Теорія Ймовір. та Матем. Статист.
Вип. 86, 2012

Theor. Probability and Math. Statist.
No. 86, 2013, Pages 1–11
S 0094-9000(2013)00885-0
Article electronically published on August 20, 2013

THE ASYMPTOTIC STABILITY OF THE MAXIMUM OF INDEPENDENT RANDOM ELEMENTS IN FUNCTION BANACH LATTICES

UDC 519.21

K. S. AKBASH AND I. K. MATSAK

Abstract. We generalize some well-known results on the asymptotic stability of the maximum of independent random variables in $\mathbb{R}^1$ to the case of q-concave Banach ideal spaces. A theorem on the relative asymptotic stability of the maximum of independent random elements in function Banach lattices is proved.

1. Introduction: Main results

Let ξ be a random variable that assumes values in $\mathbb{R}^1$ and that has the distribution function $F(x)$. Let ξ_i be independent copies of ξ. Set

$$z_n = \max_{1\le i\le n} \xi_i.$$

We say that a sequence (z_n) is relatively stable almost surely if there is a numerical sequence (a_n) such that

$$\frac{z_n}{a_n} \to 1 \quad \text{almost surely} \tag{1}$$

as $n \to \infty$. We also say that (z_n) is stable almost surely if there is a numerical sequence (a_n) such that

$$z_n - a_n \to 0 \quad \text{almost surely} \tag{2}$$

as $n \to \infty$.

Starting with the seminal Gnedenko paper [4], the (weak) convergence has been studied in the case of degenerate limit laws. The criteria for the asymptotic relations (1) and (2) are also well known for $\mathbb{R}^1$ (see [1, 3]). A survey concerning the convergence of (z_n) in distribution to degenerate laws can be found in [3].

The aim of the current paper is to obtain relations (1) and (2) for the case of infinite dimensional spaces.

The notion of the maximum of two or more random elements can be introduced in the so-called Banach lattices [8]. An important example of Banach lattices is presented by Banach ideal spaces [5].

Let (T, Λ, μ) be a measure space where μ is a σ-finite, σ-additive, and nonnegative measure. By definition, a Banach ideal space B of measurable functions defined in (T, Λ, μ) is a collection of functions such that if $y \in B$ and if $|x(t)| \le |y(t)|$ almost surely, then $x \in B$ and $\|x\| \le \|y\|$.

Throughout this paper we consider only the case of separable Banach ideal spaces.

2010 *Mathematics Subject Classification.* Primary 60B12.

Key words and phrases. Maximum of independent random elements, asymptotic stability, Banach ideal spaces.

Let B be a Banach ideal space equipped with a norm $\|\cdot\|$ and module $|\cdot|$. Assume that X is a random element that is defined on a probability space (Ω, A, P) and that assumes values in the space B. Furthermore, let X_i be independent copies of X and

$$Z_n = \max_{1 \le i \le n} X_i.$$

Finally, let

$$\begin{gathered} X = \{X(t), t \in T\}, \qquad \mathfrak{S}X = \{\sigma(t), t \in T\}, \\ X(t) = \sigma(t)\widetilde{X}(t) \quad \text{for all } t \in T, \end{gathered} \tag{3}$$

and

$$\mathsf{P}(\widetilde{X}(t) < x) = \mathsf{P}(\xi < x) = F(x)$$

for all $t \in T$. The latter two assumptions mean, in particular, that both elements $X(t)$ and $\widetilde{X}(t)$ are random variables for all $t \in T$.

Then the generalizations of relations (1) and (2) for Banach ideal space are given by

$$\lim_{n\to\infty} \left\| \frac{Z_n}{a_n} - \mathfrak{S}X \right\| = 0 \quad \text{almost surely,} \tag{4}$$

$$\lim_{n\to\infty} \|Z_n - a_n \mathfrak{S}X\| = 0 \quad \text{almost surely.} \tag{5}$$

Relations (4) and (5) can also be considered with respect to the so-called order convergence.

Recall that a sequence of elements (x_n) of a Banach lattice B is said to o-converge to an element x (we write in this case $x = o\text{-}\lim_{n\to\infty} x_n$; see [5, 6]) if there is a numerical sequence (v_n) such that $v_n \downarrow 0$ and $|x - x_n| < v_n$. The notation $v_n \downarrow 0$ means that $v_1 \ge v_2 \ge \dots$ and $\inf_{n\ge 1} v_n = 0$.

A Banach lattice B is called q-concave, $1 \le q < \infty$, if

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \le D_{(q)} \left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} \right\|$$

for some constant $D_{(q)} = D_{(q)}(B)$.

Put $\tau(F) = \sup(x \colon F(x) < 1)$,

$$\begin{gathered} \varphi(y) = \inf\left(x \ge 1 \colon \frac{1}{1 - F(x)} \ge y \right), \\ a_n = \varphi(n) = \inf\left(x \ge 1 \colon F(x) \ge 1 - \frac{1}{n} \right). \end{gathered} \tag{6}$$

In what follows we assume that $F(x)$ is a continuous increasing function and that $\tau(F) = \infty$.

Theorem 1.1. *Let X be a random element assuming values in a q-concave Banach ideal space B, $1 \le q < \infty$, for which representation (3) holds. Assume that the function $\varphi(y)$ slowly varies at infinity and that a_n is defined by equality (6).*

We further assume that there exists a number t_0 such that

$$\int_1^\infty \frac{1 - F(x)}{x\,[1 - F(x/t_0)]}\, dx < \infty \tag{7}$$

and that

$$\int_1^\infty \frac{dF(x)}{1 - F(x/t)} < \infty \tag{8}$$

for all $t > 1$. *Then*

$$o\text{-}\lim_{n\to\infty} \frac{Z_n}{a_n} = \mathfrak{S}X \quad \text{almost surely} \tag{9}$$

and

$$\mathsf{E} \left\| \sup_{n\ge 1} \frac{(Z_n)_+}{a_n} \right\|^q < \infty, \tag{10}$$

where $x_+ = \max(x, 0)$ *and* $x_- = \max(-x, 0)$.

Corollary 1.1. *Relation* (4) *holds if all assumptions of Theorem* 1.1 *are satisfied.*

Remark 1.1. Relation (4) is proved in [8] under the assumption that

$$\int_{-\infty}^{+\infty} \frac{|x|^q \, dF(x)}{1 - F(x/t)} < \infty \tag{11}$$

for all $t > 1$. Corollary 1.1 improves this result to some extent. A simple example where condition (6) does not hold but both conditions (7) and (8) hold is provided at the end of this paper.

Note also that condition (7) is necessary for (10) in the space l_q.

Theorem 1.2. *Let* X *be a random element assuming values in a* q*-concave Banach ideal space,* $1 \le q < \infty$. *Assume that representation* (3) *holds for* X *and that the sequence* a_n *is defined by equality* (6). *We further assume that there exists* x_0 *such that*

$$\int_{x_0}^{+\infty} \frac{x^q \, dF(x)}{1 - F\left((x^q - x_0^q)^{1/q}\right)} < \infty \tag{12}$$

and that

$$\int_{1}^{+\infty} \frac{dF(x)}{1 - F(x - \varepsilon)} < \infty \tag{13}$$

for all $\varepsilon > 0$. *Then*

$$o\text{-}\lim_{n\to\infty} (Z_n - a_n \mathfrak{S}X) = 0 \quad \textit{almost surely.} \tag{14}$$

Corollary 1.2. *Asymptotic relation* (5) *holds under the assumptions of Theorem* 1.2.

Corollary 1.3. *If assumption* (12) *in Theorem* 1.2 *is changed for the assumption that*

$$\lim_{x\to\infty} \frac{1 - F(x + \varepsilon)}{1 - F(x)} = 0 \tag{15}$$

for all $\varepsilon > 0$ *and if*

$$\int_{-\infty}^{+\infty} |x|^q \, dF(x) < \infty, \tag{16}$$

then the sequence Z_n *is stable in probability, that is,*

$$\|Z_n - a_n \mathfrak{S}X\| \xrightarrow{\mathsf{P}} 0. \tag{17}$$

2. Proof of Theorem 1.1

First we prove some auxiliary results for $\mathbb{R}^1$.

Lemma 2.1. *Let $\psi(s)$ be a nondecreasing Karamata slowly varying at infinity function. Put $a_n = \psi(n)$ and assume that $a_1 > 0$. Then*

$$\sum_{n=1}^{m} \frac{1}{a_n} \sim \frac{m}{a_m} \tag{18}$$

as $m \to \infty$.

Lemma 2.1 follows directly from Karamata's theorem (see, for example, [2, Chapter VIII, §9, Theorem 1(b)]). According to Karamata's theorem,

$$\int_1^y \frac{dx}{\psi(x)} \sim \frac{y}{\psi(y)}$$

as $y \to \infty$. Since

$$\frac{1}{\psi(n+1)} \le \int_n^{n+1} \frac{dx}{\psi(x)} \le \frac{1}{\psi(n)},$$

the above relation implies (18).

Lemma 2.2. *Let ξ be a nonnegative random variable with the distribution function $F(x)$ and let (ξ_i) be independent copies of ξ. Assume that (α_n) is a nondecreasing numerical sequence such that $\alpha_n \uparrow \infty$ and $\alpha_n > 0$. Put*

$$\lambda(t) = \sum_{\alpha_n < t} \frac{1}{\alpha_n}.$$

Then, for all $0 < t_0 < \infty$,

$$\mathsf{E} \sup_{n \ge 1} \frac{\xi_n}{\alpha_n} \le t_0 + \int_{\alpha_1 t_0}^{\infty} \lambda\left(\frac{t}{t_0}\right) (1 - F(t))\, dt \tag{19}$$

and

$$\mathsf{E} \sup_{n \ge 1} \frac{\xi_n}{\alpha_n} \le \frac{m}{2} + \frac{1}{2} \int_{\alpha_1 m}^{\infty} \lambda\left(\frac{t}{m}\right) (1 - F(t))\, dt, \tag{20}$$

where $m = \operatorname{med}(\sup_{n \ge 1} \xi_n / \alpha_n)$.

Lemma 2.2 is proved in [7].

Lemma 2.3. *Let ξ be a nonnegative random variable with the distribution function $F(x)$ and let (ξ_i) be independent copies of ξ. Then condition (7) is equivalent to the condition that, for all $1 \le q < \infty$,*

$$\mathsf{E} \sup_{n \ge 1} \left| \frac{\xi_n}{a_n} \right|^q < \infty, \tag{21}$$

where a_n is defined by equality (6).

Proof of Lemma 2.3. First we show that (7) $\Rightarrow$ (21). Put

$$\lambda_q(t) = \sum_{a_n^q < t} \frac{1}{a_n^q}, \qquad \mathbf{M}_q = \mathsf{E} \sup_{n \ge 1} \left| \frac{\xi_n}{a_n} \right|^q.$$

We apply estimate (19) of Lemma 2.2 with $a_1 = 1$ and $1 < t_0 < \infty$. Then

$$\mathbf{M}_q \le t_0 + \int_{t_0}^{\infty} \lambda\left(\frac{t}{t_0}\right) \left(1 - F\left(t^{1/q}\right)\right) dt. \tag{22}$$

It is clear that

$$a_n^q \le t \quad \Leftrightarrow \quad F(a_n) \le F(t^{1/q}) \quad \Leftrightarrow \quad n \le \frac{1}{1 - F(t^{1/q})}.$$

Since a_n^q is slowly varying at infinity, Lemma 2.1 with

$$m = \left[\frac{1}{1 - F(t^{1/q})}\right]$$

implies that

$$\lambda_q(t) \sim \frac{m}{a_m^q} \sim \frac{1}{t(1 - F(t^{1/q}))} \tag{23}$$

as $t \to \infty$.

Relations (22) and (23) imply that $\mathbf{M}_q$ is finite if the integral

$$\int_{t_0}^{\infty} \frac{1 - F\big(t^{1/q}\big)}{t\left[1 - F\big((t/t_0)^{1/q}\big)\right]}\, dt = \begin{bmatrix} t^{1/q} = x \\ t_0^{1/q} = x_0 \end{bmatrix} = q \int_{x_0}^{\infty} \frac{1 - F(x)}{x(1 - F(x/x_0))}\, dx \tag{24}$$

converges for some $1 < t_0 < \infty$. Clearly, this is equivalent to (7).

Now we prove the converse implication (21) $\Rightarrow$ (7). Choose

$$t_0 = m = \operatorname{med}\left(\sup_{n \ge 1} \left|\frac{\xi_n}{a_n}\right|^q\right)$$

and use estimate (20) of Lemma 2.2:

$$\infty > \mathbf{M}_q \ge \frac{t_o}{2} + \frac{1}{2}\int_{t_0}^{\infty} \lambda_q\left(\frac{t}{t_0}\right)\left(1 - F\big(t^{1/q}\big)\right)\, dt.$$

Asymptotic relation (23) implies that the convergence of the latter integral is equivalent to the convergence of the integral on the right-hand side of equality (24). This completes the proof of (7). $\square$

Proof of Theorem 1.1. Recall that (ξ_i) are independent copies of the random variable ξ, $z_n = \max_{1 \le i \le n} \xi_i$, and that the sequence (a_n) is defined by equality (6). It is clear that relation (1) follows from (8) (see [3, proof of Theorem 4.4.4]; in fact, (1) and (8) are equivalent, see [3]).

This implies that, for all $t \in T$,

$$\frac{\widetilde{Z}_n(t)}{a_n} \to 1 \quad \text{almost surely,}$$

where $\widetilde{Z}_n(t) = \max_{1 \le i \le n} \widetilde{X}_n(t)$, whence, for all $t \in T$,

$$\frac{Z_n(t)}{a_n} \to \sigma(t) \quad \text{almost surely.} \tag{25}$$

Thus,

$$\mathsf{P}\left(\frac{Z_n(t)}{a_n} \to \sigma(t) \text{ almost everywhere in } T\right) = 1 \tag{26}$$

by Fubini's theorem.

Next we prove that there exists a random element $Z \in B$ such that

$$\mathsf{P}\left(\left|\frac{Z_n(t)}{a_n}\right| \le Z(t) \text{ almost everywhere in } T\right) = 1 \tag{27}$$

for all $n \ge 1$.

It is obvious that

$$\frac{(Z_n)_+}{a_n} \le \sup_{n\ge 1} \frac{(X_n)_+}{a_n}, \qquad \frac{(Z_n)_-}{a_n} \le |X_1| \tag{28}$$

for $n \ge 1$ and hence

$$\sup_{n\ge 1} \left| \frac{Z_n}{a_n} \right| \le V + |X_1|,$$

where

$$V := \sup_{n\ge 1} \frac{(X_n)_+}{a_n}.$$

The latter inequality proves estimate (27) with $Z = V + |X_1|$ provided

$$\mathsf{E} \left\| \sup_{n\ge 1} \frac{(X_n)_+}{a_n} \right\|^q < \infty. \tag{29}$$

Then (28) and (29) imply inequality (10) of Theorem 1.1.

To derive estimate (29), we apply the following inequality obtained in the paper [8]: for an arbitrary random element Y assuming values in a q-concave Banach ideal space B, $1 \le q < \infty$,

$$\left(\mathsf{E}\,\|Y\|^q\right)^{1/q} \le D_q \left\| \left(\mathsf{E}\,|Y(\cdot)|^q\right)^{1/q} \right\|,$$

where $D_q = D_q(B)$ is the constant involved in the definition of the q-concave space B.

We have

$$\left(\mathsf{E}\,\|V\|^q\right)^{1/q} \le D_q \left\| \left(\mathsf{E}\,|V(\cdot)|^q\right)^{1/q} \right\| \le D_q \|\mathfrak{S}X\| \left(\mathsf{E} \sup_{n\ge 1} \left| \frac{(\widetilde{X}_n(t))_+}{a_n} \right|^q \right)^{1/q} < \infty. \tag{30}$$

The expectation

$$\mathsf{E} \sup_{n\ge 1} \left(\frac{(\widetilde{X}_n(t))_+}{a_n} \right)^q$$

is finite by condition (7) and Lemma 2.3. Equalities (26) and (27) imply the order convergence of the sequence (Z_n/a_n) in Banach ideal spaces (see [5]), that is, equality (9) of Theorem 1.1 is proved. □

Remark 2.1. A q-concave, $q < \infty$, Banach lattice is σ-complete and σ-order continuous (see [6, p. 83]). Moreover, such a lattice does not contain subspaces isomorphic to l_∞. Thus, any q-concave Banach ideal space has an order continuous norm ([5]). For such a space,

$$o\text{-}\lim_{n\to\infty} x_n = x \quad \Rightarrow \quad \lim_{n\to\infty} \|x_n - x\| = 0.$$

Therefore, Corollaries 1.1 and 1.2 follow from Theorems 1.1 and 1.2, respectively.

3. Proof of Theorem 1.2

In what follows we need the following elementary result.

Lemma 3.1. *Let $x > 0$ and $y > 0$. If $q \ge 1$, then*

$$|x - y|^q \le |x^q - y^q|. \tag{31}$$

Proof of Lemma 3.1. It is sufficient to consider the case of $q > 1$ only. For definiteness, let $0 < x \le y$ and put

$$0 < z = \frac{x}{y} \le 1.$$

Inequality (31) is equivalent to

$$|1 - z|^q \le 1 - z^q. \tag{32}$$

Let

$$f(z) = 1 - z^q - (1-z)^q, \qquad 0 \le z \le 1.$$

Then

$$f'(z) = -qz^{q-1} + q(1-z)^{q-1} = \begin{cases} 0, & z = \frac{1}{2}, \\ > 0, & 0 \le z < \frac{1}{2}, \\ < 0, & \frac{1}{2} < z \le 1. \end{cases}$$

Thus the function $f(z)$ increases in the interval $(0, \frac{1}{2})$ and decreases in the interval $(\frac{1}{2}, 1)$. Moreover, $f(0) = f(1) = 0$. This implies that $f(z) \ge 0$ for $z \in (0,1)$, whence inequality (32) follows. □

The proof of Theorem 1.2 is based on the following result.

Lemma 3.2. *Let ξ be a random variable with the distribution function $F(x)$ and let (ξ_i) be independent copies of ξ. Further, let the sequence a_n be defined by equality* (6) *and $z_n = \max_{1\le i\le n} \xi_i$. Assume that conditions* (12) *and* (13) *hold. Then*

$$\mathsf{E} \sup_{n\ge1} |z_n - a_n|^q < \infty. \tag{33}$$

Proof of Lemma 3.2 (*case of* $q = 1$). Since

$$\sup_{n\ge1} |z_n - a_n| \le \max\left(\sup_{n\ge1} (z_n - a_n)_+, \sup_{n\ge1} (a_n - z_n)_+ \right)$$

almost surely, it remains to prove that

$$\mathsf{E} \sup_{n\ge1} (z_n - a_n)_+ < \infty, \tag{34}$$

$$\mathsf{E} \sup_{n\ge1} (a_n - z_n)_+ < \infty. \tag{35}$$

First we establish estimate (34). By definition, a_n is a nondecreasing sequence. Thus,

$$\sup_{n\ge1} (z_n - a_n)_+ \le \sup_{n\ge1} (\xi_n - a_n)_+$$

almost surely and (34) follows from

$$\mathsf{E} \sup_{n\ge1} (\xi_n - a_n)_+ < \infty.$$

Therefore, it is sufficient to show that the integral

$$\int_1^\infty \mathsf{P}\left(\sup_{n\ge1} (\xi_n - a_n) > x \right) dx \tag{36}$$

converges (see [2]).

The expression under the sign of the integral in (36) is estimated by

$$\mathsf{P}\left(\sup_{n\ge1} (\xi_n - a_n) > x \right) \le \sum_{n\ge1} \mathsf{P}(\xi_n > x + a_n) = \sum_{n\ge1} \left(1 - F(x + a_n)\right).$$

Recall that $a_n = \varphi(n)$, $n \ge 1$, where $\varphi(y)$ is defined in (6), and that

$$1 - F(x + a_n) \le \int_{n-1}^n \big(1 - F(x + \varphi(y))\big)\, dy.$$

Hence

$$\sum_{n\ge 1}(1-F(x+a_n)) \le \int_0^\infty \big(1-F(x+\varphi(y))\big)\,dy = \int_0^\infty \left(\int_{x+\varphi(y)}^\infty dF(z)\right) dy$$
$$= \int_{x+1}^\infty \left(\int_0^{\frac{1}{1-F(z-x)}} dy\right) dF(z) \le \int_{x+1}^\infty \frac{dF(z)}{1-F(z-x)}.$$

The latter estimate and condition (12) allow us to estimate the integral in (36) as follows:

$$\int_1^\infty \left(\int_{x+1}^\infty \frac{dF(z)}{1-F(z-x)}\right) dx = \int_2^\infty dF(z)\int_1^{z-1} \frac{1}{1-F(z-x)}\,dx$$
$$\le C\int_2^\infty \frac{z\,dF(z)}{1-F(z-1)} < \infty.$$

This completes the proof of inequality (34).

Now we show that inequality (35) holds, too. It is known that

$$\lim_{n\to\infty}(z_n-a_n)=0 \quad \text{almost surely}$$

provided that condition (13) holds (see [1]).

Therefore,

$$\left\{\sup_{n\ge 1}(a_n-z_n)>x\right\} \subset \bigcup_{n\ge 1}\{a_n-z_n>x, a_{n+1}-z_{n+1}<x\}$$

for all $x>0$. Since a_n is an increasing sequence, we get

$$\begin{aligned}\mathsf{P}\left(\sup_{n\ge 1}(a_n-z_n)>x\right) &\le \sum_{n\ge 1}\mathsf{P}(z_n<a_n-x, \xi_{n+1}>a_{n+1}-x)\\ &\le \sum_{n\ge 1}F^n(a_n-x)(1-F(a_n-x)).\end{aligned} \tag{37}$$

Similarly to the proof of inequality (34), we check the convergence of the integral

$$I(x_0)=\int_{x_0}^\infty \mathsf{P}\left(\sup_{n\ge 1}(a_n-z_n)>x\right)dx$$

for some $1<x_0<\infty$:

$$\begin{aligned}I(x_0) &\le \int_{x_0}^\infty \left[\sum_{n\ge 1}F^n(a_n-x)(1-F(a_n-x))\right]dx\\ &= \sum_{n\ge 1}\int_{x_0}^\infty F^n(a_n-x)(1-F(a_n-x))\,dx\\ &= \sum_{n\ge 1}\int_{-\infty}^{a_n-x_0} F^n(y)(1-F(y))\,dy = \int_{-\infty}^\infty \left[\sum_{n\ge m}F^n(y)(1-F(y))\right]dy,\end{aligned} \tag{38}$$

where

$$m=\left[\frac{1}{1-F(x_0+y)}\right].$$

Since $1-\frac{1}{n}=F(a_n)$, the inequality $a_n-x_0\ge y$ is equivalent to

$$n\ge \frac{1}{1-F(x_0+y)}.$$

Now we choose y_0 such that $0 < F(y_0) < 1$. Then

$$\begin{aligned}\int_{-\infty}^{y_0} \sum_{n\ge m} F^n(y)(1-F(y))\,dy &\le \int_{-\infty}^{y_0} \frac{1-F(y)}{1-F(y_0)}\,dy \\ &\le \frac{1}{1-F(y_0)} \int_{-\infty}^{\infty} |y|\,dF(y) < \infty.\end{aligned} \tag{39}$$

Furthermore,

$$\int_{y_0}^{\infty} \sum_{n\ge m} F^n(y)(1-F(y))\,dy = \int_{y_0}^{\infty} (F(y))^{m+1}\,dy \le \int_{y_0}^{\infty} (F(y))^{\frac{1}{1-F(x_0+y)}}\,dy. \tag{40}$$

Put $\overline{F}(y) = 1 - F(y)$. Using the inequalities

$$(1-x)^{1/x} \le \exp(-1) \quad \text{and} \quad \exp(-x) < \frac{1}{x}, \qquad x > 0,$$

we estimate the last integral in (40) as

$$\begin{aligned}\int_{y_0}^{\infty} \left((1-\overline{F}(y))^{\frac{1}{\overline{F}(y)}}\right)^{\frac{1-F(y)}{1-F(x_0+y)}}\,dy &\le \int_{y_0}^{\infty} \frac{1-F(x_0+y)}{1-F(y)}\,dy \\ &\le C\int_{t_0}^{\infty} \frac{1-F(t)}{1-F(t-x_0)}\,dt,\end{aligned} \tag{41}$$

where $t_0 = x_0 + y_0$.

It remains to observe that the integral in condition (12) with $q = 1$ can be rewritten as follows:

$$\begin{aligned}\int_{y_0}^{\infty} \frac{y\,dF(y)}{1-F(y-y_0)} = &-\frac{y(1-F(y))}{1-F(y-y_0)}\bigg|_{y_0}^{\infty} + \int_{y_0}^{\infty} \frac{(1-F(y))}{1-F(y-y_0)}\,dy \\ &+ \int_{y_0}^{\infty} \frac{y(1-F(y))}{(1-F(y-y_0))^2}\,dF(y-y_0).\end{aligned} \tag{42}$$

The first integral on the right-hand side of (42) is bounded, while the second one coincides with the integral in (41).

The proof that the convergence of the integral in (12) yields that estimate (35) is complete in the case of $q = 1$. □

Proof of Lemma 3.2 (*case of* $q > 1$). First we assume that $\xi_k \ge 0$ almost surely for all $k \ge 1$. By Lemma 3.1,

$$\sup_{n\ge 1} |z_n - a_n|^q \le \sup_{n\ge 1} |z_n^q - a_n^q|. \tag{43}$$

Set

$$\xi_k^* = \xi_k^q, \qquad a_n^* = a_n^q, \qquad z_n^* = \max_{1\le k\le n} \xi_k^*,$$
$$F^*(x) = \mathsf{P}(\xi_k^* < x) = F\big(x^{1/q}\big).$$

The reasoning in the proof of the case of $q = 1$ implies that

$$\mathsf{E}\sup_{n\ge 1} |z_n^* - a_n^*| = \mathsf{E}\sup_{n\ge 1} |z_n^q - a_n^q| < \infty$$

if, for some $y_0 > 0$,

$$\int_{y_0}^{\infty} \frac{y\,dF^*(y)}{1-F^*(y-y_0)} = \int_{y_0}^{\infty} \frac{y\,dF(y^{1/q})}{1-F((y-y_0)^{1/q})} = \int_{t_0}^{\infty} \frac{t^q\,dF(t)}{1-F\left(\left(t^q - t_0^q\right)^{1/q}\right)} < \infty, \tag{44}$$

where $t_0 = y_0^{1/q}$. This together with bounds (12) and (43) proves (35).

Now let ξ_k be arbitrary random variables. Then

$$\sup_{n\ge 1}|z_n - a_n|^q = \sup_{n\ge 1}|(z_n)_+ - (z_n)_- - a_n|^q$$

$$\le 2^{q-1}\left(\sup_{n\ge 1}|(z_n)_+ - a_n|^q + \sup_{n\ge 1}|(z_n)_-|^q\right). \tag{45}$$

It is clear that

$$(z_n)_+ = \max_{1\le k\le n}(\xi_k)_+ \tag{46}$$

and

$$(z_n)_- \le |\xi_1|. \tag{47}$$

It is seen from the definition of a_n that the sequence (a_n) is the same for both sequences of random variables (ξ_n) and $((\xi_n)_+)$. Therefore the results obtained above and relations (46) and (47) imply

$$\mathsf{E}\sup_{n\ge 1}|(z_n)_-|^q < \infty, \qquad \mathsf{E}\sup_{n\ge 1}|(z_n)_+ - a_n|^q < \infty.$$

These bounds together with inequality (45) complete the proof of the case $q > 1$. Lemma 3.2 is proved. □

Proof of Theorem 1.2. Now the proof of Theorem 1.2 is easy. The proof follows the lines of that of Theorem 1.1. Note, however, that we will obtain

$$\mathsf{P}\left(Z_n(t) - a_n\sigma(t) \to 0 \ \text{ almost everywhere in } T\right) = 1 \tag{48}$$

and

$$\mathsf{P}\left(|Z_n(t) - a_n\sigma(t)| \le Z(t) \ \text{ almost everywhere in } T\right) = 1 \tag{49}$$

instead of relations (26) and (27), where $Z(t) = \sigma(t)\sup_{n\ge 1}|\widetilde{Z}_n(t) - a_n|$.

Equality (48) follows from some of results in [1] and Fubini's theorem.

The finiteness of $\mathsf{E}\,\|Z_n\|^q$ is derived from bound (33) of Lemma 2.3, since the Banach ideal space is q-concave. □

Corollary 1.3 is, in fact, proved in the course of the proof of Theorem 1.1. Indeed, condition (15) implies

$$z_n - a_n \xrightarrow{\mathsf{P}} 0$$

as $n \to \infty$ (see [4] or [3]). The latter relation together with inequality (16) implies the convergence of moments, as well,

$$\mathsf{E}\,|z_n - a_n|^q \to 0, \qquad n \to \infty,$$

(see [9]). Similarly to the proof of Theorem 1.1, we obtain

$$\mathsf{E}\,\|Z_n - a_n\mathfrak{S}X\|^q \to 0.$$

Examples. 1. The distribution function

$$F_1(x) = \begin{cases} 1 - \exp(-\ln^\alpha x), & \text{for } x > 1, \\ 0, & \text{for } x \le 1, \end{cases}$$

satisfies conditions (7) and (8) if $\alpha > 1$. However, these conditions do not hold if $\alpha = 1$. Note also that condition (11) does not hold for $F_1(x)$ if $\alpha = 2$ and $q = 1$.

2. The distribution function

$$F_2(x) = \begin{cases} 1 - \exp(-x^\alpha), & \text{for } x > 0, \\ 0, & \text{for } x \le 0, \end{cases}$$

satisfies conditions (12) and (13) if $\alpha > q$. Condition (12) does not hold for this distribution function if $\alpha = q$.

Bibliography

1. O. Barndorff-Nielsen, *On the limit behaviour of extreme order statistics*, Ann. Math. Statist. **34** (1963), no. 3, 992–1002. MR0150889 (27:875)
2. W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. II, John Wiley & Sons, Inc., New York–London–Sydney, 1971. MR0270403 (42:5292)
3. J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, John Wiley & Sons, New York–Chichester–Brisbane, 1978. MR489334 (80b:60040)
4. B. V. Gnedenko, *Sur la distribution limit du terme maximum d'une serie aleatoire*, Ann. Math. **44** (1943), no. 3, 423–453. MR0008655 (5:41b)
5. L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, "Nauka", Moscow, 1984; English transl., Pergamon Press, Oxford–Elmsford, N.Y., 1982. MR664597 (83h:46002)
6. J. Lindenstraus and L. Tzafriri, *Classical Banach Spaces*, Springer-Verlag, Berlin-New York, 1973. MR0415253 (54:3344)
7. I. K. Matsak, *Estimates for the moments of the supremum of normalized sums of independent random variables*, Teor. Imovirnost. Matem. Statist. **67** (2002), 104–116; English transl. in Theory Probab. Mathem. Statist. **67** (2003), 115–128. MR1956624 (2004i:60061)
8. I. K. Matsak and A. M. Plichko, *On the maxima of independent random elements in a Banach functional lattice*, Teor. Imovir. Mat. Stat. **61** (1999), 105–116; English transl. in Theory Probab. Math. Statist. **61** (2000), 109–120. MR1866964 (2002k:60020)
9. J. Pickands, *Moment convergence of sample extremes*, Ann. Math. Statist. **39** (1968), no. 3, 881–889. MR0224231 (36:7275)

Operations Research Department, Faculty for Cybernetics, Kiev National Taras Shevchenko University, Glushkov Avenue 2, Building 6, Kyiv 03127, Ukraine

E-mail address: k_m_s_kirovograd@mail.ru

Operations Research Department, Faculty for Cybernetics, Kiev National Taras Shevchenko University, Glushkov Avenue 2, Building 6, Kyiv 03127, Ukraine

E-mail address: ivanmatsak@gmail.com

Received 19/MAY/2011

Translated by N. SEMENOV