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ASYMPTOTIC PROPERTIES OF ABSOLUTELY CONTINUOUS FUNCTIONS AND STRONG LAWS OF LARGE NUMBERS FOR RENEWAL PROCESSES

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Abstract. In this paper, strong laws of large numbers for renewal processes constructed from compound counting processes are studied. In particular, a strong law of large numbers is proved for renewal processes constructed from compound Poisson processes with absolutely continuous rate functions.

1. Introduction

An important role is played by the relationship between the almost sure asymptotic behavior of a normalized underlying process and that of the corresponding renewal process constructed from the latter in continuous or discrete time. The relationship essentially depends on how the normalizing function tends to infinity; see [9, 11]. With respect to the normalization, several classes of functions are considered, namely the classes of PRV-, SQI- or POV-functions; cf. [9] and [10]–[13]. These classes are defined in terms of restrictions imposed on their (so-called) upper and lower limit functions. It is not always easy to check such restrictions in particular cases. Rather simple sufficient conditions have been found in [12] for a continuously differentiable function to belong to one of these classes.

In some problems of renewal theory there appear normalizing functions which are absolutely continuous but not continuously differentiable. It is a natural question to ask about the conditions guaranteeing that an absolutely continuous function belongs to one of the above classes. For the class of positive absolutely continuous functions a possible answer to this question is discussed in the first part of this paper. A particular example considered is given by the class of positive piecewise linear functions. The conditions obtained are similar to those for positive continuously differentiable functions in [12], but here making use of the Lebesgue density instead of the derivative. The proofs are straightforward and the results allow for obtaining new laws of large numbers for renewal processes constructed from compound counting processes. The latter results are presented in the second part of the paper.

The paper is organized as follows. Section 2 contains some necessary definitions and properties of the classes of functions to be considered later in the paper. Sufficient conditions for a positive absolutely continuous function to belong to a corresponding class are obtained in Section 3. In Section 4, these conditions are particularly specified for the class of positive piecewise linear functions. The strong law of large numbers for a renewal process constructed from a compound counting process is proved in Section 5.

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The last section, Section 6, is devoted to the strong law of large numbers for a renewal process constructed from a compound Poisson process whose rate function is locally integrable.

2. Classes of functions

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}_+$ the set of nonnegative numbers, $\mathbb{F}$ the set of real-valued functions $f = (f(t), t \ge 0)$,

$$\mathbb{F}_+ = \bigcup_{A>0} \{f \in \mathbb{F}\colon f(t) > 0, t \in [A, \infty)\},$$

and $\mathbb{F}^\infty$ the set of functions $f \in \mathbb{F}_+$ such that $\lim_{t\to\infty} f(t) = \infty$. We denote by $\mathbb{C}^\infty_{\text{inc}}$ the subspace of continuous nondecreasing functions in $\mathbb{F}^\infty$.

Measurability is understood in the sense of Lebesgue in this paper.

For $f \in \mathbb{F}_+$, we consider its *upper* and *lower limit functions*, i.e.

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} \quad \text{and} \quad f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}, \qquad c > 0,$$

assuming values in the interval $[0, \infty]$.

2.1. **RV-functions.**

Jovan Karamata introduced *regularly varying* functions and proved their basic properties in the papers [20, 21]. His results together with many other results concerning RV-functions and their generalizations have found several fruitful applications in various fields of mathematics; cf., e.g., [28, 8].

We say that a measurable function $f \in \mathbb{F}_+$ is *regularly varying* (RV) if

$$f_*(c) = f^*(c) = \varkappa(c) \in \mathbf{R}_+ \quad \text{for all } c > 0.$$

To every RV-function f there corresponds a number $\rho \in \mathbf{R}$ (called the *index of* f) such that $\varkappa(c) = c^\rho$, $c > 0$. If $\rho = 0$, we say that f is a *slowly varying* function.

2.2. **ORV-functions.**

We say that a measurable function $f \in \mathbb{F}_+$ is *O-regularly varying* (ORV) if

$$f^*(c) < \infty \quad \text{for all } c > 0. \tag{1}$$

ORV-functions were introduced by V. Avakumović in [3] and have been studied later by many mathematicians (see, for example, [22, 4, 17, 1, 2]). Monotone ORV-functions are also known under the notion of functions with *dominated variation.* Note that a monotone function $f \in \mathbb{F}_+$ is ORV if and only if relation (1) holds for some $c > 1$; see [17].

2.3. **PRV-functions.**

We say that a measurable function $f \in \mathbb{F}_+$ is *pseudo-regularly varying* (PRV) if

$$\limsup_{c\to 1} f^*(c) = 1. \tag{2}$$

PRV-functions appear in many works under different names; confer, e.g., [24, 18, 25, 26, 29, 30, 5, 6, 32, 14, 15, 16, 23, 9], [10]–[13]. A characterizing property of PRV-functions is that they preserve the asymptotic equivalence of functions (see, for example, [9]).

2.4. **SQI-functions.**

We say that a measurable function $f \in \mathbb{F}_+$ is *sufficiently quickly increasing* (SQI) if

$$f_*(c) > 1 \quad \text{for all } c > 1. \tag{3}$$

SQI-functions and their applications have been studied in [31, 16, 9], [10]–[13]. Note that any slowly varying function is not an SQI-function.

2.5. **POV-functions.** We say that a measurable function $f \in \mathbb{F}_+$ is of *positive order varying* (POV) if f is a PRV- and also a SQI-function, that is, if both conditions (2) and (3) hold. POV-functions and their applications have been studied in [9], [10]–[13].

In what follows we denote the sets of ORV-, PRV-, SQI-, and POV- functions by $\mathcal{ORV}$, $\mathcal{PRV}$, $\mathcal{SQI}$, and $\mathcal{POV}$, respectively.

2.6. **The classes of absolutely continuous functions.** We say that a *positive continuous function* $f = (f(t), t \geq t_0)$, $t_0 \geq 0$, belongs to the class of *absolutely continuous functions* $\mathbb{DL}$ if

$$f(t) = f(t_0) + \int_{t_0}^{t} \theta(u)\, du, \qquad t \geq t_0,$$

where the integral on the right-hand side is understood in the Lebesgue sense and

$$\theta = (\theta(t), t \geq t_0)$$

is a measurable and locally integrable real-valued function; that is,

$$\int_{t_0}^{t} |\theta(u)|\, du < \infty$$

for all $t > t_0$. The above function θ is called the *Lebesgue density* of the function f. A function f possessing a density θ is denoted by f_θ.

3. Integral representations and characterizing properties of absolutely continuous functions

This section contains integral representations of absolutely continuous functions. These representations imply the characterizing conditions for the corresponding properties of variation of functions in the class $\mathbb{DL}$.

Proposition 1. *Let $f \in \mathbb{DL}$ and let θ be a density of the function f. Then*

$$f(t) = f(t_0) \exp\left\{ \int_{t_0}^{t} \frac{\theta(u)}{f(u)}\, du \right\}, \qquad t \geq t_0, \tag{4}$$

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} = \exp\left\{ \limsup_{t\to\infty} \int_{t}^{ct} \frac{\theta(u)}{f(u)}\, du \right\}, \qquad c > 0, \tag{5}$$

and

$$f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)} = \exp\left\{ \liminf_{t\to\infty} \int_{t}^{ct} \frac{\theta(u)}{f(u)}\, du \right\}, \qquad c > 0. \tag{6}$$

Proof of Proposition 1. Relation (4) holds if the density θ is continuous, since $\theta(t) = f'(t)$, $t \geq t_0$, in this case, whence

$$\int_{t_0}^{t} \frac{\theta(u)}{f(u)}\, du = \int_{t_0}^{t} \frac{f'(u)}{f(u)}\, du = \ln\left(f(t)/f(t_0)\right).$$

Since relation (4) holds for continuous densities θ, the continuity of the function f implies that (4) holds for piecewise constant densities θ. To complete the proof of relation (4) we use the property that a function θ in the class $\mathbb{DL}$ is locally integrable, and thus it can be approximated in the mean sense by piecewise constant densities on any arbitrary fixed interval $[t_0, T]$.

Thus we obtain from (4) that

$$\frac{f(ct)}{f(t)} = \exp\left\{ \int_{t}^{ct} \frac{\theta(u)}{f(u)}\, du \right\}$$

for $c > 0$ and $t \geq \max\{t_0, t_0/c\}$. The latter equality implies relations (5) and (6). $\square$

The integral representations (4)–(6) allow us to derive some characterizing conditions for the corresponding properties of variation of absolutely continuous functions.

Theorem 1. *Let $f \in \mathbb{DL}$ and let θ be a density of the function f. Then*

1) *$f \in \mathcal{ORV}$ if and only if*

$$-\infty < \liminf_{t\to\infty} \int_t^{ct} \frac{\theta(u)}{f(u)}\, du \quad \text{and} \quad \limsup_{t\to\infty} \int_t^{ct} \frac{\theta(u)}{f(u)}\, du < \infty$$

for all $c > 1$;

2) *if the density θ is nonnegative, then $f \in \mathcal{ORV}$ if and only if there exists $c_0 > 1$ such that*

$$\limsup_{t\to\infty} \int_t^{c_0 t} \frac{\theta(u)}{f(u)}\, du < \infty;$$

3) *$f \in \mathcal{PRV}$ if and only if*

$$\limsup_{c\to 1}\ \limsup_{t\to\infty} \int_t^{ct} \frac{\theta(u)}{f(u)}\, du = 0; \tag{7}$$

4) *if the density θ is nonnegative, then $f \in \mathcal{PRV}$ if and only if*

$$\lim_{c\downarrow 1}\ \limsup_{t\to\infty} \int_t^{ct} \frac{\theta(u)}{f(u)}\, du = 0;$$

5) *$f \in \mathcal{SQI}$ if and only if*

$$\liminf_{t\to\infty} \int_t^{ct} \frac{\theta(u)}{f(u)}\, du > 0 \tag{8}$$

for all $c > 1$;

6) *$f \in \mathcal{POV}$ if and only if both conditions* (7) *and* (8) *hold*;

7) f is an RV-function with index ρ if and only if the limit

$$\lim_{t\to\infty} \int_t^{ct} \frac{\theta(u)}{f(u)}\, du = \rho \in (-\infty, \infty)$$

exists for all $c > 1$.

Theorem 1 implies the following sufficient conditions.

Theorem 2. *Let $f \in \mathbb{DL}$ and let θ be a density of the function f. Then*

1) *if*

$$-\infty < \liminf_{u\to\infty} \frac{u\theta(u)}{f(u)} \quad \text{and} \quad \limsup_{t\to\infty} \frac{u\theta(u)}{f(u)} < \infty$$

or, equivalently,

$$\limsup_{u\to\infty} \frac{u|\theta(u)|}{f(u)} < \infty, \tag{9}$$

then $f \in \mathcal{PRV}$;

2) *if*

$$\liminf_{u\to\infty} \frac{u\theta(u)}{f(u)} > 0,$$

then $f \in \mathcal{SQI}$;

3) *if*

$$0 < \frac{u\theta(u)}{f(u)} \quad \text{and} \quad \limsup_{u\to\infty} \frac{u\theta(u)}{f(u)} < \infty,$$

then $f \in \mathcal{POV}$;

4) *if the limit*

$$\lim_{u\to\infty}\frac{u\theta(u)}{f(u)}=\rho\in(-\infty,\infty)$$

exists, then f is an RV-function with index ρ.

Proof of Theorem 2. Let condition (9) hold. Since

$$\left|\int_t^{ct}\frac{\theta(u)}{f(u)}\,du\right|=\left|\int_t^{ct}\frac{u\theta(u)}{f(u)}\,\frac{du}{u}\right| \tag{10}$$

$$\leq\left(\sup_{u\geq t}\left|\frac{u\theta(u)}{f(u)}\right|\right)\cdot\left|\int_t^{ct}\frac{du}{u}\right|=\left(\sup_{u\geq t}\left|\frac{u\theta(u)}{f(u)}\right|\right)|\ln c| \tag{11}$$

for all $c>0$ and $t>0$,

$$\limsup_{c\to1}\ \limsup_{t\to\infty}\left|\int_t^{ct}\frac{\theta(u)}{f(u)}\,du\right|\leq\limsup_{c\to1}|\ln c|\cdot\limsup_{u\to\infty}\left|\frac{u\theta(u)}{f(u)}\right|=0.$$

Thus condition (7) holds and 1) follows from statement 2) of Theorem 1. The proofs of statements 2)–4) are the same. □

4. Properties of positive piecewise-linear functions

Let $\{a_n\}=\{a_n\}_{n\geq0}$ be a sequence of real numbers such that $a_0=0$, $a_n>0$, $n\geq1$, and let $\{t_n\}=\{t_n\}_{n\geq0}$ be another sequence such that $t_0=0$, $t_{n+1}-t_n>0$, $n\geq0$, and $t_n\to\infty$ as $n\to\infty$. Consider the *piecewise-linear function* constructed from $\{a_n\}$ and $\{t_n\}$, that is, the continuous function

$$\mathcal{L}(t)=a_n+\frac{\Delta a_n}{\Delta t_n}(t-t_n),\qquad t\in[t_n,t_{n+1}),\quad n\geq0, \tag{12}$$

where $\Delta a_n=a_{n+1}-a_n$ and $\Delta t_n=t_{n+1}-t_n$ for $n\geq0$.

Then $\mathcal{L}$ is a positive and piecewise-linear function on $(0,\infty)$. Note also that $\mathcal{L}(0)=0$. Moreover the function is absolutely continuous and possesses a density

$$\theta_{\mathcal{L}}=(\theta_{\mathcal{L}}(t),t\geq0),$$

where

$$\theta_{\mathcal{L}}(t)=\frac{\Delta a_n}{\Delta t_n},\qquad t\in[t_n,t_{n+1}),\quad n\geq0. \tag{13}$$

Hence $\mathcal{L}$ belongs to the class $\mathbb{DL}$ for all $t_0>0$.

For the sake of simplicity we assume that

$$\lim_{n\to\infty}\frac{t_{n+1}}{t_n}=1. \tag{14}$$

This condition holds if, for example, $t_n=n^\beta$, $n\geq1$, where $\beta>0$.

In view of relations (12)–(14),

$$\inf_{t\in[t_n,t_{n+1})}\frac{t\theta_{\mathcal{L}}(t)}{\mathcal{L}(t)}\sim\sup_{t\in[t_n,t_{n+1})}\frac{t\theta_{\mathcal{L}}(t)}{\mathcal{L}(t)}\sim\frac{t_n\Delta a_n}{a_n\Delta t_n},\qquad n\to\infty. \tag{15}$$

Now Theorem 2 and relation (15) imply the following assertion.

Corollary 1. *Assume that condition* (14) *holds for a piecewise-linear function $\mathcal{L}$. Then:*

1) $\mathcal{L}\in\mathcal{PRV}$ *if*

$$\liminf_{n\to\infty}\frac{t_n\Delta a_n}{a_n\Delta t_n}>-\infty\quad\textit{and}\quad\limsup_{n\to\infty}\frac{t_n\Delta a_n}{a_n\Delta t_n}<\infty;$$

2) $\mathcal{L} \in \mathcal{PRV}$ *if the sequence* $\{a_n\}$ *is nondecreasing and*

$$\limsup_{n\to\infty} \frac{t_n \Delta a_n}{a_n \Delta t_n} < \infty; \tag{16}$$

3) $L \in \mathcal{SQI}$ *if*

$$\liminf_{n\to\infty} \frac{t_n \Delta a_n}{a_n \Delta t_n} > 0; \tag{17}$$

4) $\mathcal{L} \in \mathcal{POV}$ *if conditions* (16) *and* (17) *hold*;

5) $\mathcal{L}$ is an RV-function with index ρ if the limit

$$\lim_{n\to\infty} \frac{t_n \Delta a_n}{a_n \Delta t_n} = \rho \in (-\infty, \infty)$$

exists.

5. Generalized renewal processes for compound counting processes

Let $(\nu(t), t \geq 0)$ be a random counting process whose values are nonnegative integer numbers and almost all of whose trajectories are right-continuous step functions. Assume that

$$\nu(0) = 0, \qquad \lim_{t\to\infty} \nu(t) = \infty \quad \text{a.s.},$$

and

$$\lim_{t\to\infty} \frac{\nu(t)}{f(t)} = 1 \quad \text{a.s.}, \tag{18}$$

where $f \in \mathbb{F}^\infty$. Moreover, let $\{Z_n\}_{n\geq 1}$ be a sequence of random variables being independent of the process $\nu(\cdot)$ and such that

$$\lim_{n\to\infty} \frac{Z_n}{n} = \eta \quad \text{a.s.} \tag{19}$$

for some random variable $\eta \in (0, \infty)$.

For example, if

$$Z_n = \sum_{k=1}^{n} \xi_k, \qquad n \geq 1,$$

where $\{\xi_k\}_{k\geq 1}$ is a sequence of independent identically distributed random variables with expectation $\mu \in (0, \infty)$, then (19) follows from the Kolmogorov strong law of large numbers with $\eta = \mu$.

Based on the process $\nu(\cdot)$ and the sequence $\{Z_n\}_{n\geq 1}$, we construct the *compound counting process*

$$X(t) = Z_{\nu(t)}, \qquad t \geq 0. \tag{20}$$

We agree that $X(t) = 0$ if $\nu(t) = 0$.

Relations (18) and (19) imply that

$$\lim_{t\to\infty} \frac{X(t)}{f(t)} = \lim_{t\to\infty} \frac{X(t)}{\nu(t)} \cdot \lim_{t\to\infty} \frac{\nu(t)}{f(t)} = \eta \quad \text{a.s.} \tag{21}$$

Since the trajectories of the process X are almost surely right-continuous step functions, X is a measurable, separable, real-valued stochastic process. Moreover, $X \in \mathbb{F}^\infty$ almost surely.

The following three *generalized renewal processes* are well defined for X:

$$M_X(s) = \inf\{t \ge 0 \colon X(t) \ge s\}, \qquad s \ge 0,$$
$$L_X(s) = \sup\{t \ge 0 \colon X(t) \le s\}, \qquad s \ge 0,$$
$$T_X(s) = \int_0^\infty \mathbb{1}\{X(t) \le s\}\, dt, \qquad s \ge 0.$$

We call all of them the *renewal processes*.

The above renewal processes are nondecreasing and

$$M_X(s) \le T_X(s) \le L_X(s) \tag{22}$$

for $s \ge 0$.

There are also other versions of generalized renewal processes, of course, but we concentrate on the three processes defined above. Sometimes we use the common notation R_X for any of these processes.

The renewal processes M_X, L_X, and T_X have a natural interpretation. The random variable $M_X(s)$ can be viewed as the "*first hitting time*" when the process X enters the set $[s, \infty)$ (alternatively, $M_X(s)$ is the "*first exit time*" from the set $(-\infty, s)$). Similarly, the random variables $L_X(s)$ and $T_X(s)$ can be viewed as the "*last exit time*" when the process X leaves the set $(-\infty, s]$, and the "*total sojourn time*" spent by the process in the set $(-\infty, s]$, respectively.

The following version of the strong law of large numbers follows from the assumption $f \in \mathcal{PRV}$ for renewal processes constructed from a compound counting process.

Theorem 3. *Let X be the compound counting process defined by* (20) *and $f \in \mathcal{PRV}$. Assume that relations* (18) *and* (19) *hold. Then*

1) *the strong law of large numbers*

$$\lim_{s\to\infty} \frac{f(M_X)}{s} = \lim_{s\to\infty} \frac{f(L_X(s))}{s} = \frac{1}{\eta} \quad a.s.$$

holds;

2) *if f is a nondecreasing function or if f is asymptotically equivalent to a nondecreasing function, then*

$$\lim_{s\to\infty} \frac{f(M_X(s))}{s} = \lim_{s\to\infty} \frac{f(L_X(s))}{s} = \lim_{s\to\infty} \frac{f(T_X(s))}{s} = \frac{1}{\eta} \quad a.s.$$

Proof of Theorem 3. Since $f \in \mathcal{PRV}$, $X \in \mathcal{PRV}$ almost surely in view of the strong law of large numbers (21). This together with Lemma 6.1 in [10] implies that the renewal processes M_X and L_X are almost surely quasi-inverse to the process X, that is

$$\lim_{s\to\infty} \frac{X(M_X(s))}{s} = 1 \quad \text{a.s.} \qquad \text{and} \qquad \lim_{s\to\infty} \frac{X(L_X(s))}{s} = 1 \quad \text{a.s.}$$

Using (21) again, we get

$$\lim_{s\to\infty} \frac{f(M_X(s))}{s} = \lim_{s\to\infty} \frac{f(M_X(s))}{X(M_X(s))} = \lim_{s\to\infty} \frac{f(s)}{X(s)} = \frac{1}{\eta} \quad \text{a.s.}$$

and

$$\lim_{s\to\infty} \frac{f(L_X(s))}{s} = \frac{1}{\eta} \quad \text{a.s.}$$

This proves statement 1). Now 1) and relation (22) imply statement 2). □

If f is a POV-function, then Theorem 9.2 in [11] implies that the strong law of large numbers (21) is equivalent to the following strong law of large numbers for the renewal processes:

$$\lim_{s\to\infty} \frac{M_X(s)}{f^{\sim}(s/\eta)} = \lim_{s\to\infty} \frac{L_X(s)}{f^{\sim}(s/\eta)} = \lim_{s\to\infty} \frac{T_X(s)}{f^{\sim}(s/\eta)} = 1 \quad \text{a.s.,} \tag{23}$$

where $f^{\sim}$ is an asymptotically quasi-inverse function to f. Recall (see [10]) that a function $f^{\sim}$ is called *asymptotically quasi-inverse* to a function $f \in \mathbb{F}^{(\infty)}$ if

$$f^{\sim} \in \mathbb{F}^{\infty} \quad \text{and} \quad \frac{f(f^{\sim}(s))}{s} \to \infty, \quad s \to \infty.$$

Thus the following strong law of large numbers holds for renewal processes constructed from a compound counting process, provided $f \in \mathcal{POV}$.

Theorem 4. *Let X be the compound counting process defined by* (20) *and $f \in \mathcal{POV}$. Assume that relations* (18) *and* (19) *hold. Then:*

1) *the strong law of large numbers* (23) *holds for an arbitrary asymptotically quasi-inverse function $f^{\sim}$*;
2) *if f is an RV-function of positive index ρ, then*

$$\lim_{s\to\infty} \frac{M_X(s)}{f^{\sim}(s)} = \lim_{s\to\infty} \frac{L_X(s)}{f^{\sim}(s)} = \lim_{s\to\infty} \frac{T_X(s)}{f^{\sim}(s)} = \left(\frac{1}{\eta}\right)^{1/\rho} \qquad a.s.$$

for an arbitrary asymptotically quasi-inverse function $f^{\sim}$;
3) *statements 1) and 2) hold for*

$$f^{\sim} \in \{M_f,\, L_f,\, T_f\};$$

4) *if $f \sim f_0 \in \mathbb{C}^{\infty}_{inc}$, then statements 1) and 2) hold for $f^{\sim} = f_0^{-1}$, where f_0^{-1} is the inverse function of f_0.*

6. Generalized renewal processes for a compound Poisson process

Let $(\nu(t), t \ge 0)$ be a Poisson process (not necessarily homogeneous) with a nonnegative measurable rate function $(\lambda(t), t \ge 0)$ such that

$$\int_0^t \lambda(s)\,ds < \infty \quad \text{for all } t > 0, \quad \text{and} \quad \int_0^\infty \lambda(s)\,ds = \infty. \tag{24}$$

Put

$$\Lambda(t) = \int_0^t \lambda(s)\,ds, \qquad t > 0. \tag{25}$$

Note that $(\nu(t), t \ge 0)$ is a stochastic process with independent increments such that if $0 \le t_1 < t_2$, then $\nu(t_2) - \nu(t_1)$ is a Poisson random variable with parameter $\Lambda(t_2) - \Lambda(t_1)$. The values of the process ν are nonnegative integer numbers, and its trajectories are almost surely right-continuous, nondecreasing step functions. If $\lambda(s) > 0$, $s \ge 0$, then $\Lambda \in \mathbb{C}^{\infty}_{\text{inc}}$.

Let $\{\tau_n\}_{n\ge 0}$ be a sequence such that $\tau_0 = 0$ and $\Lambda(\tau_n) = n$, $n \ge 1$. If

$$n(t) = \max\{n\colon \tau_n \le t\}, \qquad t > 0,$$

then the Kolmogorov strong law of large numbers implies that

$$\lim_{t\to\infty} \frac{\nu(\tau_{n(t)})}{\Lambda(\tau_{n(t)})} = \lim_{t\to\infty} \frac{\nu(\tau_{n(t)})}{n(t)} = 1 \quad \text{a.s.}$$

Since

$$n(t) = \Lambda(\tau_{n(t)}) \le \Lambda(t) \le \Lambda(\tau_{n(t)+1}) = n(t) + 1$$

for all $t > 0$,

$$\frac{n(t)}{n(t)+1} \cdot \frac{\nu(\tau_{n(t)})}{n(t)} \le \frac{\nu(t)}{\Lambda(t)} \le \frac{n(t)+1}{n(t)} \cdot \frac{\nu(\tau_{n(t)+1})}{n(t)+1},$$

and $\lim_{t\to\infty} n(t) = \infty$, we obtain

$$\lim_{t\to\infty} \frac{\nu(t)}{\Lambda(t)} = 1 \quad \text{a.s.,} \tag{26}$$

that is, condition (18) holds with $f = \Lambda$.

A compound counting process $X(t) = Z_{\nu(t)}$ as in (20), where $(\nu(t), t \ge 0)$ is a Poisson process as defined above, is called a *compound Poisson process.*

Applying Theorems 3–4 and relation (26) we get the following result.

Theorem 5. *Let X be a compound Poisson process as defined in* (20)*, where* $(\nu(t), t \ge 0)$ *is a Poisson process with a nonnegative measurable rate function* $(\lambda(s), s \ge 0)$ *satisfying condition* (24)*. Then:*

1) *if $\Lambda \in \mathcal{PRV}$, then*

$$\lim_{s\to\infty} \frac{\Lambda(M_X(s))}{s} = \lim_{s\to\infty} \frac{\Lambda(L_X(s))}{s} = \lim_{s\to\infty} \frac{\Lambda(T_X(s))}{s} = \frac{1}{\eta} \quad a.s.;$$

2) *if $\Lambda \in \mathcal{POV}$, then*

$$\lim_{s\to\infty} \frac{M_X(s)}{\Lambda^{\sim}(s/\eta)} = \lim_{s\to\infty} \frac{L_X(s)}{\Lambda^{\sim}(s/\eta)} = \lim_{s\to\infty} \frac{T_X(s)}{\Lambda^{\sim}(s/\eta)} = 1 \quad a.s.,$$

where $\Lambda^{\sim}$ is an arbitrary asymptotically quasi-inverse function of Λ;

3) *if $\lambda(s) > 0$, $s \ge 0$, and $\Lambda \in \mathcal{POV}$, then*

$$\lim_{s\to\infty} \frac{M_X(s)}{\Lambda^{-1}(s/\eta)} = \lim_{s\to\infty} \frac{L_X(s)}{\Lambda^{-1}(s/\eta)} = \lim_{s\to\infty} \frac{T_X(s)}{\Lambda^{-1}(s/\eta)} = 1 \quad a.s.,$$

where Λ^{-1} is the inverse function of Λ;

4) *if Λ is an RV-function with positive index ρ, then*

$$\lim_{s\to\infty} \frac{M_X(s)}{\Lambda^{\sim}(s)} = \lim_{s\to\infty} \frac{L_X(s)}{\Lambda^{\sim}(s)} = \lim_{s\to\infty} \frac{T_X(s)}{\Lambda^{\sim}(s)} = \left(\frac{1}{\eta}\right)^{1/\rho} \quad a.s.,$$

where $\Lambda^{\sim}$ is an arbitrary asymptotically quasi-inverse function of Λ;

5) *if $\lambda(s) > 0$, $s \ge 0$, and Λ is an RV-function of positive index ρ,*

$$\lim_{s\to\infty} \frac{M_X(s)}{\Lambda^{-1}(s)} = \lim_{s\to\infty} \frac{L_X(s)}{\Lambda^{-1}(s)} = \lim_{s\to\infty} \frac{T_X(s)}{\Lambda^{-1}(s)} = \left(\frac{1}{\eta}\right)^{1/\rho} \quad a.s.,$$

where Λ^{-1} is the inverse function of Λ;

6) *statements 2) and 4) hold for $\Lambda^{\sim} \in \{M_\Lambda, L_\Lambda, T_\Lambda\}$.*

Remark 1. The function Λ is an absolutely continuous function and its density is λ. Necessary and sufficient conditions for an absolutely continuous function to belong to a certain class of functions, in particular to the class of POV-functions, have been considered in Theorem 1. For example, if condition (24) holds, then $\Lambda \in \mathcal{POV}$ if and only if

$$\limsup_{c\to 1} \limsup_{t\to\infty} \int_t^{ct} \frac{\lambda(u)}{\Lambda(u)}\, du = 0$$

and

$$\liminf_{t\to\infty} \int_t^{ct} \frac{\lambda(u)}{\Lambda(u)}\, du > 0$$

for all $c > 1$.

Below we give some simple sufficient conditions for $\Lambda \in \mathcal{POV}$.

Proposition 2. *Let the rate function λ be measurable and nonnegative. Assume that condition* (24) *holds and that at least one of the following conditions holds:*

1)
$$0 < \liminf_{t\to\infty} \frac{t\lambda(t)}{\Lambda(t)} \quad \textit{and} \quad \limsup_{t\to\infty} \frac{t\lambda(t)}{\Lambda(t)} < \infty;$$

2) *there exists an $\alpha > -1$ such that*
$$0 < \liminf_{t\to\infty} \frac{\lambda(t)}{t^\alpha} \quad \textit{and} \quad \limsup_{t\to\infty} \frac{\lambda(t)}{t^\alpha} < \infty;$$

3) *$c\lambda_*(c) > 1$ for all $c > 1$ and*
$$\liminf_{c\downarrow 1} \lambda^*(c) \le 1;$$

4) *λ is an RV-function with index $\rho \in (-1, \infty)$.*

Then $\Lambda \in \mathcal{POV}$ and, moreover, statements 2), 3), and 6) of Theorem 5 *hold.*

6.1. **Renewal processes constructed from a compound Poisson process with piecewise constant rate function.** Let the Poisson process $\nu(\cdot)$ have a *positive, piecewise constant rate function* $(\lambda(t), t \ge 0)$, i.e.

$$\lambda(t) = \sum_{n=0}^{\infty} \lambda_{n+1} \mathbb{1}\{t \in [t_n, t_{n+1})\}, \qquad t \ge 0, \tag{27}$$

where

$$\lambda_n > 0, \quad n \ge 1, \quad t_0 = 0, \quad \Delta t_n = t_{n+1} - t_n > 0, \quad n \ge 0, \quad \lim_{n\to\infty} t_n = \infty, \tag{28}$$

and

$$\sum_{n=0}^{\infty} \lambda_{n+1} \Delta t_n = \infty. \tag{29}$$

It follows from (27)–(29) that Λ as in (25) is a positive, piecewise linear function for which condition (24) holds; namely,

$$\Lambda(t) = \Lambda_n + \lambda_{n+1}(t - t_n), \qquad t \in [t_n, t_{n+1}), \quad n \ge 0,$$

with

$$\Lambda_0 = 0 \quad \text{and} \quad \Lambda_{n+1} = \sum_{k=0}^{n} \lambda_{k+1} \Delta t_k, \quad n \ge 0.$$

The properties of piecewise linear functions studied in Section 4 yield conditions imposed on the sequences $\{\lambda_n\}$ and $\{t_n\}$ under which $\Lambda \in \mathcal{POV}$.

Example 1. If

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1$$

and

$$0 < \liminf_{n\to\infty} \frac{t_n \lambda_{n+1}}{\Lambda_n}, \qquad \limsup_{n\to\infty} \frac{t_n \lambda_{n+1}}{\Lambda_n} < \infty,$$

then Corollary 1 implies that $\Lambda \in \mathcal{POV}$ and statement 3) of Theorem 5 holds.

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