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AN INVESTIGATION OF AN $M^\theta/G/1/m$ QUEUEING SYSTEM WITH SERVICE MODE SWITCHING

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Abstract. We consider an $M^\theta/G/1/m$ queueing system with two service modes. The server switches between the modes when the number of customers present at the beginning of service is larger than h. Laplace transforms for the distributions of the number of customers in the system during the busy period and for the distribution function of the busy period are found. The average duration of the busy period is obtained. Formulas for the stationary distribution of the number of customers in the system and for the stationary characteristics of the system are established.

1. Introduction

The models of queueing systems where the service intensity depends on the number of customers present in the system as well as those with bulk arrivals are frequently used to describe telecommunication processes [1]. The results obtained in the papers [2]–[5] for queueing systems M/G/1 and $M^\theta/G/1$ with two service modes concern the stationary distribution of the number of customers in the system; some optimization problems are also solved in [2]–[5].

The method of embedded Markov chains is often used to establish the stationary distribution of the number of customers in the system. Following this method, a system of linear algebraic equations for stationary probabilities is obtained by the full probability formula and then the system is solved by using the moment generating functions [6]. The stationary distribution established with the help of embedded Markov chains corresponds to the service completion epoch and, generally speaking, does not coincide with the stationary distribution for arbitrary $t \to \infty$ in the case of a queueing system $M^\theta/G/1/m$ with a bounded queue.

Another approach is developed by Korolyuk [7]. His approach is applied in [8]–[11] to analyze the queueing systems M/G/1 and $M^\theta/G/1$ for both cases of bounded and unbounded queues. This approach allows one to develop an effective algorithm for the evaluation of the stationary distribution of the number of customers as well as to analyze both the stationary and transient modes for the system $M^\theta/G/1/m$. The Korolyuk approach is based on the method of potential for lower continuous random walks.

The systems $M^\theta/G/1/m$ and $M^\theta/G/1$ with the blocking strategy for the input flow and switches between the service modes (with r levels of switching $h_1 < h_2 < \cdots < h_r$) are studied in the papers [12, 13] by using the method of potential. The input flow is blocked if the total number of customers in the system exceeds a given threshold level $h = h_1$ at the moment when the server starts the service of a customer. Since the input

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flow is blocked during the threshold modes with the distributions $F_1(t), \dots, F_r(t)$ of the service times that differ from the main distribution F, it is possible to apply the method of potential with a single lower continuous random walk related to $F(t)$.

In the current paper, we consider the queueing system $\mathrm{M}^\theta/\mathrm{G}/1/\mathrm{m}$ with two service modes. We also assume that the input flow is blocked if the total number of customers in the system exceeds a fixed number m. The study of such a system is based on the method of potential for two lower continuous random walks corresponding to the main and threshold modes of service with the distributions $F(t)$ and $\widetilde{F}(t)$ of the service duration, respectively.

2. Description of the model

Let $\{\alpha_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$, $n \geq 1$, be sequences of independent identically distributed random variables, where α_n is the interarrival time between arrivals of the $(n-1)$-st and n-th groups of customers, θ_n is the number of customers in an n-th group, and β_n is the service duration of a customer n. We assume that

$$\mathsf{P}\{\alpha_n < x\} = 1 - e^{-\lambda' x}, \qquad \lambda' > 0,$$

and that $\mathsf{P}\{\theta_n = i\} = a'_i$, $i \geq 0$, where

$$a'_0 < 1, \qquad \sum_{i=0}^{\infty} a'_i = 1.$$

If $\mathsf{P}\{\theta_n > 1\} = 0$, then each group contains only one customer. If $a'_0 > 0$, then the input flow may be thinned.

Only one customer can be served at each moment. A customer leaves the system after its service is complete. After a service of a customer is complete and if there is at least one customer in the queue, the server chooses another customer from the queue and starts the service; otherwise, the server waits until the next group of customers arrives to the system. The system follows the FIFO service discipline. The queue discipline within a group can be arbitrary.

Let m be the maximal number of customers that are allowed to stay in the queue. This means that if there are $k \in [0, m+1]$ customers in the system and a group of θ_n customers arrives, then only $\min\{\theta_n, m+1-k\}$ new customers are allowed to join the queue, while the rest of them disappear.

Denote by $\xi(t)$ the total number of customers present in the system at time t. Introduce the so-called threshold level h, $h = 1, \dots, m-1$, for the process $\xi(t)$. Namely, if t is the time when the service of an n-th customer starts and $\xi(t) \leq h$, then

$$\mathsf{P}\{\beta_n < x\} = F(x), \qquad x \geq 0, \quad F(0) = 0.$$

Otherwise, that is, if $\xi(t) > h$, then

$$\mathsf{P}\{\beta_n < x\} = \widetilde{F}(x), \qquad x \geq 0, \quad \widetilde{F}(0) = 0.$$

We denote the queueing system described above by $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ (the subscript for G means the number of existing threshold modes of service). The stochastic process describing the functioning of a system $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ belongs to the class of the so-called switching processes [1].

3. Main notation and auxiliary results

Denote by P_n the conditional probability given that there are $n \geq 0$ customers in the system at the initial time and by E (P) the conditional expectation (conditional probability) given that the system starts functioning at the moment when the first group of customers arrives to the system.

It is more convenient to deal with an input flow such that $\mathsf{P}\{\theta_n = i\} = a_i$, $i \geq 1$, where

$$\sum_{i=1}^{\infty} a_i = 1.$$

One can always transform the input process and reduce it to the latter form by putting $a_i = a'_i/(1-a'_0)$. Then the intensity of the input flow is $\lambda = \lambda'(1-a'_0)$, where λ' denotes the intensity of the input flow that can be thinned.

We also introduce the following notation: $\eta(x)$ is the total number of customers arriving to the system during the time interval $[0;x)$; a_i^{k*} is the k-tuple convolution of the sequence a_i with itself; ρ_k is the stationary distribution of the number of customers in the system; and $a(s,z) = s + \lambda\bigl(1-\alpha(z)\bigr)$.

Let

$$f(s) = \int_0^\infty e^{-sx}\,dF(x), \qquad m_1 = \int_0^\infty x\,dF(x) < \infty, \qquad \overline{F}(x) = 1 - F(x);$$

$$b_1 = \sum_{k=1}^{\infty} k a_k < \infty; \qquad \alpha(z) = \sum_{k=1}^{\infty} z^k a_k; \qquad \overline{a}_n = \sum_{k=n}^{\infty} a_k,$$

$$\overline{p}_n(s) = \sum_{k=n}^{\infty} p_k(s), \qquad \overline{q}_n(s) = \sum_{k=n}^{\infty} q_k(s), \qquad \sum_{k=1}^{0} b_k = 0.$$

We define the sequences $p_i(s)$ $(i = -1, 0, 1, \dots)$ and $q_i(s)$ $(i = 0, 1, \dots)$ for $\operatorname{Re} s \geq 0$ as follows

$$\sum_{i=-1}^{\infty} z^i p_i(s) = \frac{f(a(s,z))}{z f(s)}, \qquad \sum_{i=0}^{\infty} z^i q_i(s) = \frac{1 - f(a(s,z))}{a(s,z)}, \tag{1}$$

that is,

$$\begin{aligned} p_i(s) &= \frac{1}{f(s)} \int_0^\infty e^{-sx}\,\mathsf{P}\{\eta(x) = i+1\}\,dF(x) \\ &= \frac{1}{f(s)} \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!}\,dF(x); \end{aligned} \tag{2}$$

$$q_i(s) = \int_0^\infty e^{-sx}\,\mathsf{P}\{\eta(x) = i\}\overline{F}(x)\,dx = \sum_{k=0}^{i} a_i^{k*} \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!} \overline{F}(x)\,dx.$$

We introduce the functions $R_k(s)$, $k = 1, 2, 3, \dots$, by

$$\sum_{k=1}^{\infty} z^k R_k(s) = \frac{z}{f\bigl(a(s,z)\bigr) - z}, \qquad |z| < \nu_-(s), \tag{3}$$

where $\nu_-(s)$ is a unique root of the equation $f\bigl(a(s,z)\bigr) = z$ in the interval $[0;1]$.

According to [10], one can treat $p_i(s)$ as the distribution of jumps of a certain lower continuous random walk that depends on a parameter $s \geq 0$ and corresponds to the distribution function $F(x)$ of the main mode of service.

Let

$$\begin{aligned} &\rho = \lambda m_1 b_1, \qquad \nu_- = \lim_{s\to+0,\rho>1} \nu_-(s); \\ &p_i = \lim_{s\to+0} p_i(s), \qquad R_i = \lim_{s\to+0} R_i(s), \qquad q_i = \lim_{s\to+0} q_i(s). \end{aligned} \tag{4}$$

Then equalities (1)–(3) imply

$$\sum_{i=-1}^{\infty} z^i p_i = \frac{f\big(\lambda(1-\alpha(z))\big)}{z}; \qquad p_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!}\, dF(x);$$

$$\sum_{k=1}^{\infty} z^k R_k = \frac{z}{f\big(\lambda(1-\alpha(z))\big) - z}, \qquad |z| < \min\{1, \nu_-\};$$

$$\sum_{i=0}^{\infty} z^i q_i = \frac{1 - f\big(\lambda(1-\alpha(z))\big)}{\lambda(1-\alpha(z))}, \qquad q_i = \sum_{k=0}^{i} a_i^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} \overline{F}(x)\, dx,$$

$$\sum_{k=0}^{\infty} q_k = m_1.$$

The functions and constants corresponding to the lower continuous random walk that are related to the distribution $\widetilde{F}(x)$ of the service duration for the threshold mode are evaluated similarly. These functions and constants are equipped with a tilde, namely

$$\widetilde{f}(s) = \int_0^\infty e^{-sx}\, d\widetilde{F}(x), \qquad \widetilde{m}_1 = \int_0^\infty x\, d\widetilde{F}(x) < \infty, \qquad \overline{\widetilde{F}}(x) = 1 - \widetilde{F}(x);$$

$$\widetilde{p}_i(s) = \frac{1}{\widetilde{f}(s)} \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!}\, d\widetilde{F}(x);$$

$$\widetilde{q}_i = \sum_{k=0}^{i} a_i^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} \overline{\widetilde{F}}(x)\, dx; \qquad \sum_{k=0}^{\infty} \widetilde{q}_k = \widetilde{m}_1,$$

and so on.

4. Distribution of the number of customers in a busy system

Let

$$\tau = \inf\{t \ge 0 \colon \xi(t) = 0\}$$

denote the first busy period for the system $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ and let

$$\varphi_n(t,k) = \mathsf{P}_n\{\xi(t) = k, \tau > t\}, \qquad 1 \le n, k \le m+1,$$

$$\Phi_n(s,k) = \int_0^\infty e^{-st} \varphi_n(t,k)\, dt, \qquad \operatorname{Re} s > 0.$$

It is clear that $\varphi_0(t,k) = 0$. The full probability formula implies

$$\begin{aligned} \varphi_n(t,k) &= \sum_{j=0}^{m-n} \int_0^t \mathsf{P}\{\eta(x) = j\} \varphi_{n+j-1}(t-x,k)\, dF(x) \\ &+ \int_0^t \mathsf{P}\{\eta(x) \ge m+1-n\} \varphi_m(t-x,k)\, dF(x) \\ &+ \big(\mathsf{P}\{\eta(t) = k-n\} + I\{k = m+1\}\, \mathsf{P}\{\eta(t) \ge m+2-n\}\big)\, \overline{F}(t), \\ &\qquad 1 \le n \le h, \end{aligned} \tag{5}$$

and

$$\begin{aligned}\varphi_n(t,k) &= \sum_{j=0}^{m-n} \int_0^t \mathsf{P}\{\eta(x)=j\}\varphi_{n+j-1}(t-x,k)\,d\widetilde{F}(x)\\ &+ \int_0^t \mathsf{P}\{\eta(x)\ge m+1-n\}\varphi_m(t-x,k)\,d\widetilde{F}(x)\\ &+ \left(\mathsf{P}\{\eta(t)=k-n\} + I\{k=m+1\}\,\mathsf{P}\{\eta(t)\ge m+2-n\}\right)\overline{\widetilde{F}}(t),\\ &\qquad h+1\le n\le m.\end{aligned}$$

Here $I\{A\}$ is the indicator of a random event A; it equals 1 or 0 depending on whether or not the event A occurs.

Let

$$\widetilde{A}_n(s) = 1 + \bigl(1-\widetilde{f}(s)\bigr)\sum_{i=1}^{m-n}\widetilde{R}_i(s), \qquad C_n(s) = R_{h-n}(s) - f(s)\sum_{i=1}^{h-n} R_i(s)p_{h-n-i}(s),$$
$$f_n(s,k) = q_{k-n}(s) + I\{k=m+1\}\overline{q}_{m+2-n}(s),$$
$$\widetilde{f}_n(s,k) = \widetilde{q}_{k-n}(s) + I\{k=m+1\}\overline{\widetilde{q}}_{m+2-n}(s).$$

Passing to the Laplace transform on both sides of equalities (5) and taking into account relations (2) and similar relations with $\widetilde{p}_i(s)$ and $\widetilde{q}_i(s)$ instead of $p_i(s)$ and $q_i(s)$, we obtain the system of equations with respect to the functions $\Phi_n(s,k)$:

$$\Phi_n(s,k) = f(s)\sum_{j=0}^{m-n} p_{j-1}(s)\Phi_{n+j-1}(s,k) + f(s)\overline{p}_{m-n}(s)\Phi_m(s,k) + f_n(s,k), \qquad 1\le n\le h, \tag{6}$$

$$\Phi_n(s,k) = \widetilde{f}(s)\sum_{j=0}^{m-n} \widetilde{p}_{j-1}(s)\Phi_{n+j-1}(s,k) + \widetilde{f}(s)\overline{\widetilde{p}}_{m-n}(s)\Phi_m(s,k) + \widetilde{f}_n(s,k), \qquad h+1\le n\le m, \tag{7}$$

and the boundary condition

$$\Phi_0(s,k) = 0. \tag{8}$$

For a moment, we consider (7) as a separate system of equations with respect to the functions $\Phi_n(s,k)$, $k=h,\dots,m$. Now we apply Theorem 2 of the paper [10] for the solutions of an equation in an interval together with the equality

$$\widetilde{f}(s)\sum_{j=1}^{n}\widetilde{R}_j(s)\overline{\widetilde{p}}_{n-j}(s) = \widetilde{R}_n(s) - \bigl(1-\widetilde{f}(s)\bigr)\sum_{j=1}^{n}\widetilde{R}_j(s) - 1, \qquad n\ge 1, \tag{9}$$

that follows from a relation similar to (3) but with $\widetilde{R}_i(s)$ instead of $R_i(s)$. Then solutions of system (7) are given by

$$\Phi_n(s,k) = \widetilde{A}_n(s)\Phi_m(s,k) - \sum_{i=1}^{m-n}\widetilde{R}_i(s)\widetilde{f}_{n+i}(s,k), \qquad h\le n\le m. \tag{10}$$

The system of equations (6) is rewritten as

$$\Phi_n(s,k) - f(s)\sum_{j=-1}^{h-n-1} p_j(s)\Phi_{n+j}(s,k) = f(s)\sum_{j=h-n}^{m-n-1} p_j(s)\Phi_{n+j}(s,k) + f(s)\overline{p}_{m-n}(s)\Phi_m(s,k) + f_n(s,k), \quad 1 \le n \le h. \tag{11}$$

We again use Theorem 2 of [10] and deduce from (11) that

$$\Phi_n(s,k) = R_{h-n}(s)\Phi_h(s,k) - \sum_{i=1}^{h-n} R_i(s)\Bigg(f(s)\sum_{j=h}^{m-1} p_{j-n-i}\Phi_j(s,k) + f(s)\overline{p}_{m-n-i}(s)\Phi_m(s,k) + f_{n+i}(s,k)\Bigg), \qquad 1 \le n \le h. \tag{12}$$

Substituting the expressions obtained from equality (10) to representation (12), we get for the functions $\Phi_n(s,k)$ that

$$\Phi_n(s,k) = \widetilde{D}_n(s)\Phi_m(s,k) - D_n(s,k), \qquad 1 \le n \le h-1, \tag{13}$$

for $h \le n \le m-1$, where

$$\widetilde{D}_n(s) = C_n(s)\widetilde{A}_h(s) - f(s)\sum_{i=1}^{h-n} R_i(s)\Bigg(\sum_{j=h+1}^{m-1} p_{j-n-i}(s)\widetilde{A}_j(s) + \overline{p}_{m-n-i}(s)\Bigg),$$

$$D_n(s,k) = C_n(s)\sum_{i=1}^{m-h} \widetilde{R}_i(s)\widetilde{f}_{h+i}(s,k) - f(s)\sum_{u=1}^{h-n} R_u(s)\sum_{j=h+1}^{m-1} p_{j-n-u}(s)\sum_{i=1}^{m-j} \widetilde{R}_i(s)\widetilde{f}_{j+i}(s,k) + \sum_{i=1}^{h-n} R_i(s) f_{n+i}(s,k). \tag{14}$$

Considering equality (13) with $n = 0$, we derive from the boundary condition (8) that

$$\Phi_m(s,k) = \frac{D_0(s,k)}{\widetilde{D}_0(s)}. \tag{15}$$

Therefore we proved the following result.

Theorem 4.1. *For arbitrary* $1 \le k \le m+1$ *and* $\operatorname{Re} s > 0$,

$$\int_0^\infty e^{-st}\,\mathsf{P}_n\{\xi(t) = k, \tau > t\}\,dt = \widetilde{D}_n(s)\Phi_m(s,k) - D_n(s,k), \qquad 1 \le n \le h-1;$$

$$\int_0^\infty e^{-st}\,\mathsf{P}_n\{\xi(t) = k, \tau > t\}\,dt = \widetilde{A}_n(s)\Phi_m(s,k) - \sum_{i=1}^{m-n} \widetilde{R}_i(s)\widetilde{f}_{n+i}(s,k), \quad h \le n \le m-1,$$

where the function $\Phi_m(s,k)$ *is defined by equality* (15).

5. Busy period and stationary distribution

If the system starts functioning at the moment when the first group of customers arrives, then

$$\int_0^\infty e^{-st}\,\mathsf{P}\{\xi(t)=k,\tau>t\}\,dt=\sum_{n=1}^{m}a_n\Phi_n(s,k)+\overline{a}_{m+1}\Phi_{m+1}(s,k) \tag{16}$$

for all $1\le k\le m+1$ by the full probability formula. Since

$$\varphi_{m+1}(t,k)=\int_0^t\varphi_m(t-x,k)\,d\widetilde{F}(x)+I\{k=m+1\}\overline{\widetilde{F}}(t),$$

$$\Phi_{m+1}(s,k)=\widetilde{f}(s)\Phi_m(s,k)+I\{k=m+1\}\frac{1-\widetilde{f}(s)}{s},$$

we rewrite equality (16) as

$$\begin{aligned}\int_0^\infty & e^{-st}\,\mathsf{P}\{\xi(t)=k,\tau>t\}\,dt\\ &=\left(\sum_{n=1}^{h-1}a_n\widetilde{D}_n(s)+\sum_{n=h}^{m}a_n\widetilde{A}_n(s)+\overline{a}_{m+1}\widetilde{f}(s)\right)\Phi_m(s,k)-\sum_{n=1}^{h-1}a_nD_n(s,k)\\ &\quad-\sum_{n=h}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i(s)\widetilde{f}_{n+i}(s,k)+\overline{a}_{m+1}I\{k=m+1\}\frac{1-\widetilde{f}(s)}{s}\end{aligned} \tag{17}$$

by using relations (10) and (13).

To obtain a representation for

$$\int_0^\infty e^{-st}\,\mathsf{P}\{\tau>t\}\,dt$$

we sum up equalities (17) for k running from 1 to $m+1$.

It is easy to check that

$$\sum_{k=1}^{m+1}f_n(s,k)=\sum_{k=0}^{\infty}q_k(s)=\frac{1-f(s)}{s},$$

$$\sum_{k=1}^{m+1}\widetilde{f}_n(s,k)=\sum_{k=0}^{\infty}\widetilde{q}_k(s)=\frac{1-\widetilde{f}(s)}{s}.$$

Denoting the sum

$$\sum_{k=1}^{m+1}D_n(s,k)$$

by $D_n(s)$, we establish from (14) that

$$\begin{aligned}D_n(s)&=\frac{1-\widetilde{f}(s)}{s}\left(C_n(s)\sum_{i=1}^{m-h}\widetilde{R}_i(s)-f(s)\sum_{u=1}^{h-n}R_u(s)\sum_{j=h+1}^{m-1}p_{j-n-u}(s)\sum_{i=1}^{m-j}\widetilde{R}_i(s)\right)\\ &\quad+\frac{1-f(s)}{s}\sum_{i=1}^{h-n}R_i(s).\end{aligned} \tag{18}$$

Therefore representation (17) implies the following result.

Theorem 5.1. *The Laplace transform of the distribution function of a busy period in the system* $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ *is given by*

$$\begin{aligned}\int_0^\infty e^{-st}\,\mathsf{P}\{\tau>t\}\,dt &= \Bigg(\sum_{n=1}^{h-1} a_n\widetilde{D}_n(s)+\sum_{n=h}^{m} a_n\widetilde{A}_n(s)+\overline{a}_{m+1}\widetilde{f}(s)\Bigg)\frac{D_0(s)}{\widetilde{D}_0(s)}\\ &\quad-\sum_{n=1}^{h-1} a_nD_n(s)-\frac{1-\widetilde{f}(s)}{s}\Bigg(\sum_{n=h}^{m-1} a_n\sum_{i=1}^{m-n}\widetilde{R}_i(s)-\overline{a}_{m+1}\Bigg).\end{aligned}\tag{19}$$

Now our aim is to justify the passing to the limit as $s\to+0$ in (19). We use the sequences $\{p_i\}$, $\{q_i\}$, and $\{R_i\}$ introduced in (4) and the sequences $\{\widetilde{p}_i\}$, $\{\widetilde{q}_i\}$, and $\{\widetilde{R}_i\}$ defined by

$$\widetilde{p}_i=\lim_{s\to+0}\widetilde{p}_i(s),\qquad \widetilde{R}_i=\lim_{s\to+0}\widetilde{R}_i(s),\qquad \widetilde{q}_i=\lim_{s\to+0}\widetilde{q}_i(s).$$

Since

$$f(0)=\widetilde{f}(0)=1,\qquad \lim_{s\to+0}\frac{1-f(s)}{s}=m_1,\qquad \lim_{s\to+0}\frac{1-\widetilde{f}(s)}{s}=\widetilde{m}_1,$$

$$\widetilde{A}_n(0)=1,\qquad C_n(0)=R_{h-n}-\sum_{i=1}^{h-n}R_ip_{h-n-i}=p_{-1}R_{h+1-n},\qquad R_n-\sum_{i=1}^{n}R_i\overline{p}_{n-i}=1,$$

we deduce from (14) and (18) that

$$\widetilde{D}_n(0)=C_n(0)-\sum_{i=1}^{h-n}R_i\Bigg(\sum_{j=h+1}^{m-1}p_{j-n-i}+\overline{p}_{m-n-i}\Bigg)=R_{h-n}-\sum_{i=1}^{h-n}R_i\overline{p}_{h-n-i}=1,$$

$$D_n=D_n(0)=\widetilde{m}_1\Bigg(C_n(0)\sum_{i=1}^{m-h}\widetilde{R}_i-\sum_{u=1}^{h-n}R_u\sum_{j=h+1}^{m-1}p_{j-n-u}\sum_{i=1}^{m-j}\widetilde{R}_i\Bigg)+m_1\sum_{i=1}^{h-n}R_i.$$

Passing to the limit in (19) as $s\to+0$ we derive the formula for the mean duration of a busy period.

Theorem 5.2. *The mean duration of a busy period in a queueing system* $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ *is given by*

$$\begin{aligned}\mathsf{E}\,\tau &= D_0-\sum_{n=1}^{h-1}a_nD_n-\widetilde{m}_1\Bigg(\sum_{n=h}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i-\overline{a}_{m+1}\Bigg)\\ &= m_1\Bigg(\sum_{i=1}^{h}R_i-\sum_{n=1}^{h-1}a_n\sum_{i=1}^{h-n}R_i\Bigg)\\ &\quad+\widetilde{m}_1\Bigg(p_{-1}R(h)\sum_{i=1}^{m-h}\widetilde{R}_i-\sum_{j=h+1}^{m-1}r_j(h)\sum_{i=1}^{m-j}\widetilde{R}_i-\sum_{n=h}^{m-1}a_n\sum_{i=1}^{m-n}\widetilde{R}_i+\overline{a}_{m+1}\Bigg),\end{aligned}\tag{20}$$

where

$$R(h)=R_{h+1}-\sum_{n=1}^{h-1}a_nR_{h+1-n},\qquad r_j(h)=\sum_{u=1}^{h}R_up_{j-u}-\sum_{n=1}^{h-1}a_n\sum_{u=1}^{h-n}R_up_{j-n-u}.$$

Since the queue is assumed to be bounded and since the input flow is Poissonian, the stationary distribution of the number of customers in a system $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ exists if the expectations of the random variables θ_n, β_n, and $\widetilde{\beta}_n$ are finite, that is, if $b_1<\infty$, $m_1<\infty$, and $\widetilde{m}_1<\infty$.

Now we are going to derive a formula for the stationary distribution of the number of customers in the system. Denote by τ_i, $i = 0, 1, 2, \dots$, $\tau_0 = 0$, the sequential moments when the system becomes idle. It is clear that $\tau_1 = \tau$.

Let ξ_i, $i = 0, 1, 2, \dots$, $\xi_0 = 0$, be the durations of the sequential idle intervals of the system after the moments τ_i until the next group of customers arrives to the system. It is obvious that the ξ_i are exponential random variables with parameter λ and that the moments $\xi_i + \tau_i$, $i = 0, 1, 2, \dots$, form a renewal process. Denote by $H(x)$ the renewal function corresponding to the latter renewal process, that is,

$$H(x) = \sum_{i=0}^{\infty} \Phi^{*i}(x),$$

where $\Phi(x) = \mathsf{P}\{\tau + \xi_1 < x\}$ and where $\Phi^{*k}(x)$ is the convolution of order k of the distribution function $\Phi(x)$ with itself.

Then, for $0 < k \le m + 1$,

$$\mathsf{P}\{\xi(t) = k\} = \mathsf{P}\{\xi(t) = k, \tau \ge t\} + \int_0^t \mathsf{P}\{\xi(t-u) = k\}\, d\Phi(u),$$
$$\mathsf{P}\{\xi(t) = 0\} = \mathsf{P}\{\tau < t, \tau + \xi_1 \ge t\} + \int_0^t \mathsf{P}\{\xi(t-u) = 0\}\, d\Phi(u).$$

Hence

$$\mathsf{P}\{\xi(t) = k\} = \int_0^t \mathsf{P}\{\xi(t-u) = k, \tau \ge t-u\}\, dH(u), \qquad k = 1, \dots, m+1,$$
$$\mathsf{P}\{\xi(t) = 0\} = \int_0^t \mathsf{P}\{\tau < t-u, \tau + \xi_1 \ge t-u\}\, dH(u).$$

Applying the key renewal theorem (see [14]), we evaluate the limits

$$\lim_{t\to\infty} \mathsf{P}\{\xi(t) = k\} = \frac{\lambda}{1 + \lambda \mathsf{E}\,\tau} \int_0^\infty \mathsf{P}\{\xi(u) = k, \tau \ge u\}\, du, \qquad k = 1, \dots, m+1,$$
$$\lim_{t\to\infty} \mathsf{P}\{\xi(t) = 0\} = \frac{\lambda}{1 + \lambda \mathsf{E}\,\tau} \int_0^\infty \mathsf{P}\{\tau < u, \tau + \xi_1 \ge u\}\, du.$$

Since $\mathsf{P}\{\tau < u, \tau + \xi_1 \ge u\} = \mathsf{P}\{\tau + \xi_1 \ge u\} - \mathsf{P}\{\tau \ge u\}$,

$$\int_0^\infty \mathsf{P}\{\tau < u, \tau + \xi_1 \ge u\}\, du = \frac{1}{\lambda}. \tag{21}$$

Passing to the limit in (17) as $s \to +0$ we get

$$\begin{aligned}
&\int_0^\infty \mathsf{P}\{\xi(t) = k, \tau > t\}\, dt \\
&\quad = p_{-1} R_{h+1} \sum_{i=1}^{m-h} \widetilde{R}_i \widetilde{f}_{h+i}(k) - \sum_{u=1}^{h} R_u \sum_{j=h+1}^{m-1} p_{j-u} \sum_{i=1}^{m-j} \widetilde{R}_i \widetilde{f}_{j+i}(k) + \sum_{i=1}^{h} R_i f_i(k) \\
&\quad - \sum_{n=1}^{h-1} a_n \Bigg(p_{-1} R_{h+1-n} \sum_{i=1}^{m-h} \widetilde{R}_i \widetilde{f}_{h+i}(k) \\
&\qquad - \sum_{u=1}^{h-n} R_u \sum_{j=h+1}^{m-1} p_{j-n-u} \sum_{i=1}^{m-j} \widetilde{R}_i \widetilde{f}_{j+i}(k) + \sum_{i=1}^{h-n} R_i f_{n+i}(k) \Bigg) \\
&\quad - \sum_{n=h}^{m-1} a_n \sum_{i=1}^{m-n} \widetilde{R}_i \widetilde{f}_{n+i}(k) + I\{k = m+1\} \widetilde{m}_1 \overline{a}_{m+1},
\end{aligned} \tag{22}$$

where

$$f_n(k) = q_{k-n} + I\{k = m+1\}\overline{q}_{m+2-n}, \qquad \widetilde{f}_n(k) = \widetilde{q}_{k-n} + I\{k = m+1\}\overline{\widetilde{q}}_{m+2-n}.$$

After simple algebra we derive the following result by using equalities (21)–(22).

Theorem 5.3. *The stationary distribution of the number of customers in a system* $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ *is given by*

$$\rho_0 = \frac{1}{1+\lambda\,\mathsf{E}\,\tau}, \qquad \rho_k = \frac{\lambda}{1+\lambda\mathsf{E}\tau}\left(\sum_{i=1}^{k} R_i q_{k-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{k-n} R_i q_{k-n-i}\right) \tag{23}$$

for $k = 1, \dots, h$, *or*

$$\rho_k = \frac{\lambda}{1+\lambda\mathsf{E}\tau}\Bigg(\sum_{i=1}^{h} R_i q_{k-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i q_{k-n-i} + p_{-1}R(h)\sum_{i=1}^{k-h}\widetilde{R}_i\widetilde{q}_{k-h-i} - \sum_{j=h+1}^{k-1} r_j(h)\sum_{i=1}^{k-j}\widetilde{R}_i\widetilde{q}_{k-j-i} - \sum_{n=h}^{k-1} a_n \sum_{i=1}^{k-n}\widetilde{R}_i\widetilde{q}_{k-n-i}\Bigg)$$

for $k = h+1, \dots, m$, *or*

$$\rho_{m+1} = \frac{\lambda}{1+\lambda\mathsf{E}\tau}\Bigg(\sum_{i=1}^{h} R_i\overline{q}_{m+1-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i\overline{q}_{m+1-n-i} + p_{-1}R(h)\sum_{i=1}^{m-h}\widetilde{R}_i\overline{\widetilde{q}}_{m+1-h-i} - \sum_{j=h+1}^{m-1} r_j(h)\sum_{i=1}^{m-j}\widetilde{R}_i\overline{\widetilde{q}}_{m+1-j-i} - \sum_{n=h}^{m-1} a_n\sum_{i=1}^{m-n}\widetilde{R}_i\overline{\widetilde{q}}_{m+1-n-i} + \widetilde{m}_1\overline{a}_{m+1}\Bigg).$$

6. Evaluation of stationary characteristics

Some customers may disappear without service (provided the total number of customers arrived exceeds the number $m+1$) if a system has a bounded capacity, if the customers arrive to the system in groups, and if the input flow is not blocked. Therefore the equality for the probability of service

$$\mathsf{P}_{\mathrm{sv}} = 1 - \rho_{m+1}$$

does not hold in the case of a queueing system $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ if $a_1 < 1$.

We derive a formula for P_{sv} by passing to the limit as $t \to \infty$ in the formula for the ratio of the number of customers served in the system and the total number of customers arrived to the system over time t. The mean number of customers arrived to the system over time t is equal to $\lambda b_1 t$, while the mean number of customers served in the system over the same time equals $\big(1-\rho_0\big)t/M_1$, where M_1 is the mean service time for a single customer. We evaluate the mean service time as

$$\frac{1}{M_1} = \frac{\mathsf{E}\,\omega}{m_1\,\mathsf{E}\,\tau} + \frac{\mathsf{E}\widetilde{\omega}}{\widetilde{m}_1\,\mathsf{E}\,\tau}.$$

Here $\mathsf{E}\,\omega$ and $\mathsf{E}\,\widetilde{\omega}$ are the mean durations of the mean busy period corresponding to the mean and threshold modes, respectively. Applying relation (20) for $\mathsf{E}\,\tau$, we obtain the following equality for the probability of service:

$$\begin{aligned}\mathsf{P}_{\rm sv} &= \frac{\widetilde{m}_1\,\mathsf{E}\,\omega + m_1 \mathsf{E}\widetilde{\omega}}{b_1 m_1 \widetilde{m}_1 \big(1+\lambda \mathsf{E}\tau\big)} \\ &= \frac{1}{b_1\big(1+\lambda\,\mathsf{E}\,\tau\big)} \\ &\quad\times \Bigg(\sum_{i=1}^{h} R_i - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i + p_{-1} R(h) \sum_{i=1}^{m-h} \widetilde{R}_i \\ &\qquad - \sum_{j=h+1}^{m-1} r_j(h) \sum_{i=1}^{m-j} \widetilde{R}_i - \sum_{n=h}^{m-1} a_n \sum_{i=1}^{m-n} \widetilde{R}_i + \overline{a}_{m+1}\Bigg).\end{aligned} \tag{24}$$

The stationary characteristics of the queue, namely the mean length of the queue $\mathsf{E}Q$ and mean waiting time $\mathsf{E}\,w$ are evaluated from the relations

$$\mathsf{E}\,Q = \sum_{k=1}^{m} k\rho_{k+1}, \qquad \mathsf{E}\,w = \frac{\mathsf{E}Q}{\lambda b_1 \mathsf{P}_{\rm sv}}. \tag{25}$$

Note that the above equality for $\mathsf{E}\,w$ follows from the Little formula for queueing system with losses.

The mean number of customers $\mathsf{E}\,S$ in the queueing system $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$ functioning in a stationary mode is given by

$$\mathsf{E}\,S = \mathsf{E}\,Q + 1 - \rho_0.$$

7. An example of the evaluation of the stationary distribution and characteristics of the system

Below we describe an algorithm for the evaluation of R_i, q_i and $\widetilde{R}_i$, $\widetilde{q}_i$, $i \geq 1$. It is worthwhile mentioning that the algorithm does not depend on the parameters m and h. The algorithm uses the distributions of the input flow and service time, that is, the functions $F(x)$ and $\widetilde{F}(x)$.

The definition of the quantities R_i, q_i, $\widetilde{R}_i$, and $\widetilde{q}_i$ implies the following recurrence relations: if $k \geq 1$, then

$$R_1 = \frac{1}{p_{-1}}, \quad R_{k+1} = \frac{R_k - \sum_{i=0}^{k-1} p_i R_{k-i}}{p_{-1}}, \qquad \widetilde{R}_1 = \frac{1}{\widetilde{p}_{-1}}, \quad \widetilde{R}_{k+1} = \frac{\widetilde{R}_k - \sum_{i=0}^{k-1} \widetilde{p}_i \widetilde{R}_{k-i}}{\widetilde{p}_{-1}},$$

$$q_0 = \frac{1-f(\lambda)}{\lambda}, \quad q_k = \sum_{i=1}^{k} a_i q_{k-i} - \frac{p_{k-1}}{\lambda}, \quad \widetilde{q}_0 = \frac{1-\widetilde{f}(\lambda)}{\lambda}, \quad \widetilde{q}_k = \sum_{i=1}^{k} a_i \widetilde{q}_{k-i} - \frac{\widetilde{p}_{k-1}}{\lambda},$$

where

$$p_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!}\, dF(x), \tag{26}$$

$$\widetilde{p}_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!}\, d\widetilde{F}(x), \qquad i = -1, 0, 1, \dots. \tag{27}$$

Assume that the customers arrive in groups whose sizes do not exceed 2 and that the input flow may be thinned, that is, $a'_0 + a'_1 + a'_2 = 1$. We also assume that the service times in the main and threshold modes are distributed according to the Erlang distribution of the second order with parameters μ and $\widetilde{\mu}$, respectively. This means that

$$F(x) = 1 - (1+\mu x)e^{-\mu x}, \qquad \widetilde{F}(x) = 1 - (1+\widetilde{\mu} x)e^{-\widetilde{\mu} x}$$

for $x \geq 0$. Then the mean service times are equal to $m_1 = 2/\mu$ and $\widetilde{m}_1 = 2/\widetilde{\mu}$, respectively.

We consider an example where $a_0' = 0.1$, $a_1' = 0.675$, $a_2' = 0.225$, $m = 5$, $h = 2$, $\lambda' = 20/9$, $\mu = 3$, and $\widetilde{\mu} = 6$. Then $a_1 = 0.75$, $a_2 = 0.25$, $\lambda = 2$, $m_1 = 2/3$, $\widetilde{m}_1 = 1/3$, and $b_1 = 1.25$.

According to (27)

$$
\begin{gathered}
p_{-1} = \frac{\mu^2}{(\lambda+\mu)^2}, \qquad p_0 = \frac{2a_1\mu^2\lambda}{(\lambda+\mu)^3}, \qquad p_1 = \frac{3a_1^2\mu^2\lambda^2}{(\lambda+\mu)^4} + \frac{2a_2\mu^2\lambda}{(\lambda+\mu)^3}, \\
p_2 = \frac{4a_1^3\mu^2\lambda^3}{(\lambda+\mu)^5} + \frac{6a_1a_2\mu^2\lambda^2}{(\lambda+\mu)^4}, \qquad p_3 = \frac{5a_1^4\mu^2\lambda^4}{(\lambda+\mu)^6} + \frac{12a_1^2a_2\mu^2\lambda^3}{(\lambda+\mu)^5} + \frac{3a_2^2\mu^2\lambda^2}{(\lambda+\mu)^4}, \\
p_4 = \frac{6a_1^5\mu^2\lambda^5}{(\lambda+\mu)^7} + \frac{20a_1^3a_2\mu^2\lambda^4}{(\lambda+\mu)^6} + \frac{12a_1a_2^2\mu^2\lambda^3}{(\lambda+\mu)^5}.
\end{gathered}
\tag{28}
$$

We obtain the corresponding results for $\widetilde{p}_i$ by substituting $\widetilde{\mu}$ for μ in equalities (28).

The mean duration of the busy period $\mathsf{E}\,\tau$ found from equality (20) is equal to 8.64295 for our numerical data.

The line "ρ_k" of Table 1 contains the probabilities ρ_k evaluated according to equalities (24). For the sake of comparison, the same table contains the corresponding probabilities evaluated by the simulation system GPSS World [15, 16] for $t = 500,000$.

Similarly, Table 2 contains the stationary characteristics of the system evaluated by relations (25) and (26) and by using GPSS World, as well.

TABLE 1. Stationary distribution of the number of customers ($h = 2$)

Number of customers (k)	0	1	2	3	4	5	6
ρ_k	0.0547	0.0972	0.1789	0.1876	0.1857	0.1690	0.1269
ρ_k (GPSS World, $t = 5 \cdot 10^5$)	0.0548	0.0968	0.1788	0.1879	0.1858	0.1688	0.1277

TABLE 2. Stationary characteristics ($h = 2$)

Characteristic	P_{sv}	$\mathsf{E}Q$	$\mathsf{E}\,w$
Evaluated by (25) and (26)	0.8393	2.4216	1.1541
Evaluated by GPSS World ($t = 5 \cdot 10^5$)	0.839	2.422	1.155

8. The problem of the optimal choice of the threshold h

Consider the criterion

$$
Pr(h) = c_+\lambda b_1 \mathsf{P}_{\mathrm{sv}} - c_{\mathrm{sv}}T(h) - \widetilde{c}_{\mathrm{sv}}\widetilde{T}(h) - c_0\rho_0 \tag{29}
$$

used to estimate the quality of functioning of a queueing system $\mathrm{M}^\theta/\mathrm{G}_1/1/\mathrm{m}$, where c_+ is the price paid for the service of a single customer; c_{sv} and $\widetilde{c}_{\mathrm{sv}}$ are the prices paid for functioning the system during one time unit for the main and threshold modes, respectively; $T(h) = \lambda\rho_0\,\mathsf{E}\,\omega$ and $\widetilde{T}(h) = \lambda\rho_0\,\mathsf{E}\,\widetilde{\omega}$ are the relative mean durations of the main and threshold modes, respectively; c_0 is the price paid for a time unit during an idle period.

Fix the parameter m. We want to determine a threshold level h that maximizes the mean gain $Pr(h)$.

TABLE 3. Stationary characteristics for different h

h	$\mathsf{P}_{\rm sv}$	$T(h)$	$\widetilde{T}(h)$	ρ_0
1	0.8759	0.3508	0.5545	0.0947
2	0.8393	0.4917	0.4536	0.0547
3	0.7880	0.6174	0.3480	0.0346
4	0.7093	0.7718	0.2052	0.0230

Using the same data as in Section 7 we evaluate criterion (29) for different threshold levels h. Table 3 contains the values of the stationary characteristics used to evaluate $Pr(h)$. Table 4 contains the mean gain $Pr(h)$ computed for different threshold levels h by using criterion (29) for different h and $\widetilde{c}_{\rm sv}$ with the coefficients $c_+ = 8$, $c_{\rm sv} = 10$, and $c_0 = 5$.

TABLE 4. The mean gain for different h and $\widetilde{c}_{\rm sv}$

h	$Pr(h)$ ($\widetilde{c}_{\rm sv} = 18$)	$Pr(h)$ ($\widetilde{c}_{\rm sv} = 20$)	$Pr(h)$ ($\widetilde{c}_{\rm sv} = 22$)
1	**3.5555**	2.4465	1.3375
2	3.4307	**2.5235**	1.6163
3	3.1490	2.4530	1.7570
4	2.6594	2.2490	**1.8386**

We see that the maximal gain corresponds to $h = 1$ if $\widetilde{c}_{\rm sv} = 18$; to $h = 2$ if $\widetilde{c}_{\rm sv} = 20$; and to $h = 4$ if $\widetilde{c}_{\rm sv} = 22$.

9. Concluding remarks

The system of equations (6)–(8) with respect to unknown functions $\Phi_n(s,k)$ includes the coefficients $p_j(s)$ and $\widetilde{p}_j(s)$ corresponding to two lower continuous random walks. These coefficients appear in the system of equations because of two service modes present in the queueing system $M^\theta/G_1/1/m$. Solving the system of equations (6)–(8) allows us to apply Korolyuk's method of potential to study both stationary and transient modes in a system with two service modes. Analytical results obtained in the paper are checked with the help of computer simulation. We also considered an example of the optimization problem of choice of the threshold level for switching between the modes in the queueing system.

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