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LOCAL PROPERTIES OF A MULTIFRACTIONAL STABLE FIELD

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Abstract. An anisotropic harmonizable multifractional stable field is defined. Its continuity is proved. Existence and square integrability of local time are established. It is proved that the local time is jointly continuous in the Gaussian case.

1. Introduction

Fractional processes are among the most popular tools to model long-range dependence phenomenon in natural sciences, financial mathematics, etc. For this reason, fractional Brownian motion and fractional Brownian fields are studied in many research papers. A comprehensive review of the literature on this topic can be found in [11].

Note however that fractional Brownian motion and field as models for natural phenomena have several drawbacks. First, they have stationary increments, so they do not allow one to model processes and fields that are essentially non-stationary. Second, they are self-similar, while the majority of processes on financial markets are not self-similar according to empirical observations; that is, they are "smoother" on a long time interval than on a short one.

In view of the drawbacks mentioned above, multifractional processes and fields has gained popularity recently. In particular, several multifractional definitions of a fractional Brownian motion based on different representations of a fractional Brownian motion are proposed, namely the linear multifractional Brownian motion [12], Volterra type multifractional Brownian motion [13], harmonizable multifractional Brownian motion [2]. Various multifractional random fields are studied in [1, 6, 8].

The current paper is devoted to the anisotropic harmonizable multifractional stable (N, d)-fields. The paper continues the research initiated in the paper [4] concerning the real harmonizable multifractional process and those of the papers [15, 16] concerning the fractional Brownian and stable fields.

The main focus of the paper is on the sample local properties of the field, namely on the continuity of the field and on existence and regularity of the local time.

The paper is organized as follows. Below in this section we provide basic information about stable random variables and introduce the corresponding notation. In Section 2, an anisotropic harmonizable multifractional stable field is defined and its sample continuity is studied. In Section 3, we prove the existence of the local time of the random field under consideration. The property of the sector local non-determinism is established in the Gaussian case; the joint continuity of the local time is proved as a corollary.

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1.1. **Stable random variables and families.** Below we consider only symmetric α-stable (SαS) random variables with $\alpha \in (1,2]$. More detail on stable variables can be found in [14]. Recall that a random variable ξ is SαS with the scale parameter σ^α, $\sigma > 0$, if its characteristic function is given by

$$\mathsf{E}\left[e^{i\lambda\xi}\right] = e^{-|\sigma\lambda|^\alpha}.$$

For a stable random variable ξ, define a quasinorm $\|\xi\|_\alpha^\alpha = -\ln \mathsf{E}\left[e^{i\xi}\right]$.

To construct stable random variables, one frequently uses the *complex rotationally invariant* SαS measure μ on $\mathbb{R}^N$. By definition, this is a σ-additive measure on $\mathbb{R}^N$ with the following properties:

(1) for any Borel set $A \in \mathcal{B}(\mathbb{R}^N)$, the random variable $\operatorname{Re}\mu(A)$ is SαS with the scale parameter $\lambda_N(A)$, where $\lambda_N(A)$ stands for the Lebesgue measure of A;
(2) for any $A \in \mathcal{B}(\mathbb{R}^N)$, the random variable $\mu(A)$ is *rotationally invariant*; that is, the distribution of $e^{i\theta}\mu(A)$ is the same as that of $\mu(A)$ for each $\theta \in \mathbb{R}$;
(3) for disjoint sets $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^N)$, the values $\mu(A_1), \dots, \mu(A_n)$ are independent.

The stochastic integral

$$I(f) = \int_{\mathbb{R}^N} f(x)\,\mu(dx)$$

is well defined for a function $f\colon \mathbb{R}^N \to \mathbb{C}$ with

$$\|f\|_{L^\alpha(\mathbb{R}^N)}^\alpha = \int_{\mathbb{R}^N} |f(x)|^\alpha\,dx < \infty.$$

Note that $\operatorname{Re} I(f)$ is a random SαS variable with the scale parameter $\|f\|_{L^\alpha(\mathbb{R}^N)}^\alpha$. In other words, $\|\operatorname{Re} I(f)\|_\alpha^\alpha = \|f\|_{L^\alpha(\mathbb{R}^N)}^\alpha$; that is, the real part of the stochastic integral $I(\cdot)$ maps $L^\alpha(\mathbb{R}^N)$ isometrically to a certain subset of SαS random variables.

Let $\mathbf{T}$ be a set of parameters. The random field $\{X(t), t \geq 0\}$ is well defined by

$$X(t) = \operatorname{Re}\int_{\mathbb{R}^N} f(t,x)\,\mu(dx) \tag{1.1}$$

for a measurable function $f\colon \mathbf{T} \times \mathbb{R}^N \to \mathbb{C}$ such that $f(t,\cdot) \in L^\alpha(\mathbb{R}^N)$ for all $t \in \mathbf{T}$.

We recall the following LePage representation [7] (see also [10]). Let φ be an arbitrary positive density of a probability distribution on $\mathbb{R}^N$, and let $\{\Gamma_k, k \geq 1\}$, $\{\xi_k, k \geq 1\}$, and $\{g_k, k \geq 1\}$ be three independent sets of random elements such that

(a) $\{\Gamma_k, k \geq 1\}$ is the sequence of moments when a Poisson process with the unit intensity jumps;
(b) $\{\xi_k, k \geq 1\}$ are independent random vectors with the density φ;
(c) $\{g_k, k \geq 1\}$ are independent rotationally invariant complex-valued Gaussian random variables with $\mathsf{E}\left[|\operatorname{Re} g_k|^\alpha\right] = 1$.

Then the field $\{X(t), t \geq 0\}$ defined by (1.1) has the same finite-dimensional distributions as

$$X'(t) = C_\alpha \operatorname{Re}\sum_{k=1}^\infty \Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha} f(t,\xi_k) g_k, \tag{1.2}$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin x\,dx\right)^{1/\alpha}$; the series on the right-hand side of (1.2) converges almost surely for all t.

1.2. **Notation.** For two points $s = (s_1, \dots, s_n) \in \mathbb{R}^n$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, we write $s \le t$ if $s_i \le t_i$ for all $i = 1, \dots, n$. For $a, b \in \mathbb{R}^n$ denote

$$[a, b] = \{s \in \mathbb{R}^n \colon a \le s \le b\}, \qquad (0, b] = \{s \in (0, \infty)^n \colon s \le b\}.$$

Let $\langle x, y\rangle$ denote the inner product in $\mathbb{R}^n$. Depending on the context, $|\cdot|$ will denote both the absolute value of a complex number and the Euclidian norm in $\mathbb{R}^n$, and 0 will denote both the origin in $\mathbb{R}^n$ and the zero vector in $\mathbb{R}^n$. The linear span of vectors $x^1, \dots, x^m$ will be denoted by $\operatorname{span}\{x^1, \dots, x^m\}$.

All unessential constants, whose values depend on the parameters fixed in the paper, are denoted by C.

2. Definition and path properties of a real harmonizable multifractional stable field

Let $M = (\mu^1, \dots, \mu^d)$, where $\{\mu^i, i = 1, \dots, d\}$ are independent complex rotationally invariant SαS measures on $\mathbb{R}^N$.

Fix some $T \in \mathbb{R}^N$. For a given continuous function $H : [0, T] \to (0, 1)^N$, an *anisotropic harmonizable multifractional stable* (N, d)*-field* on $[0, T]$ with the Hurst function H and stability parameter α is defined as

$$Z(t) = \operatorname{Re} \int_{\mathbb{R}^N} \prod_{k=1}^{N} \frac{e^{it_k x_k} - 1}{|x_k|^{1/\alpha + H_k(t)}} \, M(dx). \tag{2.1}$$

The case of $\alpha = 2$, where the distribution of M is Gaussian, is worth special consideration. In this case, we call Z an *anisotropic harmonizable multifractional Brownian* (N, d)*-field* with the Hurst function H.

We assume in what follows that the function H is fixed. This function is assumed to satisfy the Hölder condition

$$|H(t) - H(s)| \le C\,|t - s|^{\gamma}$$

with the exponent $\gamma > \max\limits_{\substack{t \in [0,T] \\ k=1,\dots,N}} H_k(t)$.

Let $H'_k = \min_{t \in [0,T]} H_k(t)$ and $H''_k = \max_{t \in [0,T]} H_k(t)$. Furthermore, let

$$l_h(s, y) = \frac{e^{isy} - 1}{|y|^{1/\alpha + h}}, \qquad f(t, x) = \prod_{k=1}^{N} l_{H_k(t)}(t_k, x_k),$$

$$f_j(x) = \prod_{\substack{k=1,\dots,n, \\ k \ne j}} \frac{|x_k| \wedge 1}{|x_k|^{1/\alpha}} \left(|x_k|^{-H'_k} \vee |x_k|^{-H''_k} \right),$$

$$\rho'(t, s) = \sum_{k=1}^{N} |t_k - s_k|^{H''_k}, \qquad \rho''(t, s) = \sum_{k=1}^{N} |t_k - s_k|^{H'_k}$$

for $h \in (0, 1)$, $s \in [0, \infty)$, $y \in \mathbb{R}$, $t \in [0, T]$, and $x \in \mathbb{R}^N$.

There is a certain "inconsistence" in the last notation, which comes from the fact that

$$\rho'(t, s) \le C\rho''(t, s).$$

2.1. Estimates for increments of the field.

Theorem 2.1. *There exist constants $C_1, C_2 > 0$ such that*

$$C_1 \rho'(t, s) \le \|Z_k(t) - Z_k(s)\|_{\alpha} \le C_2 \rho''(t, s) \tag{2.2}$$

for all sufficiently close $s, t \in [0, T]$ and $k = 1, \dots, N$.

Proof. Without loss of generality we can restrict the consideration to the case of $k = 1$. For $H = (H_1, \dots, H_N)$, denote

$$X^H(t) = \operatorname{Re} \int_{\mathbb{R}^N} \prod_{k=1}^N \frac{e^{it_k x_k} - 1}{|x_k|^{1/\alpha + H_k}} \, \mu^1(dx),$$

so that $Z_1(t) = X^{H(t)}(t)$.

In order to establish an upper bound, write

$$\|Z_1(t) - Z_1(s)\|_\alpha \le \left\| X^{H(s)}(t) - X^{H(s)}(s) \right\|_\alpha + \left\| X^{H(t)}(t) - X^{H(s)}(t) \right\|_\alpha .$$

Then we estimate

$$\begin{aligned}
\left\| X^{H(t)}(t) - X^{H(s)}(t) \right\|_\alpha^\alpha &\le C \int_{\mathbb{R}^N} \sum_{j=1}^N \frac{\left| e^{it_j x_j} - 1 \right|^\alpha}{|x_j|} \left| |x_j|^{-H_j(t)} - |x_j|^{-H_j(s)} \right|^\alpha f_j(x) \, dx \\
&\le C \int_{\mathbb{R}^N} \sum_{j=1}^N \frac{(|x_j| \wedge 1)^\alpha}{|x_j|} \left| \ln |x_j| \right| \left(|x_j|^{-H_j'} \vee |x_j|^{-H_j''} \right)^\alpha |H_j(t) - H_j(s)|^\alpha f_j(x) \, dx \\
&\le C \, |H_j(t) - H_j(s)|^\alpha \le C \, |t - s|^{\gamma\alpha} .
\end{aligned}$$

The expression $X^{H(s)}(t) - X^{H(s)}(s)$ is an increment of a harmonizable (non-multi) fractional stable field considered in [15]. Theorem 3.5 in [15] claims that

$$c_1 \sum_{k=1}^N |t_k - s_k|^{H_k} \le \left\| X^H(t) - X^H(s) \right\|_\alpha \le c_2 \sum_{k=1}^N |t_k - s_k|^{H_k}$$

for all sufficiently close $t, s \in [0, T]$. The constants in the latter inequality depend on H but it is easily seen from the proof of this fact in [15] that the constants are bounded away from zero under the assumption that the Hurst parameters are separated from 0 and 1. Thus we can assume that these constants are universal. Therefore

$$\|Z_1(t) - Z_1(s)\|_\alpha \le C\rho''(t, s) + C \, |t - s|^\gamma \le C\rho''(t, s)$$

for all sufficiently close $t, s \in [0, T]$.

Similarly,

$$\begin{aligned}
\|Z_1(t) - Z_1(s)\|_\alpha &\ge \left\| X^{H(s)}(t) - X^{H(s)}(s) \right\|_\alpha - \left\| X^{H(t)}(t) - X^{H(s)}(t) \right\|_\alpha \\
&\ge C \left(\rho'(t, s) - C \, |t - s|^\gamma \right) \ge C\rho'(t, s). \quad \square
\end{aligned}$$

2.2. **Continuity.** The proof of the following result uses some ideas from [7] and is similar to the proof of continuity of a real harmonizable multifractional stable process given in [4]. Put $H' = \min_{k=1,\dots,N} H_k'$.

Theorem 2.2. *The random field X has a modification with almost surely continuous paths. Moreover,*

$$|Z(t) - Z(s)| \le C_\omega \, |t - s|^{H'} \left| \ln |t - s| \right|^{1/\alpha + 1/2 + \varepsilon}$$

almost surely for all $\varepsilon > 0$ and for all $t, s \in [0, T]$.

Proof. We use the LePage representation (1.2). For an arbitrary fixed $\eta > 0$, put

$$\varphi(x) = K_\eta \prod_{k=1}^N |x_k|^{-1} \left| \ln |x_k| + 1 \right|^{-1-\eta} ,$$

where K_η is a normalizing constant. Also let $\{\Gamma_k, k \geq 1\}$ and $\{\xi_k, k \geq 1\}$ be the same as (1.2). Furthermore, let $\{G_k = (G_{k,1}, \ldots, G_{k,d}, k \geq 1\}$ be some vectors with independent complex-valued rotationally invariant Gaussian coordinates such that $\mathsf{E}\left[|\operatorname{Re} G_{k,i}|^\alpha\right] = 1$ for $i = 1, \ldots, d$.

Then

$$Z'(t) = C_\alpha \operatorname{Re} \sum_{k \geq 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f(t, \xi_k) G_k$$

has the same distribution as X. To simplify the notation we will assume that $Z' = Z$.

The conditional distribution of the field Z given Γ and ξ is Gaussian. Moreover,

$$\begin{aligned}\mathsf{E}\left[|Z(t) - Z(s)|^2 \,\middle|\, \Gamma, \xi\right] &= C_\alpha^2 \sum_{k \geq 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \left|f(t, \xi_k) - f(s, \xi_k)\right|^2 \mathsf{E}\left[|G_k|^2\right] \\ &\leq C a(u),\end{aligned}$$

where

$$a(u) = \sum_{k \geq 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \sup_{|t-s|<u} \left|f(t, \xi_k) - f(s, \xi_k)\right|^2.$$

Hence

$$\begin{aligned}&\sup_{|t-s|<u} |f(t,x) - f(s,x)| \\ &\quad \leq \sup_{|t-s|<u} \sum_{j=1}^N \frac{1}{|x_j|^{1/\alpha}} \\ &\qquad \times \left(\frac{\left|e^{it_j x_j} - e^{is_j x_j}\right|}{|x_j|^{H_j(t)}} + \left|e^{is_j x_j} - 1\right| \left| \frac{1}{|x_j|^{H_j(t)}} - \frac{1}{|x_j|^{H_j(s)}} \right| \right) f_j(x) \\ &\quad \leq C \sum_{j=1}^N \frac{|x_j|^{-H_j'} \vee |x_j|^{-H_j''}}{|x_j|^{1/\alpha}} \\ &\qquad \times \big((u|x_j|) \wedge 1 + (|x_j| \wedge 1)\big|\ln|x_j|\big| \sup_{|t-s|<u} |H_j(t) - H_j(s)|\big) f_j(x) \\ &\quad \leq C \sum_{j=1}^N \frac{|x_j|^{-H_j'} \vee |x_j|^{-H_j''}}{|x_j|^{1/\alpha}} \big((u|x_j|) \wedge 1 + (|x_j| \wedge 1)\big|\ln|x_j|\big| u^\gamma\big) f_j(x).\end{aligned} \tag{2.3}$$

Using this estimate, take the expectation $\mathsf{E}_\xi\left[a(u)\right]$ with respect to ξ:

$$\mathsf{E}_\xi\left[a(u)\right] \leq C(I_1 + I_2) \sum_{k \geq 1} \Gamma_k^{-2/\alpha},$$

where

$$I_1 = \int_{\mathbb{R}^N} \sum_{j=1}^N \frac{|x_j|^{-2H_j'} \vee |x_j|^{-2H_j''}}{|x_j|^{2/\alpha}} \big((u|x_j|) \wedge 1\big)^2 f_j(x)^2 \varphi(x)^{1-2/\alpha}\, dx,$$

$$I_2 = u^{2\gamma} \int_{\mathbb{R}^N} \sum_{j=1}^N \frac{|x_j|^{-2H'} \vee |x_j|^{-2H''}}{|x_j|^{2/\alpha}} \big(|x_j| \wedge 1\big)^2 \big|\ln|x_j|\big|^2 f_j(x)^2 \varphi(x)^{1-2/\alpha}\, dx.$$

Then we write

$$I_1 \le C \sum_{j=1}^{N} \int_{\mathbb{R}^N} \frac{|x_j|^{-2H_j'} \vee |x_j|^{-2H_j''}}{|x_j| \, |\ln |x_j| + 1|^{(1+\eta)(1-2/\alpha)}} \big((u\,|x_j|) \wedge 1\big)^2 \\ \times \prod_{k \ne j} \frac{(|x_k| \wedge 1)^2 \left(|x_k|^{-2H_k'} \vee |x_k|^{-2H_k''}\right)}{|x_k| \, |\ln |x_k| + 1|^{(1+\eta)(1-2/\alpha)}} \, dx$$

and integrate over all variables except for x_j. We also change the variable $z = ux_j$. Then we get

$$I_1 \le C u^{2H_j'} \int_{\mathbb{R}} \left(|z|^{-2H_j'-1} \vee |z|^{-2H_j''-1}\right) |\ln |z/u| + 1|^{(1+\eta)(2/\alpha-1)} \left(|z| \wedge 1\right)^2 dz \\ \le C u^{2H_j'} \, |\ln u|^{(1+\eta)(2/\alpha-1)} .$$

Similarly,

$$I_2 \le C u^{2\gamma} \sum_{j=1}^{N} \int_{\mathbb{R}^N} \frac{|x_j|^{-2H_j'} \vee |x_j|^{-2H_j''}}{|x_j| \, |\ln |x_j| + 1|^{(1+\eta)(1-2/\alpha)}} \big(|x_j| \wedge 1\big)^2 \big|\ln |x_j|\big|^2 \\ \times \prod_{k \ne j} \frac{(|x_k| \wedge 1)^2 \left(|x_k|^{-2H_k'} \vee |x_k|^{-2H_k''}\right)}{|x_k| \, |\ln |x_k| + 1|^{(1+\eta)(1-2/\alpha)}} \, dx \\ \le C u^{2\gamma}.$$

Thus,

$$\mathsf{E}_\xi \left[a(u)\right] \le C u^{2H'} \, |\ln u|^{(1+\eta)(2/\alpha-1)} \sum_{k \ge 1} \Gamma_k^{-2/\alpha}.$$

By the strong law of large numbers, $\Gamma_j/j \to 1$ almost surely as $j \to \infty$, whence $\sum_{k\ge1} \Gamma_k^{-2/\alpha} < \infty$, since $2/\alpha > 1$. Thus

$$\mathsf{E}_\xi \left[a(u)\right] \le C(\Gamma) u^{2H'} \, |\ln u|^{(1+\eta)(2/\alpha-1)}$$

almost surely.

Put $b(u) = u^{2H'} \, |\ln u|^{2(1+\eta)/\alpha}$. Then

$$\mathsf{E}_\xi \left[\sum_{n\ge1} \frac{a(2^{-n})}{b(2^{-n})}\right] \le C(\Gamma) \sum_{n\ge1} n^{-1-\eta}.$$

Therefore $a(2^{-n})/b(2^{-n}) \to 0$ as $n \to \infty$ for all ξ and Γ. It is easy to see that

$$b(2t) \le C b(t),$$

and $a(u)$ is a non-decreasing function. This together with the latter convergence implies that $a(u)/b(u) \to 0$, $u \to 0+$. Hence

$$\mathsf{E}\left[(Z(t) - Z(s))^2 \;\middle|\; \Gamma, \xi\right] \le |t-s|^{2H'} \, \big|\ln |t-s|\big|^{2(1+\eta)/\alpha}$$

for all sufficiently close t and s. Since the field Z has a normal distribution for all fixed ξ and Γ,

$$|Z(t) - Z(s)| \le C_\omega \, |t-s|^{H'} \, \big|\ln |t-s|\big|^{1/\alpha+\eta/\alpha+1/2}$$

(see, for example, [9]). This completes the proof. □

3. Local time for multifractional fields

Recall that the *local time* or *occupation density* of a random field

$$X\colon [0,T]\times\Omega\to\mathbb{R}^d,$$

where $B\subset[0,T]$ is a Borel set, is the Radon–Nikodym derivative

$$L(B,x)=\frac{dV(B,dx)}{\lambda_d(dx)}$$

of the occupation measure

$$V_B(A)=\lambda_N(s\in B\colon X(s)\in A),\qquad A\in\mathcal{B}\left(\mathbb{R}^d\right).$$

Following the tradition, we denote $L(t,x)=L([0,t],x)$.

Theorem 3.1. *If $d<\sum_{k=1}^N 1/H_k''$, then the random field Z has a local time $L(t,x)$ such that*

$$\int_{\mathbb{R}^d}\mathsf{E}\left[L(t,x)^2\right]\,dx<\infty.$$

Proof. Consider a sufficiently small set $I\subset[0,T]$ where estimates (2.2) hold. According to a known formula (see [5])

$$\begin{aligned}
\int_{\mathbb{R}^d}\mathsf{E}\left[L(I,x)^2\right]dx&=\frac{1}{(2\pi)^d}\int_{I^2}\int_{\mathbb{R}^d}\mathsf{E}\left[\exp\left\{i\left\langle u,Z(s^1)-Z(s^2)\right\rangle\right\}\right]du\,ds^1\,ds^2\\
&=\frac{1}{(2\pi)^d}\int_{I^2}\int_{\mathbb{R}^d}\prod_{k=1}^d\mathsf{E}\left[\exp\left\{iu_k\big(Z_k(s^1)-Z_k(s^2)\big)\right\}\right]du\,ds^1\,ds^2\\
&=\frac{1}{(2\pi)^d}\int_{I^2}\int_{\mathbb{R}^d}\prod_{k=1}^d\exp\left\{-\left\|u_k\big(Z_k(s^1)-Z_k(s^2)\big)\right\|_\alpha^\alpha\right\}du\,ds^1\,ds^2 \qquad (3.1)\\
&\le C\int_{I^2}\int_{\mathbb{R}^d}\prod_{k=1}^d\exp\left\{-\left|u_k\right|^\alpha\rho'(s^1,s^2)^\alpha\right\}du\,ds^1\,ds^2\\
&\le C\int_{[0,t]^2}\rho'\left(s^1,s^2\right)^{-d}ds^1\,ds^2.
\end{aligned}$$

We used estimate (2.2) for increments of the field and the independence of the components. By the generalized Cauchy inequality,

$$\rho'(s^1,s^2)=\sum_{k=1}^N\left|s_k^1-s_k^2\right|^{H_k''}\ge\varkappa\prod_{k=1}^N\left|s_k^1-s_k^2\right|^{1/\varkappa}\left(H_k''\right)^{\frac{1}{\varkappa H_k''}}\ge C\prod_{k=1}^N\left|s_k^1-s_k^2\right|^{1/\varkappa},$$

where $\varkappa=\sum_{k=1}^N 1/H_k''$. Thus,

$$\int_{\mathbb{R}^d}\mathsf{E}\left[L(I,x)^2\right]dx\le C\int_{[0,t]^2}\prod_{k=1}^N\left|s_k^1-s_k^2\right|^{-d/\varkappa}ds^1ds^2<\infty.$$

Splitting $[0,t]$ into sufficiently small parts and using the additivity of the local time $L(B,x)$ with respect to B we complete the proof of the theorem. □

3.1. **Properties of the local time in the Gaussian case.** We consider the Gaussian case, that is, the case of $\alpha = 2$. Then M is a complex-valued independently scattered Gaussian measure. Introduce the following version of the random field:

$$Y(t) = \operatorname{Re} \int_{\mathbb{R}} g(t,x)\, M(dx), \tag{3.2}$$

where

$$g(t,x) = \prod_{k=1}^{N} \frac{1 - e^{-it_k x_k}}{(-ix_k)^{H_k(t)+1/2}}, \qquad (-ix)^K = |x|^K e^{-i\pi K \operatorname{sign} x/2}.$$

It is easy to check that all the above results for Z are valid for Y as well.

The following property, called the *sector local non-determinism*, is a bridge to numerous results concerning the random field Y.

Lemma 3.2. *For all $n \geq 1$ and $t^0 \in (0,T]$,*

$$\left\| Y_1(t^n) - \operatorname{span}\left\{ Y_1\left(t^1\right), \ldots, Y_1\left(t^{n-1}\right) \right\} \right\|_2 \geq C_{t^0,n} \sum_{j=1}^{N} \min_{k=1,\ldots,n-1} \left| t_j^n - t_j^k \right|^{H_j''} \tag{3.3}$$

if $t^1, t^2, \ldots, t^n \in [t^0, T]$ are sufficiently close to each other.

Proof. Below we will assume n and the point t^0 to be fixed. It is convenient to denote all constants that depend only on n, t^0, and other fixed parameters by the same symbol C.

The Fourier transform of the function $g(t,x)$ with respect to the variable x is given by

$$\widehat{g}(t,x) = \prod_{k=1}^{N} \frac{2\pi}{\Gamma(H_k(t)+1/2)} m_{H(t_k)}(t_k, x_k), \tag{3.4}$$

where $m_H(t,x) = (t_k - x_k)_+^{H_k - 1/2} - (-x_k)_+^{H_k-1/2}$ (see, for example, [4, Appendix A]).

Our aim is to estimate from below the expression

$$\left\| Y_1(t^n) - \sum_{k=1}^{n-1} u_k Y_1\left(t^k\right) \right\|_2 = \left\| g(t^n, \cdot) - \sum_{k=1}^{n-1} u_k g\left(t^k, \cdot\right) \right\|_{L^2(\mathbb{R}^N)}$$

for all $u_1, \ldots, u_{n-1} \in \mathbb{R}$ and $t^1, \ldots, t^n$.

By the Parseval identity,

$$\left\| g(t^n, \cdot) - \sum_{k=1}^{n-1} u_k g\left(t^k, \cdot\right) \right\|_{L^\alpha(\mathbb{R}^N)} = (2\pi)^{-N} \left\| \widehat{g}(t^n, \cdot) - \sum_{k=1}^{n-1} u_k \widehat{g}\left(t^k, \cdot\right) \right\|_{L^2(\mathbb{R}^N)}.$$

Thus

$$\left\| \widehat{g}(t^n, \cdot) - \sum_{k=1}^{n-1} u_k \widehat{g}\left(t^k, \cdot\right) \right\|_{L^2(\mathbb{R}^N)} \geq C \left\| \widehat{g}(t^n, \cdot) - \sum_{k=1}^{n-1} u_k \widehat{g}\left(t^k, \cdot\right) \right\|_{L^2(A)},$$

where

$$A_j = \left(\prod_{k=1}^{j-1} [0, T_k] \right) \times \mathbb{R} \times \left(\prod_{k=j+1}^{N} [0, T_k] \right), \qquad j \in \{1, \ldots, n\}.$$

Denote by L_j the subspace of functions $f(x)$ of the space $L^2(A)$ that depend only on the variable x_j. We have

$$\left\| \widehat{g}(t^n, \cdot) - \sum_{k=1}^{n-1} u_k \widehat{g}\left(t^k, \cdot\right) \right\|_{L^2(A)} \geq C \left\| \operatorname{pr}_{L_j} \left(\widehat{g}(t^n, \cdot) - \sum_{k=1}^{n-1} u_k \widehat{g}\left(t^k, \cdot\right) \right) \right\|_{L^2(A)},$$

where pr is the orthogonal projection to L_k. Now

$$\begin{aligned}
&\left(\mathrm{pr}_{L_j} g\left(t^k,\cdot\right)\right)(x_j)\\
&\quad= \frac{1}{\prod_{l\neq j} T_l}\int_0^{T_N}\cdots\int_0^{T_{j+1}}\int_0^{T_{j-1}}\cdots\int_0^{T_1} g\left(t^k,x\right)\,dx_1\cdots dx_{j-1}\,dx_{j+1}\cdots dx_N\\
&\quad= C\left(t^k,j\right) m_{H_j(t^k)}\left(t_j^k,x_j\right),
\end{aligned}$$

where the function $C(t^k,j)$ is uniformly bounded away from zero for $t^k \geq t^0$. Introducing the notation

$$a_k = u_k C(t^k,j)/C(t^n,j),$$

we obtain

$$\begin{aligned}
&\left\|\mathrm{pr}_{L_j}\left(\widehat{g}(t^n,\cdot) - \sum_{k=1}^{n-1} u_k \widehat{g}\left(t^k,\cdot\right)\right)\right\|_{L^2(\mathbb{R}^N)}\\
&\qquad\geq C\left\|m_{H_j(t^n)}\left(t_j^n,\cdot\right) - \sum_{k=1}^{n-1} a_k m_{H_j(t^k)}\left(t_j^k,\cdot\right)\right\|_{L^2(\mathbb{R})} \geq C \min_{k=1,\ldots,n-1}\left|t_j^n - t_j^k\right|^{H_j''}.
\end{aligned}$$

The latter inequality for sufficiently close points $t_j^1,\ldots,t_j^n$ follows from the local non-determinism property for the multifractional Brownian motion (see [3]). Therefore, for all $j=1,\ldots,n$, we have

$$\left\|Y_1(t^n) - \mathrm{span}\left\{Y_1(t^1),\ldots,Y_1(t^{n-1})\right\}\right\|_2 \geq C \min_{k=1,\ldots,n-1}\left|t_j^n - t_j^k\right|^{H_j''}.$$

Adding up these inequalities for $j=1,\ldots,n$ and then dividing the sum by n, we get the desired estimate. □

The sector local non-determinism together with some of the results of [15] imply the following assertions.

Corollary 3.3. *If $d < \sum_{k=1}^{N}(H_k'')^{-1}$, then the local time $L(t,x)$ is continuous on $(0,T]\times\mathbb{R}^d$ almost surely.*

Corollary 3.4. *Let τ be such that $\sum_{k=1}^{\tau-1} 1/H_k'' < d < \sum_{k=1}^{\tau} 1/H_k''$, and let*

$$\beta_\tau = \sum_{l=1}^{\tau} H_\tau''/H_l'' + N - \tau - H_\tau'' d.$$

Then, for any $\eta\in(0,\beta_\tau)$ and any $t^0\in(0,T]$, there exists a constant C_{η,t^0} such that if $t\in[t^0,T]$, then the following inequality holds almost surely:

$$L(B(t,r),x) \leq C_{\eta,t^0} r^\eta$$

for all sufficiently small $r>0$ and all $x\in\mathbb{R}^n$, where $B(t,r)$ is a ball of radius r centered at t.

This corollary allows us to estimate the Hausdorff dimension of the field image and that of the level sets of the field (see [16] for detail).

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