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OPTIMAL STOPPING TIME PROBLEM FOR RANDOM WALKS WITH POLYNOMIAL REWARD FUNCTIONS

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Abstract. The optimal stopping time problem for random walks with a drift to the left and with a polynomial reward function is studied by using the Appel polynomials. An explicit form of optimal stopping times is obtained.

1. Introduction

The stopping time problem for stochastic processes has several important applications for the modelling of the optimal behavior of brokers and dealers trading securities in financial markets.

The classical approach to solving the optimal stopping time problem is based on the excessive functions needed to determine the so-called reference set that defines explicitly the optimal stopping time [1–3].

An entirely different approach to the solution of the optimal stopping time problem is used in this paper. Namely, our method is based on an application of the Appel polynomials (see [4, 5]). This paper is a continuation of studies initiated in [6] and is devoted to a generalization of some results obtained in [4].

Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of independent identically distributed random variables defined on a probability space $(\Omega, \Im, \mathsf{P})$ and such that $\mathsf{E}\,\xi < 0$. Consider a homogeneous Markov chain $X = (X_1, X_2, X_3, \dots)$ related to the sequence $\{\xi_i\}$ as follows:

$$X_0 = x \in \mathbf{R}, \qquad X_k = x + S_k, \qquad S_0 = 0, \qquad S_k = \sum_{i=1}^{k} \xi_i, \qquad k \geq 1.$$

We denote by P_x the probability distribution generated by the process X. Thus, P_x, $x \in \mathbf{R}$, together with X defines a Markov family with respect to the filtration $(\Im_k)_{k \geq 0}$, where $\Im_0 = \{\varnothing, \Omega\}$ and $\Im_k = \sigma\{\xi_1, \dots, \xi_k\}$, $k \geq 1$.

The optimal stopping problem is to determine a reward function

$$V(x) = \sup_{\tau \in \bar{\mathfrak{M}}_0^\infty} \mathsf{E}_x\, g(X_\tau) I\{\tau < \infty\}, \qquad x \in \mathbf{R},$$

where $g(x)$ is a given measurable function, $I\{\cdot\}$ is an indicator function, and $\bar{\mathfrak{M}}_0^\infty$ is the class of all Markov times τ assuming values in $[0, \infty]$.

The random variable

$$\tau^* = \operatorname*{argmax}_{\tau \in \bar{\mathfrak{M}}_0^\infty} \mathsf{E}_x\, g(X_\tau) I\{\tau < \infty\}$$

is called the optimal stopping time.

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The power reward function $g(x) = (x^+)^k$ is considered in the paper [4]. In the current paper, we consider polynomial reward functions $g(x)$ of the following form:

$$g(x) = \sum_{k=1}^{N} C_k (x^+)^k, \qquad C_k \in \mathbf{R}.$$

The optimal stopping time problem for a random walk X is considered in the paper [6] for the case of polynomial reward functions of an arbitrary order and with nonnegative coefficients; an explicit expression for the optimal stopping time and that for the price function are also found in [6] for this case. The aim of this paper is to determine the optimal stopping time for a random walk $X = (X_1, X_2, X_3, \ldots)$, where both a reward function and a price function are polynomial. Since the solution of the optimal stopping time problem is cumbersome for general coefficients of the polynomial reward and price functions, we state the result below for the general case and provide the detailed proof for a particular case where the order of the polynomial reward function does not exceed 3.

2. Auxiliary results and definitions

We need several auxiliary results and definitions for the proof of the main result of this paper.

Definition 1. Let η be a random variable such that $\mathsf{E}\exp(\lambda|\eta|) < \infty$ for some $\lambda > 0$. The polynomials defined by

$$\frac{\exp(uy)}{\mathsf{E}\exp(u\eta)} = \sum_{k=0}^{\infty} \frac{u^k}{k!} Q_k(y)$$

are called the Appel (or Sheffer; see [5]) polynomials, $Q_k(y) = Q_k(y, \eta)$, $k = 0, 1, 2, \ldots$.

The polynomials $Q_k(y)$ are expressed in terms of the cumulants $\chi_1, \chi_2, \ldots$ of the random variable η as follows:

$$Q_0(y) = 1, \qquad Q_1(y) = y - \chi_1, \qquad Q_2(y) = (y - \chi_1)^2 - \chi_2,$$
$$Q_3(y) = (y - \chi_1)^3 - 3\chi_2(y - \chi_1) - \chi_3,$$

where $\chi_1 = \mu_1$, $\chi_2 = -\mu_1^2 + \mu_2$, and $\chi_3 = 2\mu_1^3 - 3\mu_1\mu_2 + \mu_3$; here $\mu_k = \mathsf{E}\,\eta^k$.

Note that the polynomials $Q_k(y)$, $k = 1, \ldots, n$, are uniquely defined if one assumes that $\mathsf{E}\,|\eta|^n < \infty$. Moreover,

$$\frac{d}{dy} Q_k(y) = k Q_{k-1}(y), \qquad k \le n, \tag{1}$$

in this case. This equality is sometimes used to define the Appel polynomials recursively [4].

Note that $Q_k(y)$ is a polynomial whose order equals k; see [5]. In particular, this means that every set of n Appel polynomials $Q_1(y), Q_2(y), \ldots, Q_n(y)$ is a system of linearly independent functions.

Throughout in this paper we deal with Appel polynomials generated by the random variable $M = \sup_{k\ge 0} S_k$, that is, $Q_k(y) = Q_k(y, M)$, $k = 0, 1, 2, \ldots$. The cases where the distribution of the random variable M can be written explicitly are considered in [7–10].

The following result is proved in [4]. Lemma 1 provides us with sufficient conditions for the Appel polynomials generated by the random variable M to be well defined.

Lemma 1. a) *Let* $\mathsf{E}\,e^{\lambda\xi} < 1$ *for some* $\lambda > 0$. *Then*

$$\mathsf{E}\,e^{uM} < \infty$$

for all $u \le \lambda$.

b) *For an arbitrary* $p > 0$,

$$\mathsf{E}(\xi^+)^{p+1} < \infty \quad \Longrightarrow \quad \mathsf{E}\, M^p < \infty.$$

Remark 1. It is proved in [4] that

$$M \geq 0, \qquad \mathsf{P}\{M < \infty\} = 1, \qquad \mathsf{P}\{M = 0\} > 0, \quad \text{and} \quad M \stackrel{\text{law}}{=} (M + \xi)^+.$$

The proof of the following result is also given in [4].

Lemma 2. 1) *Let* $\mathsf{E}(\xi^+)^{n+1} < \infty$. *Then*
a) $\mathsf{E}\, Q_n(M + x) = x^n$;
b) *if* $\tau_a = \inf\{k \geq 0\colon X_k \geq a\}$, *then*

$$\mathsf{E}_x\, I\{\tau_a < \infty\} X^n_{\tau_a} = \mathsf{E}\, I\{M + x \geq a\} Q_n(M + x)$$

for all $a \geq 0$.
2) *The polynomial* $Q_n(y)$, $n \geq 1$, *has a unique positive root* a^*_n; *moreover* $Q_n(y) \leq 0$ *for* $0 \leq y < a^*_n$, *and* $Q_n(y)$ *increases for* $y \geq a^*_n$.
3) *Let* $f(x) = \mathsf{E}\, I\{M + x \geq a^*\} G(M + x) < \infty$, *where the function* $G(x)$ *is such that* $G(y) \geq G(x) \geq G(a^*) = 0$ *for all* $y \geq x \geq a^* \geq 0$. *Then* $f(x) \geq \mathsf{E}\, f(\xi + x)$ *for all* x.
4) *Let* $f(x)$ *and* $g(x)$ *be two nonnegative functions such that*

$$f(x) \geq g(x) \tag{2}$$

and $f(x) \geq \mathsf{E}\, f(\xi + x)$ *for all* x. *Then*

$$f(x) \geq \sup_{\tau \in \overline{\mathfrak{M}}^\infty_0} \mathsf{E}\, I\{\tau < \infty\} g(S_\tau + x)$$

for all x.

It is proved in [4] that the roots a^*_n of the Appel polynomials $Q_n(y)$ are increasing, namely $0 < a^*_1 < a^*_2 < a^*_3 < \dots$.

The representation of the Appel polynomials in terms of the cumulants and property 2) of Lemma 2 imply that

$$\chi_1 \geq 0, \qquad \chi_2 \geq 0, \qquad \chi_3 \geq 0, \qquad \chi_2 \geq \chi_1^2, \qquad \chi_1^3 - 3\chi_2\chi_1 + \chi_3 \geq 0.$$

The case of $\chi_1 = 0$ is degenerate in the sense that the random walk moves to the left in this case and hence $\eta = 0$. This implies that the Appel polynomials are power functions in the case of $\chi_1 = 0$. Thus throughout below we assume that $\chi_1 > 0$.

Definition 2. We say that $P(y)$ is a function of type $\mathbf{A}(a)$ if there exists a number $a > 0$ such that

$$\begin{cases} P(a) = 0, \quad P(y) \leq 0 \text{ in } [0, a), \\ P(y) \text{ increases in } [a, \infty). \end{cases} \tag{3}$$

The following result contains a simple criteria for a function

$$P_2(y) = -C_1 Q_1(y) + C_2 Q_2(y), \qquad C_1,\ C_2 > 0,$$

to be of type $\mathbf{A}(a)$.

Lemma 3. *Let* $\chi_2 > \chi_1^2$. *The polynomial*

$$P_2(y) = -C_1 Q_1(y) + C_2 Q_2(y), \qquad C_1,\ C_2 > 0,$$

is a function of type $\mathbf{A}(a)$ *if and only if the coefficients* C_1 *and* C_2 *are such that*

$$\frac{C_1}{C_2} \leq \frac{\chi_2}{\chi_1} - \chi_1.$$

In this case, the number a for which the polynomial $P_2(y)$ satisfies property (3) *is given by*

$$a = a_2^* = \frac{1}{2}\frac{C_1}{C_2} + \sqrt{\frac{1}{4}\frac{C_1^2}{C_2^2} + \chi_2} + \chi_1.$$

Proof. We represent the polynomial $P_2(y)$ in terms of the cumulants

$$P_2(y) = C_2(y-\chi_1)^2 - C_1(y-\chi_1) - C_2\chi_2.$$

Then the roots of the polynomial $P_2(y)$ are given by

$$a_{1,2}^* = \chi_1 + \frac{1}{2}\frac{C_1}{C_2} \pm \sqrt{\frac{1}{4}\frac{C_1^2}{C_2^2} + \chi_2}.$$

Since the maximal root of this polynomial is positive and $P_2(y)$ is increasing in the semiaxis on the right of the maximal root, the system of conditions (3) is equivalent to the inequality $a_1^* \le 0$. We write the latter condition explicitely as

$$\chi_1 + \frac{1}{2}\frac{C_1}{C_2} - \sqrt{\frac{1}{4}\frac{C_1^2}{C_2^2} + \chi_2} \le 0,$$

or, equivalently,

$$C_2(\chi_2 - \chi_1^2) \ge C_1\chi_1. \tag{4}$$

The left hand side of inequality (4) is positive by the assumptions of the lemma. This implies the following restrictions on the coefficients C_1 and C_2:

$$\frac{C_1}{C_2} \le \frac{\chi_2}{\chi_1} - \chi_1.$$

Lemma 3 is proved. □

Theorem 1. *Let $\mathsf{E}\,\xi < 0$ and $\mathsf{E}(\xi^+)^k < \infty$ for $k \in \{1, 2\}$. Assume that $\chi_2 > \chi_1^2$ and that the coefficients C_1 and C_2 satisfy the assumptions of Lemma* 3. *Then the stopping time $\tau_2^* = \inf\{k \ge 0 \mid X_k \ge a_2^*\}$ is optimal for the random walk $X = (X_0, X_1, X_2, \ldots)$ with the reward function $g(x) = -C_1x^+ + C_2(x^+)^2$.*

The proof of this result is given in [6].

3. Main result

Consider the reward function

$$g(x) = C_1x^+ + C_2(x^+)^2 + \ldots + C_n(x^+)^n, \qquad C_n > 0.$$

This function is related to a linear combination of the Appel polynomials

$$P_n(y) = C_1Q_1(y) + C_2Q_2(y) + \ldots + C_nQ_n(y), \qquad C_n > 0.$$

We choose the coefficient C_n to be positive, since we will assume throughout that the polynomial $P_n(y)$ is a function of type $\mathbf{A}(a)$ and this is not the case if the coefficient is negative.

We want to establish the restrictions imposed on the coefficients $C_1, C_2, \ldots, C_n$ of the polynomial $P_n(y)$ under which $P_n(y)$ is a function of type $\mathbf{A}(a)$. Another aim is to find an explicit expression for the price function and the optimal stopping time for the random walk $X = (X_1, X_2, X_3, \ldots)$ with the reward function $g(x)$.

The following result contains necessary and sufficient conditions to be imposed on the coefficients in order that the polynomial $P_n(y)$ is a function of type $\mathbf{A}(a)$. We omit the proof of this result.

Theorem 2. *The polynomial $P_n(y)$ is a function of type $\mathbf{A}(a)$ if and only if one of the following conditions holds:*

a) *The derivative of the polynomial $P_n(y)$ does not have roots in the semiaxis $[0,\infty)$, and $P_n(0)<0$.*
b) *The derivative of the polynomial $P_n(y)$ does have roots in the semiaxis $[0,\infty)$ but these points are not local extremums of the polynomial $P_n(y)$, and $P_n(0)<0$.*
c) *The derivative of the polynomial $P_n(y)$ does have roots in the semiaxis $[0,\infty)$ and some of them are local extremums of the polynomial $P_n(y)$; the first of the local extremums in $[0,\infty)$ is a point of minimum; the function $P_n(y)$ is negative at each of the local maximums $y\in[0,\infty)$ if they exist; and $P_n(0)\le 0$.*
d) *The derivative of the polynomial $P_n(y)$ does have roots in the semiaxis $[0,\infty)$ and some of them are local extremums of the polynomial $P_n(y)$; the first of the local extremums in $[0,\infty)$ is a point of maximum of the function $P_n(y)$ and $P_n(y)$ is negative at each of the local maximums $y\in[0,\infty)$.*

The assumptions of Theorem 2 imposed on the coefficients C_1, C_2, $\dots$, C_n of the polynomial $P_n(y)$ look cumbersome if $n>3$. Thus we restrict the consideration to the case of $n=3$. The following result provides necessary and sufficient conditions on the coefficients C_1, C_2, and C_3 for the case of $n=3$ under which the polynomial $P_3(y)$ is a function of type $\mathbf{A}(a)$.

Theorem 3. *A polynomial $P_3(y)=C_1Q_1(y)+C_2Q_2(y)+C_3Q_3(y)$, $C_3>0$, is a function of type $\mathbf{A}(a)$ with some $a>0$ if and only if one of the following four conditions holds*:

$$\begin{cases} \frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2\le 0,\\ \frac{C_2}{C_3}[\chi_1^2-\chi_2]-\frac{C_1}{C_3}\chi_1\\ \qquad <\chi_1^3-3\chi_2\chi_1+\chi_3, \end{cases} \qquad \begin{cases} \frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2> 0,\\ \sqrt{\frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2}<\frac{C_2}{C_3}-3\chi_1,\\ \frac{C_2}{C_3}[\chi_1^2-\chi_2]-\frac{C_1}{C_3}\chi_1<\chi_1^3-3\chi_2\chi_1+\chi_3, \end{cases}$$

$$\begin{cases} \frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2> 0,\\ \sqrt{\frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2}>3\chi_1-\frac{C_2}{C_3},\\ \sqrt{\frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2}\ge\frac{C_2}{C_3}-3\chi_1,\\ \frac{C_2}{C_3}[\chi_1^2-\chi_2]-\frac{C_1}{C_3}\chi_1\le\chi_1^3-3\chi_2\chi_1+\chi_3, \end{cases} \qquad \begin{cases} \frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2> 0,\\ \sqrt{\frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2}\le 3\chi_1-\frac{C_2}{C_3},\\ 2\left(\frac{C_2^2}{C_3^2}-3\frac{C_1}{C_3}+9\chi_2\right)^{\frac{3}{2}}\\ \qquad <9\frac{C_1}{C_3}\frac{C_2}{C_3}-2\frac{C_2^3}{C_3^3}+27\chi_3. \end{cases}$$

Proof. We use property (1) of Appel polynomials and the representation of the Appel polynomials in terms of cumulants to write the derivative of the function $P_3(y)$ in an explicit form. We have

$$\frac{dP_3(y)}{dy}=C_1+2C_2Q_1(y)+3C_3Q_2(y)=3C_3(y-\chi_1)^2+2C_2(y-\chi_1)+C_1-3C_3\chi_2.$$

Since $C_3>0$, the graph of the derivative as a function of the argument $(y-\chi_1)$ is a parabola going to infinity with its argument. The roots of the derivative are given by

$$y_{1,2}=\chi_1+\frac{-C_2\pm\sqrt{C_2^2-3C_1C_3+9C_3^2\chi_2}}{3C_3}.$$

Denote by $D=C_2^2-3C_1C_2+9C_3^2\chi_2$ the discriminant of the derivative $dP_3(y)/dy$.

Now we rewrite the assumptions of Theorem 2 for the case of a polynomial $P_3(y)$ of the third order.

Assumption a) holds if and only if one of the following conditions holds:

1) either the discriminant D of the derivative $dP_3(y)/dy$ is negative and $P_3(0)<0$;

2) or the discriminant D of the derivative $dP_3(y)/dy$ equals zero, $y_1 < 0$, and $P_3(0) < 0$;
3) or the discriminant D of the derivative $dP_3(y)/dy$ is positive, $y_2 < 0$, and $P_3(0) < 0$.

Assumption b) is equivalent to the following one:

4) the discriminant D of the derivative $dP_3(y)/dy$ equals zero, $y_1 \geq 0$, and $P_3(0) < 0$.

Assumption c) is equivalent to the following one:

5) the discriminant D of the derivative $dP_3(y)/dy$ is positive, $y_1 < 0$, $y_2 \geq 0$, and $P_3(0) \leq 0$.

Finally, assumption d) holds if and only if

6) the discriminant D of the derivative $dP_3(y)/dy$ is positive, $y_1 \geq 0$, and $P_3(y_1) < 0$.

Merging conditions 1), 2), and 4), we obtain the system

$$\begin{cases} D \leq 0, \\ P_3(0) < 0, \end{cases}$$

which is equivalent to the first system in the statement of Theorem 2.

Transforming conditions 3), 5), and 6), we see that they coincide with the second, third, and fourth systems in the statement of Theorem 2, respectively.

Therefore all the possible cases are considered and Theorem 2 is proved. □

Now we are going to investigate the solvability of the system in Theorem 3 for different cases.

Remark 2. a) Let $\chi_2 - \chi_1^2 = 0$ and $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$. The domains where the systems of inequalities in Theorem 3 have solutions are depicted in Figure 1(a). Our current aim is to show that there are no polynomials $P_3(y)$ of type $\mathbf{A}(a)$ with $C_1 < 0$. Indeed, the last inequalities of the first and second systems in Theorem 3 reduce to $C_1/C_3 > 0$, while the last inequality of the third system is given by $C_1/C_3 \geq 0$. We have shown numerically that the last inequality of the fourth system does not hold if C_1 and C_3 are such that $C_1/C_3 < 0$.

b) Let $\chi_2 - \chi_1^2 > 0$ and $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$. The domains where a solution of the systems of inequalities in Theorem 3 exists are depicted in Figure 1(b). Note that there are no polynomials $P_3(y)$ of type $\mathbf{A}(a)$ with two restrictions $C_1 < 0$ and $C_2 < 0$ in this case. This result follows, since the last inequalities of the first and second systems in Theorem 3 reduce to $C_2 > \chi_1/(\chi_1^2 - \chi_2)C_2$. The last inequality of the third system transforms to $C_2 \geq \chi_1/(\chi_1^2 - \chi_2)C_2$. We have shown numerically that the last inequality of the fourth system also does not hold if $C_1 < 0$ and $C_2 < 0$.

c) Let $\chi_2 - \chi_1^2 = 0$ and $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 > 0$. The domains where solutions of the systems of inequalities in Theorem 3 exist are depicted in Figure 1(c). Note that there are polynomials $P_3(y)$ of type $\mathbf{A}(a)$ in this case whose coefficients C_1 and C_2 both are positive or both are negative or such that $C_1 \cdot C_2 < 0$. This follows from the first and third system in Theorem 3.

d) Let $\chi_2 - \chi_1^2 > 0$ and $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 > 0$. The domains where the systems of inequalities in Theorem 3 have solutions are depicted in Figure 1(d) for this case. Similarly to the preceding case, there are polynomials $P_3(y)$ of type $\mathbf{A}(a)$ whose coefficients C_1 and C_2 both are positive or both are negative or such that $C_1 \cdot C_2 < 0$.

Note that domains I, II, and III are depicted in Figure 1 with the help of precise restrictions, while approximate restrictions are used for domain IV.

The domains whose points satisfy the systems of inequalities in Theorem 3 are depicted in Figure 1 for the cases discussed in Remark 2.

By the roman numbers I, II, III, and IV we denote the numbers of the appropriate systems corresponding to the domains in Figure 1.

We have used the following data to depict Figure 1.

a) $\chi_2 - \chi_1^2 = 0$, $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$, $(\chi_1 = 1,\ \chi_2 = 1,\ \chi_2 = 2)$.
b) $\chi_2 - \chi_1^2 > 0$, $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$, $(\chi_1 = 1,\ \chi_2 = 2,\ \chi_2 = 5)$.
c) $\chi_2 - \chi_1^2 = 0$, $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 > 0$, $(\chi_1 = 1,\ \chi_2 = 1,\ \chi_2 = 3)$.
d) $\chi_2 - \chi_1^2 > 0$, $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 > 0$, $(\chi_1 = 1,\ \chi_2 = 2,\ \chi_2 = 6)$.

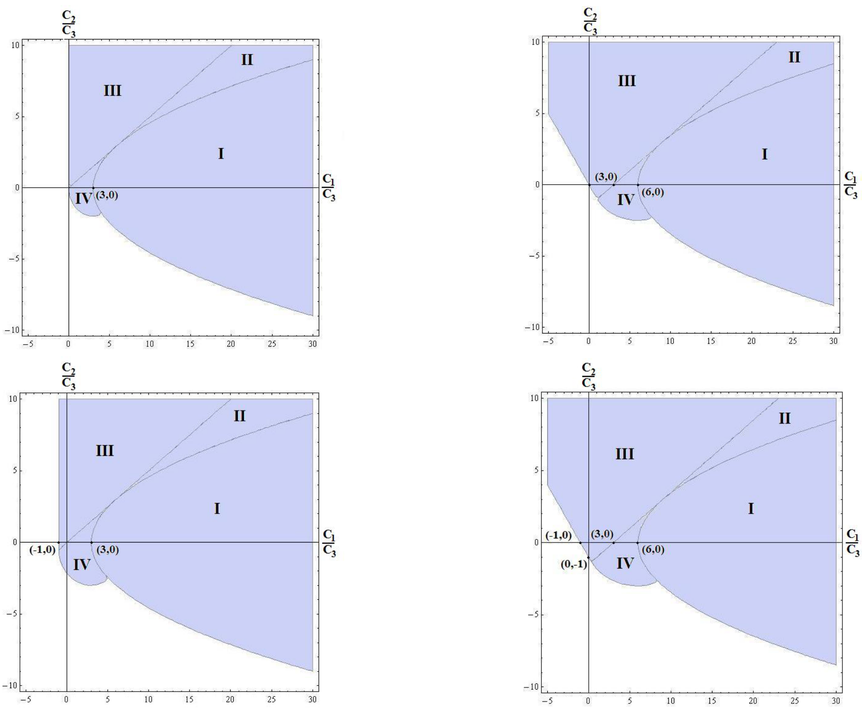

FIGURE 1. Domains of solutions of systems in Theorem 3 for different cases

Remark 3. It is proved in the paper [6] that if $C_1 > 0$, $C_2 > 0$, and $C_3 > 0$, then the polynomial $P_3(y) = C_1Q_1(y)+C_2Q_2(y)+C_3Q_3(y)$ is a function of type $\mathbf{A}(a)$. This result is compatible with that of Theorem 3. In particular, it is seen from Figure 1 that the domain of solutions of the systems of inequalities in Theorem 3 covers the right upper quadrant in all the cases a)–d).

The following result provides an explicit form of the optimal stopping times and price function for the random walk $X = (X_1, X_2, X_3, \ldots)$ with a polynomial reward function of order n.

Theorem 4. *Let* $\mathsf{E}\,\xi < 0$ *and* $\mathsf{E}(\xi^+)^{k+1} < \infty$ *for* $k \in \{1, 2, \ldots, n\}$. *Assume that the coefficients* $C_1, C_2, \ldots, C_n$ *of the polynomial* $g(x) = C_1x^+ + C_2(x^+)^2 + \ldots + C_n(x^+)^n$ *are such that* $C_n > 0$ *and that the polynomial* $P_n(y) = C_1Q_1(y) + C_2Q_2(y) + \ldots + C_nQ_n(y)$ *is a function of type* $\mathbf{A}(a)$. *Let* a_n^* *be a positive root of the polynomial* $P_n(y)$.

Then the stopping time $\tau_n^* = \inf\{k \ge 0 \mid X_k \ge a_n^*\}$ *is optimal for the random walk* $X = (X_0, X_1, X_2, \dots)$ *and the reward function* $g(x)$. *Moreover*

$$V_n(x) := \sup_{\tau \in \bar{\mathfrak{M}}_0^\infty} \mathsf{E}_x\, g(X_\tau) I\{\tau < \infty\} = \mathsf{E}_x\, g(X_{\tau_n^*}) I\{\tau_n^* < \infty\}$$

and

$$V_n(x) = \mathsf{E}\, P_n(M + x) I\{M + x \ge a_n^*\}.$$

Proof. Consider the functions $\widehat{g}(x) = C_1 x + C_2 x^2 + \dots + C_n x^n$ with $C_n > 0$ and

$$\widehat{V}_n(x) := \sup_{\tau \in \hat{\mathfrak{M}}_0^\infty} \mathsf{E}_x\, \widehat{g}(X_\tau) I\{\tau < \infty\},$$

where $\hat{\mathfrak{M}}_0^\infty$ is the class of stopping times of a special form, namely $\hat{\tau} = \tau_a$, $a \ge 0$. Here $\tau_a = \inf\{k \ge 0 | X_k \ge a\}$.

It is clear that $\hat{g}(X_{\tau_a}) = g(X_{\tau_a})$ in the set $\{\tau_a < \infty\}$, whence $\hat{V}_n(x) \le V_n(x)$, since $V_n(x)$ is defined with respect to a wider class of stopping times $\bar{\mathfrak{M}}_0^\infty$.

Now we show that

$$\hat{V}_n(x) = \mathsf{E}\, P_n(M + x) I\{M + x \ge a_n^*\}. \tag{5}$$

Indeed, the case 1), b) of Lemma 2 implies that

$$\mathsf{E}_x\, I\{\tau_a < \infty\} X_{\tau_a}^k = \mathsf{E}\, I\{M + x \ge a\} Q_k(M + x), \qquad k \in \{1, 2, \dots, n\}.$$

Multiplying the preceding equality for the corresponding Appel polynomials by the coefficients $C_1, C_2, \dots, C_n$ and summing up these results, we obtain

$$\mathsf{E}_x\, I\{\tau_a < \infty\} \hat{g}(X_{\tau_a}) = \mathsf{E}\, I\{M + x \ge a\} P_n(M + x),$$

where $P_n(M + x) \ge 0$ in the set $\{M + x \ge a\}$ for all $a \in [a_n^*, \infty)$. Thus

$$\mathsf{E}\, P_n(M + x) I\{M + x \ge a\}$$

is a decreasing function in $[a_n^*, \infty)$.

Now let $a \in [0, a_n^*]$. Then

$$\begin{aligned} \mathsf{E}\, P_n(M + x) I\{M + x \ge a\} = \mathsf{E}\, P_n(M + x) - \mathsf{E}\, P_n(M + x) I\{M + x < 0\} \\ - \mathsf{E}\, P_n(M + x) I\{0 \le M + x < a\}. \end{aligned}$$

The case 1), a) of Lemma 2 implies that

$$\begin{aligned} \mathsf{E}\, P_n(M + x) &= \mathsf{E}(C_1 Q_1(M + x) + C_2 Q_2(M + x) + \dots + C_n Q_n(M + x)) \\ &= C_1 x + C_2 x^2 + \dots + C_n x^n. \end{aligned}$$

It follows from Theorem 2 that $P_n(M + x) I\{0 \le M + x < a\} \le 0$, whence we deduce that

$$P_n(M + x) I\{0 \le M + x < a\}$$

is a decreasing function and thus the function $P_n(M + x) I\{M + x \ge a\}$ increases in $[0, a_n^*]$. Since the $P_n(M + x) I\{0 \le M + x < a\}$ is continuous and decreases in $[a_n^*, \infty)$, it attains the maximal value at the point $a = a_n^*$. Therefore equality (5) is proved and the stopping time $\widehat{\tau}_n = \tau_{a_n^*}$ is optimal in the class $\hat{\mathfrak{M}}_0^\infty$ for the reward function $\widehat{g}(x)$.

It remains to show that $\hat{V}_n \ge V_n(x)$ to complete the proof. Consider the function $f(x) = \hat{V}_n(x) = \mathsf{E}\, P_n(M + x) I\{M + x \ge a_n^*\}$ and apply statement 40 of Lemma 2 with $g(x) = C_1 x^+ + C_2 (x^+)^2 + \dots + C_n (x^+)^n$. First we check condition (2) for $x \in (0, a_n^*)$. Observe that

$$I\{M + x \ge a_n^*\} P_n(M + x) = (P_n(M + x))^+$$

by Theorem 3 for $x \in (0, a_n^*)$.

The Jensen inequality together with the case 1), a) of Lemma 2 imply that

$$f(x) = \mathsf{E}(P_n(M+x))^+ \geq (\mathsf{E}\, P_n(M+x))^+ = C_1x^+ + C_2(x^+)^2 + \ldots + C_n(x^+)^n = g(x).$$

The second condition in statement 4) of Lemma 2 holds with the function $G(y) = P_n(y)$ in view of statement 3) of Lemma 2.

Therefore, the function $f(x)$ is an excessive majorant for

$$g(x) = C_1x^+ + C_2(x^+)^2 + \ldots + C_n(x^+)^n,$$

whence we conclude that $f(x) \geq V_n(x)$. Since $f(x) = \hat{V}_n(x)$, we prove that $\hat{V}_n \geq V_n(x)$. Therefore, $\hat{V}_n = V_n(x)$ and hence the stopping time $\widehat{\tau}_n = \tau_{a_n^*}$ is optimal in the class $\mathfrak{M}_0^\infty$ for the reward function $g(x)$.

Theorem 4 is proved. □

4. Examples

The conditions $\chi_2 - \chi_1^2 > 0$, $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 > 0$ and $\chi_2 - \chi_1^2 = 0$, $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$ imposed on the cumulants χ_1, χ_2, and χ_3 of the supremum of the random walk $M = \sup_{k\geq 0} S_k$, where $S_0 = 0$, $S_k = \sum_{i=1}^k \xi_i$, $k \geq 1$, appear in Lemma 3 and also in the investigation of the solvability of the systems of inequalities in Theorem 3 (see Remark 2).

The main attention in the papers that are devoted to the studies of the distributions of supremums of random walks is paid to the tail behavior of distributions [11] or to the asymptotic behavior of distributions [12–14]. Only a few results are known where the distributions of maximums of random walks [7–10] can be written in a closed form. Examples of random walks possessing the properties needed for the purposes of this paper are constructed in [10] by using the distribution of the maximum of an integer-valued random walk with the exponential distribution that is widely used in the queueing theory [7].

Let $\nu_1, \nu_2, \nu_3 \ldots$ be independent identically distributed random variables assuming two values 2 and 0 with probabilities p and $1-p$, respectively, where $p \in [0, \frac{1}{2})$.

Consider the random walks $N_r = \sum_{i=1}^r \nu_i$ and $S_r = N_r - r = \sum_{i=1}^r (\nu_i - 1)$, $S_0 = 0$. The increments of the random walk S_r are equal to 1 and -1 with probabilities p and $1-p$, respectively. The expectation of the jump is equal to $\mathsf{E}[\nu_i - 1] = 2p - 1$ and is negative if $p \in [0, \frac{1}{2})$. This means that the random walk has a drift to the left.

The distribution of the random walk N_r, $r \geq 1$, is given by

$$\mathsf{P}(N_r = k) = \begin{cases} 0 & \text{if } k < 0 \text{ or if } k \text{ is odd}, \\ (1-p)^r & \text{if } k = 0, \\ 0 & \text{if } k \text{ is even and if } r < \frac{k}{2}, \\ p^{\frac{k}{2}}(1-p)^{r-\frac{k}{2}} & \text{if } k \text{ is even and if } r \geq \frac{k}{2}. \end{cases}$$

It is proved in the paper [10] that the distribution of the random variable $M = \sup_{k\geq 1} S_k$ is given by

$$\mathsf{P}(M < k) = \begin{cases} 1 - (1-2p)\sum_{j=1}^\infty \mathsf{P}(N_j = j+k) & \text{if } k > 0, \\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

Since the probabilistic characteristics of the maximum M of the random walk depend on the probability p, we denote the cumulants and moments of this maximum by $\chi_i(p)$, $i \geq 1$, and $\mu_i(p)$, $i \geq 1$, respectively.

Theorem 5. *There exists $p_0 \in (0, \frac{1}{2})$ such that the cumulants $\chi_1(p)$, $\chi_2(p)$, and $\chi_3(p)$ of the supremum $M = \sup_{k\geq 0} S_k$ of the random walk $\{S_k\}$ are such that*

$$\chi_2(p) - \chi_1^2(p) > 0, \quad \chi_1^3(p) - 3\chi_1(p)\chi_2(p) + \chi_3(p) > 0$$

for all $p \in (0, p_0]$.

Proof. It is easy to check that

$$\mu_1(p) = \sum_{k=1}^{\infty} k\,\mathsf{P}(M = k) = \sum_{k=1}^{\infty} \mathsf{P}(M \geq k),$$

$$\mu_2(p) = \sum_{k=1}^{\infty} k^2\,\mathsf{P}(M = k) = 2\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} \mathsf{P}(M \geq j),$$

$$\mu_3(p) = \sum_{k=1}^{\infty} k^3\,\mathsf{P}(M = k) = 6\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=j}^{\infty} \mathsf{P}(M \geq l).$$

If $p = 0$, then the supremum M of the random walk S_k equals zero, whence we derive that all the moments and cumulants of this supremum are zero.

Consider the function $f_1(p) = \chi_2(p) - \chi_1^2(p) = \mu_2(p) - 2\mu_1^2(p)$. Its derivative at the point $p = 0$ is given by

$$\frac{df_1(p)}{dp} = \frac{d\mu_2(p)}{dp} - 4\mu_1(p) \cdot \frac{d\mu_1(p)}{dp}. \tag{7}$$

The second term of the right hand side of (7) equals zero at the point $p = 0$, since $\mu_1(0) = 0$ in this case.

Then we consider the first term. Using formula (6),

$$\frac{d\mu_2(p)}{dp} = \left(2\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} \mathsf{P}(M \geq j)\right)'_p = 2\left(\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=1}^{\infty} (1-2p)\,\mathsf{P}(N_l = l+j)\right)'_p. \tag{8}$$

The expression on the right hand side of (8) is a power series with respect to p. The theorem on the differentiation of power series allows one to exchange the summation and the differentiation in the domain of convergence.

To evaluate the derivative $d\mu_2(p)/dp$ at the point $p = 0$, we write

$$\begin{aligned}\frac{d\mu_2(p)}{dp} &= 2\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\left((1-2p)\sum_{l=1}^{\infty} \mathsf{P}(N_l = l+j)\right)'_p \\ &= 2\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}(1-2p)\sum_{l=1}^{\infty}(\mathsf{P}(N_l = l+j))'_p - 4\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=1}^{\infty}\mathsf{P}(N_l = l+j).\end{aligned}$$

Using the explicit form of the distribution of N_r, $r \geq 1$, we prove that the second term on the right hand side of the latter equality equals zero at $p = 0$.

Except for one occasion, all the terms of the first sum on the right hand side equal zero at $p = 0$. The nonzero term (being equal to one) corresponds to the case of $k = l = j = 1$. Thus $f_1'(p)\big|_{p=0} = 2$.

Therefore the function $f_1(p) = \chi_2(p) - \chi_1^2(p)$ equals zero at $p = 0$. Since

$$f_1'(p)\big|_{p=0} = 2 > 0,$$

the function $f_1(p)$ increases in a neighborhood of zero. Hence there exists a number $p_0^1 \in (0, \frac{1}{2})$ such that $f_1(p) > 0$ for all $p \in (0, p_0^1]$.

Now we consider the function

$$f_2(p) = \chi_1^3(p) - 3\chi_1(p)\chi_2(p) + \chi_3(p).$$

Note that $f_2(0) = 0$. Next we evaluate the derivative of the function $f_2(p)$ at zero:

$$\frac{df_2(p)}{dp} = 3\chi_1^2(p)\frac{d\chi_1(p)}{dp} - 3\left(\chi_1(p)\frac{d\chi_2(p)}{dp} + \chi_2(p)\frac{d\chi_1(p)}{dp}\right) + \frac{d\chi_3(p)}{dp}.$$

The first two terms equal zero at $p = 0$, since all the moments of the random variable M equal zero. The third term is rewritten as

$$\begin{aligned}\frac{d\chi_3(p)}{dp} &= \left(2\mu_1^3(p) - 3\mu_1(p)\mu_2(p) + \mu_3(p)\right)_p' \\ &= 6\mu_1^2(p)\frac{d\mu_1(p)}{dp} - 3\frac{\mu_1(p)}{dp}\mu_2(p) - 3\frac{\mu_2(p)}{dp}\mu_1(p) + \frac{d\mu_3(p)}{dp}.\end{aligned}$$

All the terms on the right hand side, except the last one, equal zero at the point $p = 0$.

We rewrite the last term $d\mu_3(p)/dp$ as

$$\frac{d\mu_3(p)}{dp} = \left(6\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=j}^{\infty}\mathsf{P}(M \geq l)\right)_p' = 6\left(\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=j}^{\infty}\sum_{h=1}^{\infty}(1-2p)\,\mathsf{P}(N_h = l+h)\right)_p'.$$

Again using the theorem on the differentiation of power series, we justify the interchange of the summation and differentiation. Thus

$$\begin{aligned}\frac{d\mu_3(p)}{dp} &= 6\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=j}^{\infty}\sum_{h=1}^{\infty}\left((1-2p)\,\mathsf{P}(N_h = l+h)\right)_p' \\ &= 6\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=j}^{\infty}\sum_{h=1}^{\infty}(1-2p)\left(\mathsf{P}(N_h = l+h)\right)_p' - 12\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\sum_{l=j}^{\infty}\sum_{h=1}^{\infty}\mathsf{P}(N_h = l+h).\end{aligned}$$

The last term on the right hand side of the latter equality equals zero at $p = 0$. All the terms of the first sum on the right hand side, except one term, equal zero at the point $p = 0$. The nonzero term equals one and corresponds to the case of $k = j = l = h = 1$.

Thus we conclude that the function

$$f_2(p) = \chi_1^3(p) - 3\chi_1(p)\chi_2(p) + \chi_3(p)$$

equals zero at $p = 0$ and $f_2'(p)\big|_{p=0} > 0$. Hence this function increases in a neighborhood of zero and there exists a number $p_0^2 \in (0, \frac{1}{2})$ such that $f_2(p) > 0$ for all $p \in (0, p_0^2]$.

Finally we put $p_0 = \min\{p_0^1, p_0^2\}$. The proof of Theorem 5 is complete. $\square$

One of the cases discussed in Remark 2 involves the condition

$$\chi_2 - \chi_1^2 = 0, \qquad \chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$$

imposed on the cumulants of the maximum M of a random walk.

Remark 4. It is proved in the monograph [7] that if the increments ξ of a random walk S_k can be written as $\xi = \xi^+ - \xi^-$ and if the distribution of ξ^+ is exponential, that is,

$$\mathsf{P}(\xi^+ > x) = c \cdot \exp(-\alpha x), \qquad \alpha > 0,\ x > 0,$$

then the distribution of the supremum $M = \sup_{k\geq 0} S_k$ of a random walk is also exponential, namely $\mathsf{P}(M > x) = \text{Const} \cdot \exp(-\lambda_1 x)$. In this case, $\chi_2 - \chi_1^2 = 0$ and $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$. These equalities are seen from the graphs of the Appel polynomials Q_2 and Q_3. The cumulants of the maximum of a random walk in the case of the exponential distribution are given by

$$\chi_1 = \frac{1}{\lambda_1}, \qquad \chi_2 = \frac{1}{\lambda_1^2}, \qquad \chi_3 = \frac{2}{\lambda_1^3}.$$

The Appel polynomials Q_2 and Q_3 are depicted in Figure 2 for the case of $\lambda_1 = 10$. We see that $Q_2(0) = Q_3(0) = 0$ and this is equivalent to the set of two equalities $\chi_2 - \chi_1^2 = 0$ and $\chi_1^3 - 3\chi_1\chi_2 + \chi_3 = 0$.

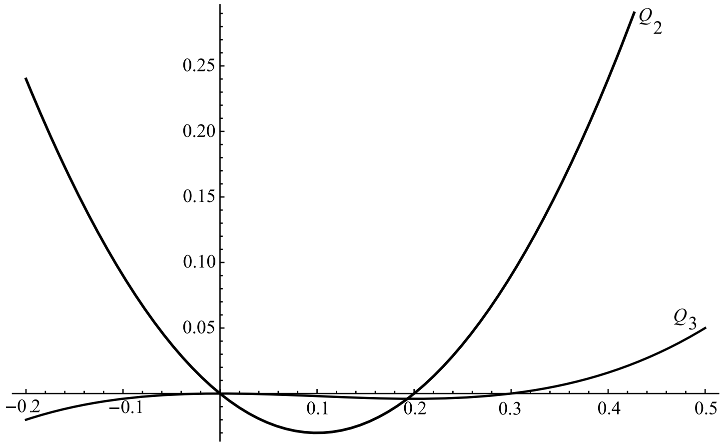

FIGURE 2. Appel polynomials for the exponential distribution of the supremum of a random walk

5. Concluding remarks

The optimal stopping problem for a random walk with a drift to the left is considered in this paper for a polynomial reward function. Necessary and sufficient conditions are established under which a linear combination of the Appel polynomials that corresponds to the reward function has a unique positive root. We found an explicit form for the optimal stopping time and the price function of a random walk in terms of the maximal positive root of the linear combination of the Appel polynomials. It turns out that the optimal stopping time is the moment when the random walk crosses a barrier for the first time; the latter moment can be evaluated as the maximal positive root of the corresponding linear combination of the Appel polynomials. Some examples of random walks are considered for which the results of the current paper are applied.

Bibliography

[1] Ĭ. Ī. Gīhman and A. V. Skorohod, *Controlled stochastic processes*, Springer-Verlag, New York, 1979. Translated from the Russian by Samuel Kotz. MR544839 (80h:60081)

[2] Evgenii B. Dynkin and Aleksandr A. Yushkevich, *Markov processes: Theorems and problems*, Translated from the Russian by James S. Wood, Plenum Press, New York, 1969. MR0242252 (39 #3585a)

[3] D. A. Darling, T. Liggett, and H. M. Taylor, *Optimal stopping for partial sums*, Ann. Math. Statist. **43** (1972), 1363–1368. MR0312564 (47 #1121)

[4] A. A. Novikov and A. N. Shiryaev, *On an effective case of the solution of the optimal stopping problem for random walks*, Teor. Veroyatn. Primen. **49** (2004), no. 2, 373–382, DOI 10.1137/S0040585X97981093 (Russian, with Russian summary); English transl., Theory Probab. Appl. **49** (2005), no. 2, 344–354. MR2144307 (2005m:60084)

[5] Wim Schoutens, *Stochastic processes and orthogonal polynomials*, Lecture Notes in Statistics, vol. 146, Springer-Verlag, New York, 2000. MR1761401 (2001f:60095)
1co done

[6] V. V. Tomashik and Yu. S. Mishura, *Optimal stopping times for random walks for polynomial reward functions*, Applied Statistics. Actuarial and Financial Mathematics (2008), no. 1–2, 101–110. (Ukrainian)

[7] A. A. Borovkov, *Veroyatnostnye protsessy v teorii massovogo obsluzhivaniya*, Izdat. "Nauka", Moscow, 1972 (Russian). MR0315800 (47 #4349)

[8] O. V. Viskov, *A random walk with an upper-continuous component, and the Lagrange inversion formula*, Teor. Veroyatnost. i Primenen. **45** (2000), no. 1, 166–175, DOI 10.1137/S0040585X97978105 (Russian, with Russian summary); English transl., Theory Probab. Appl. **45** (2001), no. 1, 164–172. MR1810980 (2001j:60089)

[9] Wolfgang Stadje, *An iterative approximation procedure for the distribution of the maximum of a random walk*, Statist. Probab. Lett. **50** (2000), no. 4, 375–381, DOI 10.1016/S0167-7152(00)00124-3. MR1802232 (2001m:60105)

[10] Lajos Takács, *Combinatorial methods in the theory of stochastic processes*, John Wiley & Sons Inc., New York, 1967. MR0217858 (36 #947)

[11] D. Korshunov, *On distribution tail of the maximum of a random walk*, Stochastic Process. Appl. **72** (1997), no. 1, 97–103, DOI 10.1016/S0304-4149(97)00060-4. MR1483613 (98i:60027)

[12] A. A. Borovkov, *On subexponential distributions and the asymptotics of the distribution of the maximum of sequential sums*, Sibirsk. Mat. Zh. **43** (2002), no. 6, 1235–1264, DOI 10.1023/A:1021109132124 (Russian, with Russian summary); English transl., Siberian Math. J. **43** (2002), no. 6, 995–1022. MR1946226 (2004a:62031)

[13] S. Zakhari and S. G. Foss, *On the exact asymptotics of the maximum of a random walk with increments in a class of light-tailed distributions*, Sibirsk. Mat. Zh. **47** (2006), no. 6, 1265–1274, DOI 10.1007/s11202-006-0112-8 (Russian, with Russian summary); English transl., Siberian Math. J. **47** (2006), no. 6, 1034–1041. MR2302843 (2008i:60040)

[14] D. A. Korshunov, *The critical case of the Cramér-Lundberg theorem on the asymptotics of the distribution of the maximum of a random walk with negative drift*, Sibirsk. Mat. Zh. **46** (2005), no. 6, 1335–1340, DOI 10.1007/s11202-005-0102-2 (Russian, with Russian summary); English transl., Siberian Math. J. **46** (2005), no. 6, 1077–1081. MR2195033 (2006j:60047)

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