Теорія Ймовір. та Матем. Статист.
Вип. 86, 2012

Theor. Probability and Math. Statist.
No. 86, 2013, Pages 123–136
S 0094-9000(2013)00893-X
Article electronically published on August 20, 2013

STRONG MARKOV APPROXIMATION OF LÉVY PROCESSES AND THEIR GENERALIZATIONS IN A SCHEME OF SERIES

UDC 519.21

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Abstract. The notion of the strong Markov approximation that generalizes the notion of the Markov approximation is introduced. We consider an infinitesimal scheme of series that satisfies the assumptions of a Gnedenko theorem. Under these assumptions, we prove that a sequence of step processes constructed from a corresponding random walk is a strong Markov approximation for a Lévy process. The same result is obtained for a sequence of difference approximations of a solution to a stochastic differential equation driven by a Lévy noise.

1. Introduction

The Skorokhod representation theorem on a common probability space allows one to study the convergence in probability of random elements in a separable metric space instead of the convergence of a sequence of distributions. For function spaces whose elements are stochastic processes, this method can be applied to pass to the limit for schemes where a sequence of continuous functionals of pre-limit processes approximates a functional of the limit process. On the other hand, some problems include the question about the limit behavior of discontinuous functionals for which the above reasoning does not apply (see relation (1) and Theorem 1 below). For such a case, an additional property for the pair of processes defined in a common probability space can be useful. An example of such a property is the Markov property. For the sake of illustration, consider the classical central limit theorem. Let a sequence of random broken lines $X_k(t)$ be constructed from a random walk $\{S_k\}$ whose jumps have zero mean and finite second moment

$$X_k(t) = \frac{S_{k-1}}{\sqrt{n}} + (nt-k+1)\left[\frac{S_k}{\sqrt{n}} - \frac{S_{k-1}}{\sqrt{n}}\right], \qquad t \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \ k \in \mathbb{N}.$$

Let X be a Wiener process. Then the classical Donsker invariance principle [1] holds, namely the random broken lines converge in $\mathbb{C}[0;T]$ to the distribution of the Wiener process. Therefore, there exist $\hat{X}_n$ and $\hat{X}$ defined in a common probability space such that the distribution of $\hat{X}_n$ is the same as that of the broken lines, the distribution of $\hat{X}$ coincides with that of the Wiener process, and the distance in $\mathbb{C}[0;T]$ between $\hat{X}_n$ and $\hat{X}$ approaches zero in probability. Note that the random broken lines possess the Markov property at the nodes with respect to the natural filtration. A natural question is whether or not it is possible to construct a pair of processes such that the distributions of the components coincide with those of broken lines and the Wiener process, respectively, and

2010 *Mathematics Subject Classification.* Primary 60J25, 60F17, 60B10.

Key words and phrases. Lévy processes, central limit theorem in a scheme of series, strong Markov approximation.

the pair of processes possesses the Markov property? An answer to this question is given in [2] in terms of the so-called *Markov approximation* introduced therein.

Definition 1. We say that a sequence X_n is a Markov approximation of a process X if, for all $\gamma > 0$ and $T < \infty$, there exist a number $K(\gamma, T)$ and a sequence of two component processes $\{\hat{Y}_n = (\hat{X}_n, \hat{X}^n)\}$ such that

(1) $\hat{X}_n \overset{d}{=} X_n$, $\hat{X}^n \overset{d}{=} X$;
(2) the processes $\hat{Y}_n$, $\hat{X}_n$, and $\hat{X}^n$ possess the Markov property at the points
$$\frac{iK(\gamma,T)}{n}, \qquad i \in \mathbb{N},$$
with respect to the filtration $\{\hat{F}_t^{\,n} = \sigma(\hat{Y}_n(s)), s \le t\}$;
(3) $\limsup_{n\to\infty} \mathsf{P}\left(\sup_{i \le \frac{Tn}{K(\gamma,T)}} \rho\left(\hat{X}_n\left(\frac{iK(\gamma,T)}{n}\right), \hat{X}^n\left(\frac{iK(\gamma,T)}{n}\right)\right) > \gamma \right) < \gamma.$

It is proved in [2] that the sequence of random broken lines is a Markov approximation of a Wiener process. If the approximation improves (that is, if $\gamma \to 0+$), then except for one trivial case of standard normal of the random variables $\{\xi_n, n \ge 1\}$, the Markov property of the pair becomes worse (that is, $K(\gamma, T) \to \infty$ for an arbitrary $T > 0$); see [2].

As explained above, Definition 1 allows one to prove limit theorems for some discontinuous functionals. The following limit result is proved in the paper [3] for functionals of the form

$$(1) \qquad \phi_n^{s,t}(X_n) = \sum_{k: s \le \frac{k}{n} < t} F_n\left(X_n\left(\frac{k}{n}\right), \dots, X_n\left(\frac{k+L-1}{n}\right)\right), \qquad 0 \le s < t,$$

where $F_n(\cdot)$ are nonnegative functions and L is a fixed integer number. Note that the Markov approximation is one of the key assumptions in [3].

Theorem 1. *Assume that the following conditions hold.*

(1) *The sequence X_n is a Markov approximation of the process X.*
(2) *$F_n(\cdot)$ are nonnegative, bounded, and uniformly converge to zero as $n \to \infty$.*
(3) *There exists a W-functional $\phi(X)$ of the limit process such that the characteristics of the functionals ϕ_n uniformly converge to the characteristic of $\phi(X)$ (see Section 6 in* [4]*).*
(4) *The characteristic of the functional $\phi(X)$ is uniformly continuous.*

Then ϕ_n converges in distribution to $\phi(X)$.

Markov approximations are considered in [3] and [5] for the classical central limit theorem and some other nonlinear generalizations are also discussed. Let Z_n be a solution of the following difference approximation of a stochastic differential equation:
$$Z_n\left(\frac{k+1}{n}\right) = Z_n\left(\frac{k}{n}\right) + a\left(Z_n\left(\frac{k}{n}\right)\right)\cdot\frac{1}{n} + b\left(Z_n\left(\frac{k}{n}\right)\right)\cdot \Delta X_n\left(\frac{k}{n}\right),$$
$Z_n(0) = z$, where
$$\Delta X_n\left(\frac{k}{n}\right) = X_n\left(\frac{k+1}{n}\right) - X_n\left(\frac{k}{n}\right)$$
is the increment of a trajectory of the broken line and where Z is a solution of the diffusion stochastic differential equation
$$dZ(t) = a(Z(t))\,dt + b(Z(t))\,dX(t), \qquad Z(0) = z.$$

Here $X(t)$ is a Wiener process. It is proved in [3] and [5] that, under appropriate assumptions, the sequence of processes $Z_n(t)$ is a Markov approximation of the process $Z(t)$. This means that the property of the Markov approximation is not "sensible", to some extent, with respect to some "nonlinear perturbations" of the initial system. This distinguishes the results on the Markov approximation from other results related to the central limit theorem, such as the Skorokhod embedding theorem [6].

In this paper, we study the Markov approximation in the framework of the general central limit theorem for triangular arrays. Namely, we consider the scheme of series

$$\{\xi_{kn}, 1 \le k \le n, n \ge 0\}$$

satisfying the conditions of the Gnedenko theorem [7]. Thus, according to the Skorokhod theorem, the process X_n constructed from this scheme of series converges in $\mathbb{D}[0;1]$ to a Lévy process without the diffusion component. We prove that X_n is a Markov approximation of the process X. Moreover, we prove that X_n is a Markov approximation in the sense explained in the following definition.

Definition 2. We say that a sequence of processes X_n is a strong Markov approximation of a process X if the condition of the Markov approximation holds with $K(\gamma, T) = 1$.

We also prove that an analogous result holds for some nonlinear generalization of the scheme described above. Namely, we consider a process defined by the following stochastic differential equation

$$Z(t) = Z_0 + \int_0^t a(Z(s))\, ds + \int_0^t b(Z(s-))\, dX(s),$$

where $X(t)$ is the Lévy process considered above. Let Z_n, $n \ge 1$, be a sequence of step processes defined at the points $\frac{k}{n}$, $1 \le k \le n$, by the difference scheme constructed from increments of the process X_n. Imposing a natural assumption (Lipschitz condition) on the coefficients we prove that the sequence of processes Z_n is a strong Markov approximation of the process Z.

2. Strong Markov approximation for a Lévy process

Consider an infinitesimal scheme of the series $\{\xi_{kn}, 1 \le k \le n, n \ge 1\}$, where ξ_{kn}, $1 \le k \le n$, are identically distributed random variables with the distribution function F_n, $n \ge 1$. Further, let a Lévy process be given by

$$X(t) = \int_0^t \int_{|u| \le 1} u[\nu(du, ds) - \Pi(du)] + \int_0^t \int_{|u|>1} u\, \nu(du, ds).$$

Denote by Π the Lévy measure of the process X and let

$$\Pi_n(dx) = n\, F_n(dx). \tag{2}$$

For $x > 0$, write

$$\Pi_n^+(x) = n[1 - F_n(x)], \qquad \Pi_n^-(-x) = nF_n(-x). \tag{3}$$

Consider the function

$$\tau(x) = \begin{cases} x, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

and denote

$$\beta_n = \mathbf{E}(\tau(\xi_{kn})), \qquad b_n = n\beta_n, \qquad B_n = n\beta_n^2.$$

The following set of conditions being a reformulation of Gnedenko's conditions can be found in Chapter XVII, §7 of [8] .

(**H**) Let

$$\Pi_n^+(x) \to \Pi^+(x) \tag{4}$$

at all points $x > 0$ of continuity of the measure Π; let

$$\Pi_n^-(-x) \to \Pi^-(-x) \tag{5}$$

at all points $-x$, $x > 0$, of continuity of the measure Π; and let

$$\int_{-s}^{s} x^2\,\Pi_n(dx) - B_n \to \int_{-s}^{s} x^2\,\Pi(dx) \tag{6}$$

for some $s > 0$ such that $\{-s, s\}$ are points of continuity of Π.

The following is the main result of this section.

Theorem 2. *If conditions (**H**) and (**C**) hold, then the process*

$$X_n(t) = \sum_{k=1}^{[nt]} \xi_{kn} - tb_n, \qquad n \geq 1,\ t \in [0;1],$$

is a strong Markov approximation of the process $X(t)$.

We construct the processes for the Markov approximation in the form of sums with independent increments over the intervals of length $\frac{1}{n}$ such that the difference satisfies the above estimate. First we consider some auxiliary constructions.

Given some fixed $R > 1$ being a point of continuity of the measure Π, consider the process

$$X^1(t) = \int_0^t \int_{|u|\leq 1} u\,\tilde{\nu}(du, ds) + \int_0^t \int_{1<|u|\leq R} u\,\nu(du, ds), \qquad t \in [0;1],$$

and independent identically distributed random variables

$$\xi_{kn}^1 = \begin{cases} \xi_{kn}, & |\xi_{kn}| \leq R, \\ 0, & |\xi_{kn}| > R, \end{cases} \qquad 1 \leq k \leq n,\ n \geq 1.$$

Write

$$X_n^1(t) = \sum_{k=1}^{[nt]} \xi_{kn}^1 - tb_n^1, \qquad t \in [0;1],$$

where the constants b_n^1 are defined similarly to the preceding case. Our aim is to estimate the acurracy of this approximation.

Lemma 1. *For an arbitrary $\gamma > 0$,*

$$\mathsf{P}\left(\sup_{1\leq k\leq n}\left|X\left(\frac{k}{n}\right) - X^1\left(\frac{k}{n}\right)\right| > \gamma\right) < 1 - \exp(-\Pi(|x| > R)) \tag{7}$$

and

$$\mathsf{P}\left(\sup_{1\leq k\leq n}\left|X_n\left(\frac{k}{n}\right) - X_n^1\left(\frac{k}{n}\right)\right| > \gamma\right) < 1 - \exp(-\Pi(|x| > R)) + \delta_n, \tag{8}$$

where

$$\delta_n = (1 - \mathsf{P}(|\xi_{kn}| > R))^n - \exp(-\Pi(|x| > R)).$$

Remark 1. Note that

$$\delta_n \to 0, \qquad n \to \infty,$$

under condition (**H**).

Proof. To prove inequality (7) note that if the random event

$$\left\{ \sup_{1\le k\le n} \left| X\left(\frac{k}{n}\right) - X^1\left(\frac{k}{n}\right)\right| > \gamma \right\}$$

occurs, then so does the event

$$\left\{ \int_0^1 \int_{|x|>R} |u|\,\nu(du,ds) \neq 0 \right\}$$

which implies that

$$\begin{aligned} \mathsf{P}\left(\sup_{1\le k\le n} \left| X\left(\frac{k}{n}\right) - X^1\left(\frac{k}{n}\right)\right| > \gamma \right) &\le \mathsf{P}\left(\int_0^1 \int_{|x|>R} |u|\,\nu(du,ds) \neq 0 \right) \\ &\le 1 - \exp(-\Pi(|x|>R)). \end{aligned}$$

Now we prove inequality (8). The definitions of the processes X_n and X_n^1 imply

$$\begin{aligned} &\mathsf{P}\left(\sup_{1\le k\le n} \left| X_n\left(\frac{k}{n}\right) - X_n^1\left(\frac{k}{n}\right)\right| > \gamma \right) \le 1 - \mathsf{P}(\xi_{kn}\mathbf{1}_{|\xi_{kn}|<\gamma}, k=1,2,\dots,n) \\ &\quad = 1 - (1 - \mathsf{P}(\xi_{kn}\mathbf{1}_{|\xi_{kn}|>\gamma}))^n \le 1 - (1 - \mathsf{P}(|\xi_{kn}|>R))^n \\ &\quad = 1 - \exp(-\Pi(|x|>R)) + \delta_n. \quad \square \end{aligned}$$

□

Let $\varepsilon < 1$ be a fixed number being a point of continuity of the measure Π. Consider the process

$$\begin{aligned} X^2(t) &= \int_0^t \int_{\varepsilon\le|u|\le 1} u\,\tilde\nu(du,ds) + \int_0^t \int_{1<|u|\le R} u\,\nu(du,ds) \\ &= \int_0^t \int_{\varepsilon\le|u|\le R} u\,\nu(du,ds) - t\int_{\varepsilon\le|u|\le 1} u\,\Pi(du), \qquad t\in[0;1], \end{aligned}$$

and independent identically distributed random variables

$$\xi_{kn}^2 = \begin{cases} 0, & |\xi_{kn}^1| < \varepsilon, \\ \xi_{kn}^1, & |\xi_{kn}^1| \ge \varepsilon, \end{cases} \qquad 1\le k\le n,\ n\ge 1.$$

Our next aim is to estimate the accuracy of this approximation.

Lemma 2. *Given an arbitrary* $\gamma > 0$,

$$\mathsf{P}\left(\sup_{1\le k\le n} \left| X\left(\frac{k}{n}\right) - X^1\left(\frac{k}{n}\right)\right| > \gamma \right) < \frac{1}{\gamma^2}\int_{-\varepsilon}^{\varepsilon} x^2\,\Pi(dx) \tag{9}$$

and

$$\mathsf{P}\left(\sup_{1\le k\le n} \left| X_n^1\left(\frac{k}{n}\right) - X_n^2\left(\frac{k}{n}\right)\right| > \gamma \right) < \frac{1}{\gamma^2}\int_{-\varepsilon}^{\varepsilon} x^2\,\Pi(dx) + \Delta_n(\varepsilon), \tag{10}$$

where

$$\Delta_n(\varepsilon) = \int_{|u|<\varepsilon} x^2\,\Pi_n(dx) - n\left(\int_{-\varepsilon}^{\varepsilon} x\,F_n(dx)\right)^2 - \int_{-\varepsilon}^{\varepsilon} x^2\,\Pi(dx).$$

Remark 2. Since $\{\xi_{kn}\}$ is an infinitesimal scheme of series and condition (**H**) holds,

$$\Delta_n \to 0, \qquad n\to\infty.$$

Proof. We prove inequality (10); the idea of the proof of inequality (9) is similar. The differences $X_n^1(k/n) - X_n^2(k/n)$ and $1 \le k \le n$, are centered and jointly independent random variables. Indeed, the definition of the processes X_n implies that

$$\mathsf{E}\left[X_n^1\left(\frac{k}{n}\right) - X_n^2\left(\frac{k}{n}\right)\right] = \mathsf{E}\left[\sum_{j=1}^{k}\left(\left(\xi_{jn}^1 - \xi_{jn}^2\right) - \left(\beta_n^1 - \beta_n^2\right)\right)\right] = 0.$$

Moreover,

$$X_n^1(1) - X_n^2(1) = \sum_{k=1}^{n}[X_n^1(k/n) - X_n^2(k/n)].$$

Consider the second moment of the difference

$$\mathsf{E}\left|X_n^1(1) - X_n^2(1)\right|^2 = n \operatorname{Var} \xi_{kn}\mathbf{1}_{|\xi_{kn}|<\varepsilon} = \int_{|u|<\varepsilon} x^2\,\Pi_n(dx) - n\left(\int_{-\varepsilon}^{\varepsilon} x\,F_n(dx)\right)^2$$
$$= \int_{-\varepsilon}^{\varepsilon} x^2\,\Pi(dx) + \Delta_n(\varepsilon).$$

Now inequality (10) follows from a Kolmogorov inequality (see [9]). $\square$

Next we consider the process

$$\tilde{X}^2(t) = \int_0^t \int_{\varepsilon \le |u| \le R} u\,\nu(du, ds), \qquad t \in [0; 1].$$

Its Lévy measure Π^2 is finite by construction, since Π^2 is the restriction of the measure Π to the set $A_{\varepsilon,R} \overset{\text{def}}{=} \{x, \varepsilon \le |x| \le R\}$. Thus $\tilde{X}^2$ is a compound Poisson process with the intensity of jumps

$$\lambda = \Pi(A_{\varepsilon,R})$$

and with the distribution of jumps

$$\Xi_{\varepsilon,R}(A) = \frac{\Pi^2(A \cap A_{\varepsilon,R})}{\Pi^2(A_{\varepsilon,R})}, \qquad A \in \mathfrak{B}(\mathbb{R}).$$

Then the increment of the process $\tilde{X}^2$ in an interval of length $\frac{1}{n}$ can be rewritten as

$$\zeta_1^2 = X^2\left(\frac{1}{n}\right) = \mathbf{1}_{\beta=1}\theta + \mathbf{1}_{\beta=2}\kappa_n,$$

where the distribution of θ equals $\Xi_{\varepsilon,R}$ and where the distribution of β is given by

$$\mathsf{P}(\beta = 0) = \exp\left(-\frac{\lambda}{n}\right), \qquad \mathsf{P}(\beta = 1) = \frac{\lambda}{n}\exp\left(-\frac{\lambda}{n}\right), \qquad \mathsf{P}(\beta = 2) = \bar{o}\left(\frac{1}{n}\right).$$

The random variable κ_n has the distribution of the compound Poisson process X^2 given that two or more jumps occur in an interval of length $\frac{1}{n}$.

Remark 3. The following bound for the second moment of the random variable κ_n is valid:

$$\mathsf{E}\,|\kappa_n|^2 \le \sum_{k\ge 2}\frac{\frac{1}{k!}(kR)^2 \cdot \exp\left\{-\frac{\lambda_{\varepsilon,R}}{n}\right\} \cdot \left(\frac{\lambda_{\varepsilon,R}}{n}\right)^k}{1 - \exp\left\{-\frac{\lambda_{\varepsilon,R}}{n}\right\} - \frac{\lambda_{\varepsilon,R}}{n} \cdot \exp\left\{-\frac{\lambda_{\varepsilon,R}}{n}\right\}} \to C, \qquad n \to \infty,$$

where C is a certain constant.

The increments of the process

$$\tilde{X}_n^2 = \sum_{k=1}^{[nt]} \xi_{kn}^2$$

over an interval of length $\frac{1}{n}$ can be written as

$$\zeta_{1n}^2 = X_n^2\left(\frac{1}{n}\right) = \mathbf{1}_{\alpha=1}\theta_n,$$

where the distribution of the random variable θ_n is

$$\Xi_{\varepsilon,R,n}(A) = \frac{F_n^2(A\cap A_{\varepsilon,R})}{F_n^2(A_{\varepsilon,R})}, \qquad A \in \mathfrak{B}(\mathbb{R}),$$

and where the distribution of α equals

$$\mathsf{P}(\alpha = 0) = 1 - F_n^2(A_{\varepsilon,R}), \qquad \mathsf{P}(\alpha = 1) = F_n^2(A_{\varepsilon,R}).$$

Since ε and R in the above construction are points of continuity of the measure Π, condition (**H**) implies that

$$\Xi_{\varepsilon,R,n} \Rightarrow \Xi_{\varepsilon,R}, \qquad n\to\infty.$$

According to the Skorokhod theorem on a common probability space [10], there exist a probability space $(\Omega_1, \mathcal{F}, \mathsf{P})$ and random variables $(\hat{\theta}_n, \hat{\theta})$ defined in $(\Omega_1, \mathcal{F}, \mathsf{P})$ such that

$$|\hat{\theta}_n - \hat{\theta}| \xrightarrow{\mathsf{P}} 0, \qquad n\to\infty,$$

where $\hat{\theta}_n$ has the distribution $\Xi_{\varepsilon,R,n}$ and $\hat{\theta}$ has the distribution $\Xi_{\varepsilon,R}$.

Remark 4. Note that $|\hat{\theta}_n| \le R$ and $|\hat{\theta}| \le R$ by construction. Hence, the convergence in probability implies that

$$\mathsf{E}\,|\hat{\theta} - \hat{\theta}_n|^p \to 0, \qquad n\to\infty,$$

for all $p \ge 1$.

One can construct a pair of random variables $(\hat{\alpha};\hat{\beta})$ in a common probability space in such a way that

$$p_{0,0} = \mathsf{P}(\hat{\alpha} = 0, \hat{\beta} = 0) = 1 - \frac{\lambda}{n} + \bar{o}\left(\frac{1}{n}\right), \qquad n\to\infty,$$

$$p_{1,1} = \mathsf{P}(\hat{\alpha} = 1, \hat{\beta} = 1) = \frac{\lambda}{n} + \bar{o}\left(\frac{1}{n}\right), \qquad n\to\infty,$$

$$p_{i,j} = \bar{o}\left(\frac{1}{n}\right), \qquad n\to\infty,\ i\ne j.$$

We consider the random vector $(\hat{\theta}, \hat{\theta}_n, \hat{\alpha}, \hat{\beta}, \hat{\kappa}_n)$ in a common probability space, where $\hat{\kappa}_n$ is independent of the pairs $(\hat{\alpha};\hat{\beta})$ and $(\hat{\theta}, \hat{\theta}_n)$. Then we define the random variables $\hat{\zeta}_1^2$ and $\hat{\zeta}_{1n}^2$ similarly to ζ_1^2 and ζ_{1n}^2. By construction,

$$\hat{\zeta}_1^2 \stackrel{d}{=} \zeta_1^2, \qquad \hat{\zeta}_{1n}^2 \stackrel{d}{=} \zeta_{1n}^2.$$

Write

$$\hat{\eta}_1^2 = \hat{\zeta}_1^2 - \frac{1}{n}\int_{\varepsilon\le|u|\le1} u\,\Pi(du),$$

$$\hat{\eta}_{1n}^2 = \hat{\zeta}_{1n}^2 - \beta_n^2.$$

Proposition 1. *The pair of random variables* $(\hat\eta_1^2, \hat\eta_{1n}^2)$ *is such that*

$$\mathsf{E}\left|\hat\eta_1^2 - \hat\eta_{1n}^2\right|^2 = \bar o\left(\frac{1}{n}\right), \qquad n \to \infty.$$

Proof. We have

$$\begin{aligned}\mathsf{E}\left|\hat\eta_1^2 - \hat\eta_{1n}^2\right|^2 &\le 2\,\mathsf{E}\left|\mathbf{1}_{\hat\beta=1}\hat\theta + \mathbf{1}_{\hat\beta=2}\hat\kappa_n - \mathbf{1}_{\hat\alpha=1}\hat\theta_n\right|^2 + 2\,\mathsf{E}\left|\frac{1}{n}\int_{\varepsilon\le|u|\le1} u\,\Pi(du) - \beta_n^2\right|^2\\ &= 2\left(\mathsf{E}\big|\hat\theta_n\big|^2 p_{0,1} + \mathsf{E}\left|\hat\kappa_n\right|^2 p_{0,2} + \mathsf{E}\big|\hat\theta\big|^2 p_{1,0} + \mathsf{E}\big|\hat\theta - \hat\theta_n\big|^2 p_{1,1} + \mathsf{E}\big|\hat\theta_n - \hat\kappa_n\big|^2 p_{1,2}\right)\\ &\quad + 2\,\mathsf{E}\left|\frac{1}{n}\int_{\varepsilon\le|u|\le1} u\,\Pi(du) - \beta_n^2\right|^2.\end{aligned}$$

Note that the last term is $\bar o(1/n)$. Each of two numbers $\mathsf{E}\,|\hat\theta_n|$ and $\mathsf{E}\,|\hat\theta|^2$ is bounded from above by the constant R^2, while $\mathsf{E}\,|\hat\kappa_n|^2$ is finite in view of Remark 3. Taking into account Remark 4 and an explicit expression,

$$p_{1,1} = \frac{1}{n}\Pi^2(A_{\varepsilon,R}) + \bar o\left(\frac{1}{n}\right),$$

we complete the proof. □

Proposition 2. *In a common probability space* $(\Omega^*, \mathcal{F}^*, P^*)$*, there exists a set of six random variables*

$$\eta_{1n}, \quad \eta_{1n}^1, \quad \eta_{1n}^2 \quad \eta_1^2, \quad \eta_1^1, \quad \eta_1$$

and a process $X(t)$*,* $t \in [0; 1/n]$*, such that*

(1) $(\eta_{1n}, \eta_{1n}^1, \eta_{1n}^1)$ *and* $(\xi_{1n}, \xi_{1n}^1, \xi_{1n}^2)$ *are identically distributed,*
(2) $(\eta_1, \eta_1^1, \eta_1^2)$ *and* $X(1/n), X^1(1/n), X^2(1/n)$ *are identically distributed,*
(3) (η_{1n}^2, η_1^2) *and* $(\hat\eta_{1n}^2, \hat\eta_1^2)$ *are identically distributed,*
(4) *the process* $X(t)$ *has the same distribution as the Lévy process, and* $X(1/n) = \eta_1$*.*

Proof. The proof of Proposition 2 is based on the following auxiliary result (see, for example, [11, Chapter 11, §8]).

Lemma 3. *Let* μ *and* ν *be two distributions in* $\mathbb{X}\times\mathbb{Y}$ *and* $\mathbb{Y}\times\mathbb{Z}$*, respectively, where* $\mathbb{X}$*,* $\mathbb{Y}$*, and* $\mathbb{Z}$ *are complete separable spaces such that*

$$\mu(\mathbb{X}, A) = \nu(A, \mathbb{Z})$$

for all $A \in \mathcal{B}(\mathbb{Y})$*. Then one can construct three random elements* ξ*,* η*, and* ζ *in a common probability space such that the joint distribution of* ξ *and* η *equals* μ*, and the joint distribution of* η *and* ζ *equals* ν*.*

Let $\mathbb{X} = \mathbb{Y} = \mathbb{Z} = \mathbb{R}$. Consider the pairs $(\hat\eta_{1n}^2, \hat\eta_1^2)$ and $(X^2(1/n), X^1(1/n))$ in $\mathbb{X}\times\mathbb{Y}$ and $\mathbb{Y}\times\mathbb{Z}$, respectively. The distributions of these pairs are known. Applying Lemma 3, we construct η_{1n}^2, η_1^2, and η_1^1 in a common probability space with the properties needed.

Then we put

$$\mathbb{X} = \mathbb{R}^2, \qquad \mathbb{Y} = \mathbb{R}, \qquad \mathbb{Z} = \mathbb{R}.$$

Applying Lemma 3 to $(\eta_{1n}^2, \eta_1^2, \eta_1^1)$ and $(X^1(1/n),\ X^2(1/n))$ we construct η_{1n}^2, η_1^2, η_1^1, and η_1 in a common probability space. Repeating similar reasoning we complete the proof. □

Now we turn to the proof of the main result. First we briefly describe the construction of the sequence of two component processes. The random variables η_{1n}, η_{1n}^1, η_{1n}^2, η_1^2, η_1^1, and η_1 and the process $X(t)$, $t \in [0;1/n]$, are defined in the same probability space Ω^*. This means that the increments of the initial processes are defined in a common probability space.

Write $\Omega := (\Omega^*)^n$, $\omega = (\omega_{1n}, \omega_{2n}, \dots, \omega_{nn})$. Then the sequence $\{\hat\eta_{kn}(\omega), 1 \le k \le n\}$ is defined as follows:

$$\hat\eta_{kn}(\omega) = \eta_{1n}(\omega_{kn}).$$

By construction,

$$\hat\eta_{kn} \stackrel{d}{=} \xi_{kn}.$$

For $\hat\eta_{kn}$, we define the constants β_{kn}, b_n, and B_n as described above. Then we define the process $\hat X_n(t) := \sum_{k=1}^{[nt]} \hat\eta_{kn} - tb_n$, $t \in [0;1]$. By construction,

$$\hat X_n(t) \stackrel{d}{=} X_n(t).$$

Now we construct the process

$$\hat X^n(t,\omega) = X\left(\frac{1}{n}, \omega_{1n}\right) + X\left(\frac{1}{n}, \omega_{2n}\right) + \dots \\ + X\left(\frac{1}{n}, \omega_{([nt])n}\right) + X\left(t - \frac{[nt]}{n}, \omega_{([nt]+1)n}\right), \qquad t \in [0;1].$$

It is clear that

$$\hat X^n(t) \stackrel{d}{=} X(t),$$

where $X(t)$, $t \in [0;1]$, is the initial Lévy process.

Thus, the sequence of two component processes $\{\hat Y_n := (\hat X^n, \hat X_n)\}$ possesses the first property of the Markov approximation. The Markov property of the processes constructed above is proved in a standard way (see [12, §10]).

Similarly to $\hat X_n(t)$, $t \in [0;1]$, we construct the processes $\hat X_n^1(t)$ and $\hat X_n^2(t)$, $t \in [0;1]$, in the space Ω. The processes $\hat X^{1,n}(t)$ and $\hat X^{2,n}(t)$, $t \in [0;1]$, are defined in the same way as X^1 and X^2 for X.

Lemma 4. *For an arbitrary* $\gamma > 0$,

$$\limsup_{n\to\infty} \mathsf{P}\left(\sup_{1\le k\le n}\left|\hat X_n^2\left(\frac{k}{n}\right) - \hat X^{2,n}\left(\frac{k}{n}\right)\right| > \gamma\right) = 0.$$

Proof. Consider

$$\left|\hat X_n^2\left(\frac{k}{n}\right) - \hat X^{2,n}\left(\frac{k}{n}\right)\right| \le \left|\bar X_n^2\left(\frac{k}{n}\right) - \bar X^{2,n}\left(\frac{k}{n}\right)\right| \\ + \left|k\int_{1<|u|<R} x\,F_n(dx) - \frac{k}{n}\int_{1<|u|<R} x\,\Pi(dx)\right|,$$

where

$$\bar X_n^2\left(\frac{k}{n}\right) - \bar X^{2,n}\left(\frac{k}{n}\right) = \hat X_n^2 - k\int_{1<|u|<R} u\,F_n(du) - \left(\hat X^{2,n} - \frac{k}{n}\int_{1<|u|<R} u\,\Pi(du)\right), \\ 1 \le k \le n.$$

The sequence $\{\bar X_n^2(k/n) - \bar X^{2,n}(k/n), 1 \le k \le n\}$ is a martingale with respect to the filtration $\{\hat F_{k/n}^n = \sigma((\hat X_n^2(s), \hat X^{2,n}(s)), s \le k/n)\}$ and thus satisfies the maximal inequality for martingales. Namely,

$$\mathsf{P}\left(\sup_{1\le k\le qn}\left|\bar{X}_n^2\left(\frac{k}{n}\right)-\bar{X}^{2,n}\left(\frac{k}{n}\right)\right|>\gamma\right)<\frac{\mathsf{E}\,|\bar{X}_n^2(1)-\bar{X}^{2,n}(1)|^2}{\gamma^2}$$

for all $\gamma>0$. Taking into account Proposition 1 and condition (**H**) we obtain

$$\begin{aligned}&\mathsf{E}\,|\bar{X}_n^2(1)-\bar{X}^{2,n}(1)|^2\\ &\quad\le 2\left(n\,\mathsf{E}\,|\hat{\eta}_1^2-\hat{\eta}_{1n}^2|^2+\left|k\int_{1<|u|<R}u\,F_n(du)-\frac{k}{n}\int_{1<|u|<R}u\,\Pi(du)\right|^2\right)\to 0,\\ &\qquad n\to\infty.\end{aligned}$$

The proof is complete. □

Proposition 3. *For an arbitrary* $\gamma>0$,

$$\limsup_{n\to\infty}\mathsf{P}\left(\sup_{1\le k\le n}\left|\hat{X}_n\left(\frac{k}{n}\right)-\hat{X}^n\left(\frac{k}{n}\right)\right|>\gamma\right)<\gamma.$$

Proof. The properties of the measure Π imply that

$$1-\exp(-\Pi(|x|>R))\to 0,\qquad R\to\infty,$$

$$\int_{-\varepsilon}^{\varepsilon}x^2\,\Pi(dx)\to 0,\qquad \varepsilon\to\infty.$$

The estimates obtained in Lemmas 1 and 2 hold for the processes $\hat{X}_n$ and $\hat{X}^n$. Fix an arbitrary $\gamma>0$. Then one can choose $R>1$ and $\varepsilon>0$ being points of continuity of the measure Π and such that the inequality of Proposition 3 holds. □

The results above mean that the third property of a Markov approximation holds. Moreover, properties (2) and (3) of a strong Markov approximation hold with $K(\gamma,T)=1$. Therefore, the processes $X_n(t)$ is a strong Markov approximation of the Lévy process $X(t)$.

3. Difference approximations of solutions of stochastic differential equations with a Lévy noise

Consider a Markov process defined by the following stochastic differential equation

$$Z(t)=Z_0+\int_0^t a(Z(s))\,ds+\int_0^t b(Z(s-))\,dX(s), \tag{11}$$

where $X(t)$ is the Lévy process considered above and Z_0 is a positive constant. Let the sequence of stochastic processes Z_n, $n\ge 1$, be defined at points $\frac{k}{n}$, $1\le k\le n$, by the difference relation

$$Z_n\left(\frac{k}{n}\right)=Z_n\left(\frac{k-1}{n}\right)+a\left(Z_n\left(\frac{k-1}{n}\right)\right)\cdot\frac{1}{n}+b\left(Z_n\left(\frac{k-1}{n}\right)\right)\cdot\Delta_{kn}, \tag{12}$$

where $\Delta_{kn}=X_n(k/n)-X_n((k-1)/n)=\xi_{kn}-\beta_n$. Define $Z_n(t)$, $n\ge 1$, at other points $t\in[0,1]$ by the linear interpolation,

$$Z_n(t)=Z_n\left(\frac{k-1}{n}\right)+(nt-k+1)\left[Z_n\left(\frac{k}{n}\right)-Z_n\left(\frac{k-1}{n}\right)\right],\qquad t\in\left[\frac{k-1}{n};\frac{k}{n}\right).$$

The following (global Lipschitz condition) is a standard condition for the existence and uniqueness of a solution of the stochastic differential equation (11):

(**L**) *Global Lipschitz condition*: There exists a constant $L > 0$ such that

$$|b(x) - b(y)| \le L|x - y|, \qquad |a(x) - a(y)| \le L|x - y|$$

for all $x, y \in \mathbb{R}$ (see, for example, [13, Chapter V, §3]).

Theorem 3. *Assume that conditions* (**H**), (**C**), *and* (**L**) *hold. Then the sequence* Z_n, $n \ge 1$, *is a strong Markov approximation of the process* Z.

Proof. Consider the pair of processes $(\hat{X}_n, \hat{X}^n)$ constructed above. We introduce the process $\hat{Z}_n$ as the functional defined by equality (12) where $\hat{X}_n$ substitutes X_n. Let $\hat{Z}^n$ be defined by the stochastic differential equation (11), where $\hat{X}^n$ stands in place of the noise.

By construction, the pair $(\hat{Z}_n, \hat{Z}^n)$ satisfies the first property of a Markov approximation. The Markov property can be checked by following the same idea as in the case of the preceding pair of the processes. A bound for the distance in the third property of a Markov approximation can be proved by using the method described in Section 2.

The processes $\hat{Z}_n^1$, $\hat{Z}^{1,n}$, $\hat{Z}_n^2$, and $\hat{Z}^{2,n}$ are constructed from the corresponding processes used to approximate $\hat{X}_n$ and $\hat{X}^n$. The estimate of the accuracy of the first approximation is the same as in Lemma 1. Below are necessary bounds for the second approximation.

Lemma 5. *Let* $\gamma > 0$ *be an arbitrary number. Then*

$$\mathsf{P}\left(\sup_{1\le k\le n}\left|\hat{Z}_n^1\left(\frac{k}{n}\right) - \hat{Z}_n^2\left(\frac{k}{n}\right)\right| > \gamma\right) < \frac{\sigma_n(\varepsilon)C(n,\varepsilon,R)}{\gamma^2},$$

where $C(n,\varepsilon,R)$ *approaches a certain constant as* $n \to \infty$ *and is bounded as* $\varepsilon \to 0$, *and where*

$$\sigma_n(\varepsilon) = \frac{1}{n}\left[\int_{-\varepsilon}^{\varepsilon} u^2\,\Pi(du) + \Delta_n(\varepsilon)\right],$$

$$\mathsf{P}\left(\sup_{1\le k\le n}\left|\hat{Z}^{1,n}\left(\frac{k}{n}\right) - \hat{Z}^{2,n}\left(\frac{k}{n}\right)\right| > \gamma\right) \le \frac{\sigma(\varepsilon)}{\gamma^2} B_1 e^{B(\varepsilon,R)}.$$

Here B_1 *is an upper bound for the moments of the processes,* $B_{\varepsilon,R}$ *is a constant being bounded as* $\varepsilon \to 0$ *if* R *is fixed, and*

$$\sigma(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} u^2\,\Pi(du).$$

Proof. We briefly describe the idea of the proof of the first bound. Write

$$\Delta_{k,n}^Z = \hat{Z}_n^1\left(\frac{k}{n}\right) - \hat{Z}_n^2\left(\frac{k}{n}\right), \qquad \Delta_{k,n}^a = a\left(\hat{Z}_n^1\left(\frac{k}{n}\right)\right) - a\left(\hat{Z}_n^2\left(\frac{k}{n}\right)\right),$$

$$\Delta_{k,n}^b = b\left(\hat{Z}_n^1\left(\frac{k}{n}\right)\right) - b\left(\hat{Z}_n^2\left(\frac{k}{n}\right)\right), \qquad \Delta_{k,n}^\xi = \left(\hat{\xi}_{kn}^1 - \hat{\xi}_{kn}^2\right) - \left(\beta_n^1 - \beta_n^2\right),$$

$$\bar{\beta}_n = \int_{1\le|u|\le R} u\,F_n(du), \qquad \tilde{\beta}_n^i = \mathsf{E}\,\hat{\xi}_{kn}^i = \beta_n^i + \bar{\beta}_n, \qquad i = 1, 2.$$

According to Chebyshev's inequality,

$$\mathsf{P}\left(\sup_{1\le k\le n}|\Delta_{k,n}^Z| > \gamma\right) \le \frac{1}{\gamma^2}\,\mathsf{E}\left(\sup_{1\le k\le n}|\Delta_{k,n}^Z|\right)^2.$$

The definition of the corresponding processes implies that

$$\begin{aligned}
&\frac{1}{\gamma^2}\,\mathsf{E}\left(\sup_{1\le k\le n}\left|\Delta^Z_{k,n}\right|\right)^2\\
&\le \frac{3}{\gamma^2}\left(\mathsf{E}\left(\sup_{1\le k\le n}\sum_{i=1}^{k}\left|\Delta^a_{i-1,n}\right|\cdot\frac{1}{n}\right)^2+\mathsf{E}\left(\sup_{1\le k\le n}\sum_{i=1}^{k}\left|\Delta^b_{i-1,n}\right|\cdot\bar{\beta}_n\right)^2\right.\\
&\qquad+\mathsf{E}\left(\sup_{1\le k\le n}\left|\sum_{i=1}^{k}\left\{b\left(\hat{Z}^1_n\left(\frac{k-1}{n}\right)\right)\cdot\left[\hat{\xi}^1_{kn}-\tilde{\beta}^1_n\right]\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.\left.-\,b\left(\hat{Z}^2_n\left(\frac{k-1}{n}\right)\right)\cdot\left[\hat{\xi}^2_{kn}-\tilde{\beta}^2_n\right]\right\}\right|\right)^2\right)\\
&=:\frac{3}{\gamma^2}(S_1+S_2+S_3).
\end{aligned}$$

Now we can apply the maximal martingale inequality for the third term on the right-hand side.

To estimate the first two terms, we prove the following inequality for the processes $\hat{Z}^1_n$ and $\hat{Z}^2_n$:

$$\mathsf{E}\left|\hat{Z}^1_n\left(\frac{k}{n}\right)-\hat{Z}^2_n\left(\frac{k}{n}\right)\right|^2\le 3B^2\sigma_n(\varepsilon)\frac{\left(1+\frac{1}{n}L^2C_{\varepsilon,R}+r_{n,\varepsilon,R}\right)^k-1}{L^2C_{\varepsilon,R}+nr_{n,\varepsilon,R}}$$

for $1\le k\le n$, where

$$C_{\varepsilon,R}=\int_{\varepsilon\le|u|\le R}u^2\,\Pi(du),\qquad r_{n,\varepsilon,R}=\frac{L^2}{n^2}-2\beta^2_n\bar{\beta}_n=\bar{o}\left(\frac{1}{n}\right).$$

The definition of the processes $\hat{Z}^1_n$ and $\hat{Z}^2_n$ yields

$$\begin{aligned}
&\mathsf{E}\left|\Delta^Z_{k,n}\right|^2\\
&\le 4\left[\mathsf{E}\left|\Delta^Z_{k-1,n}\right|^2+\frac{L^2}{n^2}\,\mathsf{E}\left|\Delta^Z_{k-1,n}\right|^2+L^2(\bar{\beta}_n)\,\mathsf{E}\left|\Delta^Z_{k-1,n}\right|^2\right.\\
&\qquad\left.+\,\mathsf{E}\left|b\left(\hat{Z}^1_n\left(\frac{k-1}{n}\right)\right)\left[\hat{\xi}^1_{kn}-\tilde{\beta}^1_n\right]-b\left(\hat{Z}^2_n\left(\frac{k-1}{n}\right)\right)\left[\hat{\xi}^2_{kn}-\tilde{\beta}^2_n\right]\right|^2\right].
\end{aligned}\tag{13}$$

Denote the last term on the right-hand side of (13) by M. Adding and subtracting the term

$$b\left(\hat{Z}^1_n\left(\frac{k-1}{n}\right)\right)\left[\hat{\xi}^2_{kn}-\tilde{\beta}^2_n\right]$$

we get

$$\begin{aligned}
M=\mathsf{E}\left|b\left(\hat{Z}^1_n\left(\frac{k-1}{n}\right)\right)\right|^2|\Delta^\xi_{k,n}|^2+\operatorname{Var}\hat{\xi}^2_{kn}\,\mathsf{E}\left|\Delta^b_{k-1,n}\right|^2\\
+\,2\,\mathsf{E}\,b\left(\hat{Z}^1_n\left(\frac{k-1}{n}\right)\right)\Delta^\xi_{k,n}\cdot\hat{\xi}^2_{kn}\Delta^b_{k-1,n}.
\end{aligned}$$

Passing to the conditional expectation with respect to the σ-algebra

$$\mathfrak{F}^Z_{k-1,n}=\sigma\left\{\left(\hat{Z}^1_n\left(\frac{l}{n}\right),\hat{Z}^2_n\left(\frac{l}{n}\right)\right),1\le l\le k-1\right\},$$

we obtain

$$2\,\mathsf{E}\,\mathsf{E}\left[b\left(\hat{Z}_n^1\left(\frac{k-1}{n}\right)\right)\Delta_{k,n}^{\xi}\cdot\hat{\xi}_{kn}^2\Delta_{k-1,n}^b\;\Big|\;\mathfrak{F}_{k-1,n}^Z\right]=0.$$

Thus the right-hand side of (13) is estimated as follows:

$$\begin{aligned}4\Bigg[\mathsf{E}\left|\Delta_{k-1,n}^Z\right|^2+\frac{L^2}{n^2}\,\mathsf{E}\left|\Delta_{k-1,n}^Z\right|^2+L^2(\bar{\beta}_n)^2\,\mathsf{E}\left|\Delta_{k-1,n}^Z\right|^2+\frac{B^2\sigma_n(\varepsilon)}{n}\\ +L^2\operatorname{Var}\hat{\xi}_{kn}^2\,\mathsf{E}\left|\Delta_{k-1,n}^b\right|^2\Bigg]\\ =4\left[\left(1+L^2(\bar{\beta}_n)^2+L^2\operatorname{Var}\hat{\xi}_{kn}^2+\frac{L^2}{n^2}\right)\cdot\mathsf{E}\left|\Delta_{k-1,n}^b\right|^2+\frac{B^2\sigma_n(\varepsilon)}{n}\right].\end{aligned}$$

Moreover,

$$(\bar{\beta}_n)^2+\operatorname{Var}\hat{\xi}_{kn}^2=(\bar{\beta}_n)^2+\mathsf{E}(\xi_{kn}^2)^2-(\beta_n)^2-(\bar{\beta}_n)^2-2\beta_n\bar{\beta}_n=\frac{1}{n}C_{\varepsilon,R}-2\beta_n\bar{\beta}_n.$$

Then

$$\mathsf{E}\left|\Delta_{k,n}^Z\right|^2\le 3\left[\left(1+\frac{1}{n}C_{\varepsilon,R}+r_{n,\varepsilon,R}\right)\mathsf{E}\left|\Delta_{k-1,n}^Z\right|^2+\frac{B^2\sigma_n(\varepsilon)}{n}\right].$$

Iterating the latter relation we prove the required inequality. This inequality implies the first inequality of the lemma. Using an analogous reasoning and Gronwall's lemma (see, for example, [12]), we prove the second bound, as well. □

The following result is similar to Lemma 4.

Lemma 6. *Let $\gamma>0$ be an arbitrary number. Then*

$$\limsup_{n\to\infty}\mathsf{P}\left(\sup_{1\le k\le n}\left|\hat{Z}_n^2\left(\frac{k}{n}\right)-\hat{Z}^{2,n}\left(\frac{k}{n}\right)\right|>\gamma\right)=0.$$

Taking into account the bounds obtained above and considering the same ε and R as in the previous construction we prove the third property in the definition of a Markov approximation. As already shown, the properties 2 and 3 of this definition hold with $K(\gamma,T)=1$. This proves that $Z_n(t)$ is a strong Markov approximation of the Lévy process $Z(t)$. □

4. Concluding remarks

We prove that

$$X_n(t)=\sum_{k=1}^{[nt]}\xi_{kn}-tb_n,\qquad n\ge 1,\quad t\in[0;1],$$

is a strong Markov approximation of a Lévy process where $\{\xi_{kn},1\le k\le n,n\ge 1\}$ is an infinitesimal scheme of series. We also prove that a sequence of difference approximations is a strong Markov approximation of a solution of a stochastic differential equation with a Lévy noise. Such results allow one to obtain limit theorems for strongly discontinuous functionals of Lévy processes and for solutions of stochastic differential equations with a Lévy noise by using the method introduced in [3].

Bibliography

1. M. Donsker, *An invariance principle for certain probability limit theorems*, Mem. Amer. Math. Soc. **6** (1951), 1–10. MR0040613 (12:723a)
2. A. M. Kulik, *Markov approximation of stable processes by random walks*, Theory Stoch. Process. **12(28)** (2006), no. 1–2, 87–93. MR2316289 (2008j:60082)
3. Yu. N. Kartashov and A. M. Kulik, *Weak convergence of additive functionals of a sequence of Markov chains*, Theory Stoch. Process. **15(31)** (2009), no. 1, 15–32. MR2603167 (2011a:60130)
4. E. B. Dynkin, *Markov processes*, "Fizmatgiz", Moscow, 1963; English transl., Springer-Verlag, Berlin–Göttingen–Heidelberg, vol. I and II, 1965. MR0193670 (33:1886)
5. A. M. Kulik, *Difference approximation of the local times of multidimensional diffusions*, Teor. Imovir. Matem. Statyst. **78** (2008), 86–102; English transl. in Theor. Probability and Math. Statist. **78** (2009), 83–95. MR2446852 (2010b:60212)
6. A. V. Skorokhod, *Studies in the Theory of Random Processes*, Kiev University, Kiev, 1961; English transl., Addison-Wesley, Reading, 1965. MR0185620 (32:3082b)
7. B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, "Gostehizdat", Moscow, 1949; English transl., Addison-Wesley (Cambridge, MA), 1954. MR0062975 (16:52d)
8. W. Feller, *An Introduction to Probability Theory and its Applications*, vol. 2, John Wiley & Sons, Inc., New York–London–Sydney, 1971. MR0270403 (42:5292)
9. A. N. Shiryaev, *Probability*, third edition, vol. 1, MCNMO, Moscow, 2004; English transl. of second Russian edition, Springer-Verlag, Berlin–New York, 1996.
10. A. V. Skorokhod, *Limit theorems for stochastic processes*, Teor. Veroyatnost. Primenen. **1** (1956), 289–319; English transl. in Theory Probab. Appl. **1**, 261–290. MR0084897 (18:943c)
11. R. M. Dudley, *Real Analysis and Probability*, Cambridge University Press, New York, 2004. MR1932358 (2003h:60001)
12. I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations*, "Naukova Dumka", Kiev, 1968; English transl., Springer-Verlag, New York, 1972.
13. P. E. Protter, *Stochastic Integration and Differential Equations*, Second edition, Springer, New York, 2004. MR2020294 (2005k:60008)

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Received 21/JUNE/2011

Translated by N. SEMENOV