}^d} \left(\sum_{k=1}^{N} \partial_k F(\eta) \rho_k(x)\right)^2 \eta(dx)\, \mathcal{G}_\theta(d\eta), \end{aligned} \tag{4.4}$$

which is positive because $\eta \in K$. By [6, Proposition I.3.3] the rest of the claim follows. □

By $(\mathcal{E}_{\text{ext}}^{\mathcal{G}_\theta}, \mathcal{D}(\mathcal{E}_{\text{ext}}^{\mathcal{G}_\theta}))$ we denote the closure w.r.t.

$$\mathcal{E}_{\text{ext},1}^K := \langle \cdot, \cdot \rangle_{L^2(K, \mathcal{G}_\theta)} + \mathcal{E}_{\text{ext}}^{\mathcal{G}_\theta}.$$

Theorem 4.7. *$\left(\mathcal{E}_{ext}^{\mathcal{G}_\theta}, \mathcal{D}(\mathcal{E}_{ext}^{\mathcal{G}_\theta})\right)$ is a conservative Dirichlet form.*

Proof. Let $\rho_\varepsilon \in C_b^\infty(\mathbb{R})$ such that

(1) $\rho_\varepsilon \colon \mathbb{R} \to [-\varepsilon, 1 + \varepsilon]$ and $\rho_\varepsilon' \le 1$,
(2) $\rho_\varepsilon(t) = t \quad \forall t \in [0, 1]$,
(3) $\forall t_1 \ge t_2 \colon \rho_\varepsilon(t_1) \le \rho_\varepsilon(t_2)$.

Then $|\rho_\varepsilon(t)| \le |\rho_\varepsilon'(t)| \cdot |t| \le |t|$. For any $F \in \mathcal{F}C_b^2(K, \mathcal{M})$ we have that $\rho_\varepsilon \circ F \in D\big(\mathcal{E}_{\text{ext}}^{\mathcal{G}_\theta}\big)$ and using (4.4) we see that

$$\limsup_{\varepsilon \to 0} \mathcal{E}_{\text{ext}}^{\mathcal{G}_\theta}(\rho_\varepsilon \circ F, \rho_\varepsilon \circ F) \le \mathcal{E}_{\text{ext}}^{\mathcal{G}_\theta}(F, F).$$

Hence, by [6, Proposition I.4.10], the closure is a Dirichlet form. That it is conservative is obvious. □

4.3. **Intrinsic Dirichlet form.** We define for

$$F, G \in \mathcal{F}C_b^2(K, C_0^2(\mathbb{R}^d))$$

the gradient bilinear form

$$\mathcal{E}_{\text{int}}^{P_\lambda}(F,G) := \int_K \left\langle \nabla_{\mathbb{R}^d}^K F(\eta), \nabla_{\mathbb{R}^d}^K G(\eta) \right\rangle_{T_{\mathbb{R}^d,\eta}^K} \mathcal{G}_\theta(d\eta). \tag{4.5}$$

By Lemma 4.3 it is well-defined.

Proposition 4.8. *For $F \in \mathcal{F}C_b^2(K, C_0^2(\mathbb{R}^d))$ we set, using its standard representation,*

$$\begin{aligned}\left(L_{int}^{\mathcal{G}_\theta} F\right)(\eta) := &- \sum_{i,j=1}^N g_{i,j}(\eta) \int_{\mathbb{R}^d} \left\langle \nabla^{\mathbb{R}^d} \rho_i(x), \nabla^{\mathbb{R}^d} \rho_j(x) \right\rangle_{\mathbb{R}^d} d\eta(x) \\ &+ \sum_{i=1}^N g_i(\eta) \int_{\mathbb{R}^d} \Delta^{\mathbb{R}^d} \rho_i(x)\, d\eta(x)\end{aligned}$$

where $\nabla^{\mathbb{R}^d}$ is the gradient in $\mathbb{R}^d$ and $\Delta^{\mathbb{R}^d}$ denotes the Laplace operator on $\mathbb{R}^d$. Then we have for all $F, G \in \mathcal{F}C_b^2(K, C_0^2(\mathbb{R}^d))$,

$$\mathcal{E}_{int}^{P_\lambda}(F,G) = \int_K \left(L_{int}^{\mathcal{G}_\theta} F\right)(\eta) G(\eta)\, \mathcal{G}_\theta(d\eta), \tag{4.6}$$

and $\mathcal{E}_{int}^{P_\lambda}$ is a well-defined, positive definite, symmetric bilinear form. Moreover, it is closable.

Proof. For the proof we use Theorem 4.1 and that $\operatorname{div}^{\mathbb{R}^d} \nabla^{\mathbb{R}^d} = \Delta^{\mathbb{R}^d}$:

$$\begin{aligned}\mathcal{E}_{\text{int}}^{P_\lambda}(F,G) &= \int_K \left\langle \nabla_{\mathbb{R}^d}^K F(\eta), \nabla_{\mathbb{R}^d}^K G(\eta) \right\rangle_{T_{\mathbb{R}^d,\eta}^K} \mathcal{G}_\theta(d\eta) \\ &= \int_K \left(L_{\text{int}}^{\mathcal{G}_\theta} F\right)(\eta) G(\eta)\, \mathcal{G}_\theta(d\eta).\end{aligned}$$

This implies the symmetry, bilinearity and the generator of $\mathcal{E}_{\text{int}}^{P_\lambda}$. We see its positive definiteness by

$$\mathcal{E}_{\text{int}}^{P_\lambda}(F,F) = \int_K \underbrace{\left\langle \nabla_{\mathbb{R}^d}^K F(\eta), \nabla_{\mathbb{R}^d}^K F(\eta) \right\rangle_{T_{\mathbb{R}^d,\eta}^K}}_{\geq 0} \mathcal{G}_\theta(d\eta) \geq 0.$$

The closability now follows by [6, Proposition I.3.3]. □

By $\left(\mathcal{E}_{\text{int}}^{P_\lambda}, \mathcal{D}(\mathcal{E}_{\text{int}}^{P_\lambda})\right)$ we denote the closure w.r.t.

$$\mathcal{E}_{\text{int},1}^K := \langle \cdot, \cdot \rangle_{L^2(K,\mathcal{G}_\theta)} + \mathcal{E}_{\text{int}}^{\mathcal{G}_\theta}.$$

Theorem 4.9. *The closure $\left(\mathcal{E}_{int}^{\mathcal{G}_\theta}, \mathcal{D}(\mathcal{E}_{int}^{\mathcal{G}_\theta})\right)$ is a Dirichlet form.*

Proof. Using ρ_ε as in the proof of Theorem 4.7 we see that

$$\begin{aligned}
\mathcal{E}^{\mathcal{G}_\theta}_{\text{int}}(\rho_\varepsilon \circ F, \rho_\varepsilon \circ F) &= \int_K \left\langle \nabla^K_{\mathbb{R}^d}(\rho_\varepsilon \circ F)(\eta), \nabla^K_{\mathbb{R}^d}(\rho_\varepsilon \circ F)(\eta) \right\rangle_{T^K_{\mathbb{R}^d,\eta}} \mathcal{G}_\theta(d\eta) \\
&= \int_K \sum_{i,j=1}^N \partial_i(\rho_\varepsilon \circ g_F)(\langle \eta, \rho_1\rangle, \dots, \langle \eta, \rho_N\rangle)\partial_j(\rho_\varepsilon \circ g_F)(\langle \eta, \rho_1\rangle, \dots, \langle \eta, \rho_N\rangle) \\
&\quad \times \int_{\mathbb{R}^d} \left\langle \nabla^{\mathbb{R}^d}\rho_i(x), \nabla^{\mathbb{R}^d}\rho_j(x) \right\rangle_{\mathbb{R}^d} \eta(dx)\, \mathcal{G}_\theta(d\eta) \\
&= \int_K (\rho'_\varepsilon(F(\eta)))^2 \sum_{i,j=1}^N \partial_i g_F(\langle \eta, \rho_1\rangle, \dots, \langle \eta, \rho_N\rangle)\partial_j g_F(\langle \eta, \rho_1\rangle, \dots, \langle \eta, \rho_N\rangle) \\
&\quad \times \int_{\mathbb{R}^d} \left\langle \nabla^{\mathbb{R}^d}\rho_i(x), \nabla^{\mathbb{R}^d}\rho_j(x) \right\rangle_{\mathbb{R}^d} \eta(dx)\, \mathcal{G}_\theta(d\eta) \\
&\le \mathcal{E}^{\mathcal{G}_\theta}_{\text{int}}(F, F).
\end{aligned}$$

Thus by [6, Propositions I.4.7 and I.4.10] the bilinear form is a Dirichlet form. □

4.4. **Dirichlet form.** We define for $F, G \in \mathcal{F}C^2_b(K, C^2_0(\mathbb{R}^d))$,

$$\mathcal{E}^{\mathcal{G}_\theta}(F, G) := \int_K \left\langle \nabla^K F, \nabla^K G \right\rangle_{T_\eta K} \mathcal{G}_\theta(d\eta).$$

It is well-defined by Lemma 4.3.

Corollary 4.10. *We have for $F, G \in \mathcal{F}C^2_b(K, C^2_0(\mathbb{R}^d))$ that*

$$\mathcal{E}^{\mathcal{G}_\theta}(F, G) = \int_K F(\eta) \left(L^{\mathcal{G}_\theta} G\right)(\eta)\, \mathcal{G}_\theta(d\eta),$$

where

$$\left(L^{\mathcal{G}_\theta} G\right)(\eta) := \left(L^{\mathcal{G}_\theta}_{ext} G\right)(\eta) + \left(L^{\mathcal{G}_\theta}_{int} G\right)(\eta).$$

Proof. This follows by Theorem 4.4 and Propositions 4.5 and 4.8. □

Proposition 4.11. $(\mathcal{E}^{\mathcal{G}_\theta}, \mathcal{F}C^2_b(K, \mathcal{M}))$ *is a closable, symmetric positive definite bilinear form.*

Proof. This follows by (3.4), Proposition 4.6 and Proposition 4.8. □

We denote its closure w.r.t.

$$\mathcal{E}^{\mathcal{G}_\theta}_1 := \langle \cdot, \cdot \rangle_{L^2(K, \mathcal{G}_\theta)} + \mathcal{E}^{\mathcal{G}_\theta}$$

by $\left(\mathcal{E}^{\mathcal{G}_\theta}, \mathcal{D}(\mathcal{E}^{\mathcal{G}_\theta})\right)$. In general this closure does not have to coincide with the ones corresponding to the Dirichlet forms $\mathcal{E}^{\mathcal{G}_\theta}_{\text{ext}}$ and $\mathcal{E}^{\mathcal{G}_\theta}_{\text{int}}$.

Theorem 4.12. $\left(\mathcal{E}^{\mathcal{G}_\theta}, \mathcal{D}(\mathcal{E}^{\mathcal{G}_\theta})\right)$ *is a conservative Dirichlet form.*

Proof. This follows by the arguments used to prove Theorems 4.7 and 4.9. □

Remark 4.13. In addition one can show the quasi-regularity of the Dirichlet forms and obtain an associated Markov process with "nice" path properties.

All the presented results hold for the more general setting of $\mathbb{R}^d$ being replaced by an arbitrary connected, separable orientated C^∞-Riemannian manifold X with the volume element v and the intensity measure being $m(dx) = \rho(x)\, v(dx)$, where $\rho^{1/2} \in H^{1,2}_{\text{loc}}(X, v)$.

These results are presented in [3], whose publication is in preparation.

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