

## THE DISTANCE BETWEEN FRACTIONAL BROWNIAN MOTION AND THE SUBSPACE OF MARTINGALES WITH “SIMILAR” KERNELS

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ABSTRACT. We study the problem of approximation of a fractional Brownian motion with the help of Gaussian martingales that can be represented as the integrals with respect to a Wiener process and with nonrandom integrands being “similar” to the kernel of the fractional Brownian motion. The “similarity” is understood in the sense that an integrand is the value of the kernel at some point. We establish analytically and evaluate numerically the upper and lower bounds for the distance between the fractional Brownian motion and the space of Gaussian martingales.

### 1. INTRODUCTION

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0, 1]\}, \mathbb{P})$  be a complete probability space. Consider a fractional Brownian motion  $B^H = \{B_t^H, \mathcal{F}_t, t \in [0, 1]\}$  with the Hurst index  $H \in (0, 1)$  defined in this space. In other words,  $B^H$  is a zero mean Gaussian stochastic process whose covariance function is given by

$$\mathbb{E} B_t^H B_s^H = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}).$$

Such a process with the Hurst index  $H > \frac{1}{2}$  is widely used for modeling various phenomena in economics and nature in the case of a long range dependence. It is well known that the fractional Brownian motion is neither semimartingale nor a Markov process with the exception of  $H = \frac{1}{2}$  (fractional Brownian motion is a standard Wiener process in this case). A natural question arises as to how far the fractional Brownian motion is from other stochastic processes of a simpler structure, in particular, how far it is from Gaussian martingales. The Gaussian martingales are defined by

$$\int_0^t a(s) d\widetilde{W}(s), \quad t \in [0, 1],$$

where  $\widetilde{W}(t)$ ,  $t \in [0, 1]$ , is some Wiener process. Therefore we face the following problem. Let  $\widetilde{W} = \{\widetilde{W}_t, \mathcal{F}_t, t \in [0, 1]\}$  be a certain Wiener process. We search for a function  $a \in L_2([0, 1])$  that minimizes the distance

$$\varrho_H^2 := \inf_{a \in L_2([0, 1])} \sup_{t \in [0, 1]} \mathbb{E} \left( B_t^H - \int_0^t a(s) d\widetilde{W}_s \right)^2.$$

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To solve this problem, we first use the integral representation obtained in [1] for a fractional Brownian motion in terms of a standard Wiener process on a finite interval. Consider an integral kernel with a weak singularity

$$K(t, s) = C_\alpha \left( t^\alpha s^{-\alpha} (t - s)^\alpha - \alpha s^{-\alpha} \int_s^t u^{\alpha-1} (u - s)^\alpha du \right) \mathbb{1}_{0 < s < t \leq 1},$$

where

$$C_\alpha = \alpha \sqrt{\frac{(2\alpha + 1)\Gamma(1 - \alpha)}{\Gamma(\alpha + 1)\Gamma(1 - 2\alpha)}}$$

and where  $\Gamma$  is the Gamma function,  $\alpha = H - \frac{1}{2}$ . Then there is a Wiener process  $W = \{W_t, \mathcal{F}_t, t \in [0, 1]\}$  considered with respect to the filtration  $\mathcal{F}$  such that  $B^H$  admits the following representation:

$$\begin{aligned} (1) \quad B_t^H &= \int_0^1 K(t, s) dW_s = \int_0^t K(t, s) dW_s \\ &= C_\alpha \int_0^t \left( t^\alpha s^{-\alpha} (t - s)^\alpha - \alpha s^{-\alpha} \int_s^t u^{\alpha-1} (u - s)^\alpha du \right) dW_s. \end{aligned}$$

If  $H \in (\frac{1}{2}, 1)$ , then the kernel  $K(t, s)$  can be reduced to the following form:

$$K(t, s) = C_\alpha s^{-\alpha} \int_s^t u^\alpha (u - s)^{\alpha-1} du \mathbb{1}_{0 < s < t \leq 1}.$$

Having in mind possible applications we further consider the fractional Brownian motions whose Hurst indices are such that  $H \in (\frac{1}{2}, 1)$ .

It is proved in [2] that the best approximation is attained if  $W = \widetilde{W}$ , that is, the problem is reduced to the problem of minimization of the expression

$$\varrho_H^2 = \inf_{a \in L_2([0,1])} \sup_{t \in [0,1]} \int_0^t (K(t, s) - a(s))^2 ds,$$

and thus the problem becomes an essentially analytical problem.

The functional

$$(2) \quad f(x) = \sup_{t \in [0,1]} \int_0^t (K(t, s) - x(s))^2 ds$$

is well defined for all  $x \in L_2([0, 1])$ . It is proved in [2] that the functional  $f$  attains its minimum at a unique point  $x \in L_2([0, 1])$ . An exact analytical expression for a function that minimizes the functional  $f$  is unknown. Hence it is worthwhile to study the functional  $f$  in some subclasses of functions of  $L_2([0, 1])$  and search for its minimum in those subclasses. This minimum is an upper bound for the distance between the fractional Brownian motion and the space of Gaussian martingales.

Some of the subclasses of functions are considered in [3]–[7]. In this paper, we consider the subclass  $\mathcal{K} \subset L_2([0, 1])$  of functions being of the form

$$a(s) = K(t_0, s) \mathbb{1}_{0 < s < t_0}$$

for some point  $t_0 \in [0, 1]$ . We find analytically a point  $t_0^*$  that minimizes the functional  $f$  in the subclass  $\mathcal{K}$  and estimate numerically the distance

$$(3) \quad \hat{\varrho}_H^2 = \sup_{t \in [0,1]} \int_0^t (K(t, s) - K(t_0^*, s))^2 ds.$$

This number is an upper bound for  $\varrho_H^2$ . We also find a lower bound for this distance.

2. FINDING A MINIMIZING FUNCTION IN THE CLASS  $\mathcal{K}$

Let  $t_0 \in [0, 1]$ . We evaluate the values of the functional  $f$  at functions belonging to the subclass  $\mathcal{K}$ , that is, at functions of the form

$$a_{t_0}(s) = \begin{cases} K(t_0, s), & \text{for } 0 < s \leq t_0, \\ 0, & \text{for } t_0 \leq s \leq 1, \end{cases}$$

and find a point  $t_0^* \in [0, 1]$  for which the function  $a_{t_0^*}$  minimizes the functional  $f$  in the subclass  $\mathcal{K}$ . Put

$$\hat{\varrho}_H^2 = \sup_{t \in [0, 1]} \int_0^t (K(t, s) - K(t_0^*, s))^2 ds \quad \text{and} \quad g_{t_0}(t) = \int_0^t (a_{t_0}(s) - K(t, s))^2 ds.$$

The function  $g_{t_0}$  is continuous in the interval  $[0, 1]$  in view of the properties of the kernel  $\mathcal{K}$ . Then

$$\hat{\varrho}_H^2 = \max_{0 \leq t \leq 1} g_{t_0^*}(t) = \min_{t_0 \in [0, 1]} f(a_{t_0}).$$

**Lemma 2.1.** *The equality*

$$\sup_{0 \leq t \leq t_0} g_{t_0}(t) = C_\alpha^2 t_0^{2H} \sup_{0 \leq p \leq 1} \int_0^p y^{2\alpha} \left( \int_{p/y}^{1/y} x^\alpha (x - 1)^{\alpha-1} dx \right)^2 dy$$

holds.

*Proof.* Let  $t < t_0$ . We have

$$\begin{aligned} g_{t_0}(t) &= \int_0^t (a_{t_0}(s) - K(t, s))^2 ds = \int_0^t (K(t_0, s) - K(t, s))^2 ds \\ &= C_\alpha^2 \int_0^t \left( s^{-\alpha} \int_s^{t_0} u^\alpha (u - s)^{\alpha-1} - s^{-\alpha} \int_s^t u^\alpha (u - s)^{\alpha-1} \right)^2 ds \\ &= C_\alpha^2 \int_0^t s^{-2\alpha} \left( \int_t^{t_0} u^\alpha (u - s)^{\alpha-1} du \right)^2 ds \\ &= C_\alpha^2 \int_0^t s^{-2\alpha} \left( s^{2\alpha} \int_{t/s}^{t_0/s} x^\alpha (x - 1)^{\alpha-1} dx \right)^2 ds \\ &= C_\alpha^2 \int_0^t s^{2\alpha} \left( \int_{t/s}^{t_0/s} x^\alpha (x - 1)^{\alpha-1} dx \right)^2 ds \\ &= C_\alpha^2 \int_0^{t/t_0} (yt_0)^{2\alpha} \left( \int_{t/t_0 y}^{1/y} x^\alpha (x - 1)^{\alpha-1} dx \right)^2 t_0 dy \\ &= C_\alpha^2 t_0^{2H} \int_0^{t/t_0} y^{2\alpha} \left( \int_{t/t_0 y}^{1/y} x^\alpha (x - 1)^{\alpha-1} dx \right)^2 dy. \end{aligned} \tag{4}$$

Put  $p = t/t_0$ . Then

$$g_{t_0}(t) = C_\alpha^2 t_0^{2H} \int_0^p y^{2\alpha} \left( \int_{p/y}^{1/y} x^\alpha (x - 1)^{\alpha-1} dx \right)^2 dy,$$

whence we derive the result needed. □

**Lemma 2.2.** *If  $t > t_0$ , then*

$$g_{t_0}(t) = (t - t_0)^{2H}. \tag{5}$$

*Proof.* Put  $M_t = \int_0^t a(s) dW_s$ . If  $t > t_0$ , then

$$M_t = \int_0^{t_0} K(t_0, s) dW_s = B_{t_0}^H,$$

whence

$$g_{t_0}(t) = \mathbb{E} (B_t^H - M_t)^2 = \mathbb{E} (B_t^H - B_{t_0}^H)^2 = (t - t_0)^{2H}.$$

□

Thus  $\sup_{t_0 \leq t \leq 1} g_{t_0}(t) = (1 - t_0)^{2H}$ . Let

$$(6) \quad D_\alpha = C_\alpha^2 \sup_{0 \leq p \leq 1} \int_0^p s^{2\alpha} \left( \int_{p/s}^{1/s} x^\alpha (x - 1)^{\alpha-1} dx \right)^2 ds.$$

**Lemma 2.3.** *The equality*

$$\hat{\varrho}_H^2 = \min_{t_0 \in [0,1]} f(a_{t_0}) = \frac{D_\alpha}{(1 + D_\alpha^{1/2H})^{2H}}$$

holds.

*Proof.* Consider two functions  $u(t) = D_\alpha t^{2H}$  and  $v(t) = (1 - t)^{2H}$  for  $t \in [0, 1]$ . Then

$$f(a_{t_0}) = \max\{u(t_0), v(t_0)\}.$$

The function  $u$  increases while the function  $v$  decreases in the interval  $[0, 1]$ . Both functions  $u$  and  $v$  are continuous. Moreover,  $0 = u(0) < v(0) = 1$  and

$$D_\alpha = u(1) > v(1) = 0.$$

Thus there exists a unique point  $t_0^* \in (0, 1)$  such that  $u(t_0^*) = v(t_0^*)$  and

$$u(t_0^*) = v(t_0^*) = \min_{t_0 \in [0,1]} \max\{u(t_0), v(t_0)\}.$$

Now we evaluate the point  $t_0^*$ . Since  $D_\alpha t_0^{*2H} = (1 - t_0^*)^{2H}$ ,

$$t_0^* = \frac{1}{1 + D_\alpha^{1/2H}}.$$

Therefore

$$\min_{t_0 \in [0,1]} f(a_{t_0}) = u(t_0^*) = \frac{D_\alpha}{(1 + D_\alpha^{1/2H})^{2H}}.$$

□

### 3. LOWER BOUNDS FOR THE DISTANCE BETWEEN THE FRACTIONAL BROWNIAN MOTION AND THE SPACE OF GAUSSIAN MARTINGALES

A lower bound for the distance  $\varrho_H^2$  is obtained in [2]. According to Lemma 4 in [2],

$$(7) \quad \sup_{t \in [0,1]} \int_0^t (K(t, s) - a(s))^2 ds \geq \frac{1}{4} \int_0^{t_1} (K(t_2, s) - K(t_1, s))^2 ds$$

for an arbitrary function  $a \in L_2([0, 1])$  and for all  $0 \leq t_1 < t_2 \leq 1$ .

This means that

$$\varrho_H^2 \geq \frac{1}{4} \sup_{0 \leq t \leq 1} \int_0^t (K(1, s) - K(t, s))^2 ds.$$

Put

$$\hat{\varrho}_H^2 = \frac{1}{4} \sup_{0 \leq t \leq 1} \int_0^t (K(1, s) - K(t, s))^2 ds.$$

Hence  $\check{\varrho}_H^2$  is a lower bound for the distance between a fractional Brownian motion and the space of Gaussian martingales. Now we study the behavior of the function

$$g(t) = \int_0^t (K(1, s) - K(t, s))^2 ds.$$

**Lemma 3.1.** (1) *The function  $g$  defined above has a unique point of maximum in the interval  $[0, 1]$ .*

(2) *The equality*

$$\check{\varrho}_H^2 = \frac{D_\alpha}{4}$$

*holds if  $\frac{1}{2} < H < 1$ .*

*Proof.* 1) It is obvious that  $g(0) = g(1) = 0$ . Thus the points of maximum of the function  $g$  belong to the open interval  $(0, 1)$ . The derivative of  $g$  is given by

$$g'(t) = K^2(1, t) - 2 \int_0^t (K(1, s) - K(t, s))K'_t(t, s) ds.$$

Consider the integral  $I(t) = \int_0^t (K(1, s) - K(t, s))K'_t(t, s) ds$  on the left hand side of the preceding equality. Obviously the integral is equal to

$$\begin{aligned} I(t) &= \int_0^t (K(1, s) - K(t, s))K'_t(t, s) ds \\ &= t^\alpha \int_0^t (t-s)^{\alpha-1} s^{-2\alpha} \int_t^1 u^\alpha (u-s)^{\alpha-1} du ds \\ (8) \quad &= t^\alpha \int_t^1 u^\alpha \int_0^t s^{-2\alpha} (t-s)^{\alpha-1} (u-s)^{\alpha-1} ds du \\ &= t^{\alpha-1} \int_t^1 u^\alpha \int_0^1 s^{-2\alpha} (1-s)^{\alpha-1} \left(\frac{u}{t} - s\right)^{\alpha-1} ds du. \end{aligned}$$

It is proved in [1] that

$$\int_0^1 s^{\mu-1} (1-s)^{\nu-1} (c-s)^{-\mu-\nu} ds = c^{-\nu} (c-1)^{-\mu} B(\mu, \nu)$$

for all  $c > 1$ ,  $\mu > 0$ , and  $\nu > 0$ , where  $B(\cdot, \cdot)$  is the beta function. We apply the latter equality in the right hand side of (8) and obtain

$$\begin{aligned} (9) \quad I(t) &= B(1-2\alpha, \alpha) t^{\alpha-1} \int_t^1 u^\alpha \left(\frac{u}{t} - 1\right)^{2\alpha-1} du = B(1-2\alpha, \alpha) \int_t^1 (u-t)^{2\alpha-1} du \\ &= \frac{B(1-2\alpha, \alpha)}{2\alpha} (1-t)^{2\alpha}. \end{aligned}$$

Using equality (9), we transform the equation  $g'(t) = 0$  to the form

$$K^2(1, t) = \frac{B(1-2\alpha, \alpha)}{2\alpha} (1-t)^{2\alpha}$$

or, equivalently, to the form

$$(10) \quad K(1, t) = C_\alpha (1-t)^\alpha,$$

where  $C_\alpha^2 = B(1-2\alpha, \alpha)/\alpha$ .

Since  $K(1, 1) = 0$ ,  $t = 1$  is a root of (10). The left hand side of (10) equals  $+\infty$  at  $t = 0$ , while the right hand side equals  $C_\alpha$  at  $t = 0$ . Then we rewrite (10) as follows:

$$t^\alpha \int_1^{1/t} u^\alpha (u-1)^{\alpha-1} du = C_\alpha (1-t)^\alpha$$

or

$$(11) \quad \int_1^{1/t} u^\alpha(u-1)^{\alpha-1} du = \left(\frac{1}{t} - 1\right)^\alpha.$$

Consider the function

$$G(t) = \int_1^{1/t} u^\alpha(u-1)^{\alpha-1} du - \left(\frac{1}{t} - 1\right)^\alpha$$

in the interval  $(0, 1)$ . It is easily seen that  $G(t) \sim t^{-2\alpha}/(2\alpha) - C_\alpha t^{-\alpha} \rightarrow +\infty$  as  $t \rightarrow 0$ , while  $G(t) \rightarrow 0$  as  $t \rightarrow 1$ . The derivative of the function  $G(t)$  is given by

$$G'(t) = (t^{-1} - 1)^{\alpha-1} t^{-2} (\alpha C_\alpha - t^{-\alpha}).$$

The derivative  $G'(t)$  equals zero at a unique point that belongs to the open interval  $(0, 1)$ , since  $\alpha B(1 - 2\alpha, \alpha) > 1$  for  $\alpha \in (0, 1/2)$ , whence  $\alpha C_\alpha > 1$  (see [3]).

Therefore, the function  $G(t)$  decreases in the left part of the interval  $(0, 1)$  and increases to zero in the right part of the same interval. Thus the function has a unique point of minimum in the interval  $(0, 1)$ , and the minimum is negative. This implies that the function equals zero at a unique point between the origin and the point of minimum. The case (1) of the lemma is proved.

2) Note that  $g(t) = g_1(t)$ ,  $t \in [0, 1]$ , where the function  $g_{t_0}$  is defined in Section 2. According to Lemma 2.1 and the definition of  $D_\alpha$ , we get

$$\sup_{0 \leq t \leq 1} g_1(t) = C_\alpha^2 \sup_{0 \leq p \leq 1} \int_0^p y^{2\alpha} \left( \int_{p/y}^{1/y} x^\alpha(x-1)^{\alpha-1} dx \right)^2 dy = D_\alpha. \quad \square$$

**Corollary 3.1.** For all  $H \in (\frac{1}{2}, 1)$ ,

$$(12) \quad \check{\varrho}_H^2 = \frac{D_\alpha}{4} \leq \varrho_H^2 \leq \frac{D_\alpha}{(1 + D_\alpha^{1/2H})^{2H}} = \hat{\varrho}_H^2.$$

#### 4. EVALUATION OF $\check{\varrho}_H^2$ AND $\hat{\varrho}_H^2$

First we evaluate  $D_\alpha$ , and then we find  $\check{\varrho}_H^2$  and  $\hat{\varrho}_H^2$  by using equality (12). The numerical procedure for the evaluation of  $D_\alpha$  is reduced to the evaluation of the integral

$$(13) \quad \int_A^B s^{2\alpha} \left( \int_{u(s)}^{v(s)} x^\alpha(x-1)^{\alpha-1} dx \right)^2 ds,$$

where  $A, B \in [0, 1]$ ,  $A \leq B$ , and  $u$  and  $v$  are some functions on  $[A, B]$  such that

$$1 \leq u(s) \leq v(s), \quad s \in [A, B].$$

We show how one can evaluate such integrals numerically. The function  $Q$  is defined by

$$Q(z) = \int_1^z x^\alpha(x-1)^{\alpha-1} dx, \quad z \in [1, \infty).$$

Then one can rewrite expression (13) as follows:

$$(14) \quad \int_A^B s^{2\alpha} (Q(v(s)) - Q(u(s)))^2 ds.$$

The main problem in the procedure of the evaluation of  $Q$  is that the integrand has a singularity at the point 1. To remove the singularity we integrate by parts:

$$\begin{aligned}
 (15) \quad Q(z) &= \int_1^z x^\alpha(x-1)^{\alpha-1} dx = \frac{x^\alpha(x-1)^\alpha}{\alpha} \Big|_1^z - \int_1^z x^{\alpha-1}(x-1)^\alpha dx \\
 &= \frac{z^\alpha(z-1)^\alpha}{\alpha} - \int_1^z x^{\alpha-1}(x-1)^\alpha dx.
 \end{aligned}$$

Put  $q(z) = x^{\alpha-1}(x-1)^\alpha, z \in [1, \infty)$ .

**The algorithm for the evaluation of  $Q(z), z \in [1, \infty)$ .** Let  $N = 100\,000$  and  $\Delta = 0.1$ . Consider a partition  $\pi_N$  of the interval  $[1, 1 + N\Delta] = [1, 10\,001]$  such that

$$\pi_N = \{1 = t_0^N < t_1^N = 1 + \Delta < \dots < t_k^N = 1 + k\Delta < \dots < t_N^N = 1 + N\Delta\}.$$

The function  $H(z), z \in [1, \infty)$ , is evaluated numerically as follows:

- (1) If there exists a number  $k$  such that  $z = t_k^N$ , then we use the Simpson formula for the numerical integration [8] and obtain

$$Q(t_k^N) \approx \frac{(t_k^N(t_k^N - 1))^\alpha}{\alpha} - \sum_{i=1}^k (q(t_{i-1}^N) + 4q((t_{i-1}^N + t_i^N)/2) + q(t_i^N)) / 6.$$

- (2) If  $t_k^N < z < t_{k+1}^N, 1 \leq k \leq N - 1$ , then we use step (1) and consider the linear interpolation

$$Q(z) \approx \frac{(t_{k+1}^N - z)}{\Delta} Q(t_k^N) + \frac{(z - t_k^N)}{\Delta} Q(t_{k+1}^N).$$

- (3) If  $z > t_N^N$ , then  $Q(z) \sim z^{2\alpha}/(2\alpha)$  as  $z \rightarrow \infty$  by the l'Hospital rule, and thus one can put

$$Q(z) \approx \frac{z^{2\alpha}}{2\alpha}.$$

**The algorithm for the evaluation of  $I(A, B, u, v) = \int_A^B s^{2\alpha}(Q(v(s)) - Q(u(s)))^2 ds$ .** Let  $N = 1\,000$ . Put  $\Delta = (B - A)/N$  and consider the following partition  $\pi$  of the interval  $[A, B]$ :

$$\pi = \{A = s_0^N < A + \Delta = s_1^N < \dots < A + k\Delta = s_k^N < \dots < B = s_N^N\}.$$

Put  $s_i^{N*} = (s_{i-1}^N + s_i^N)/2$ .

Then

$$\begin{aligned}
 (16) \quad \int_A^B s^{2\alpha}(Q(v(s)) - Q(u(s)))^2 ds &= \sum_{i=1}^N \int_{s_{i-1}^N}^{s_i^N} s^{2\alpha}(Q(v(s)) - Q(u(s)))^2 ds \\
 &\approx \sum_{i=1}^N (Q(v(s_i^{N*})) - Q(u(s_i^{N*})))^2 \int_{s_{i-1}^N}^{s_i^N} s^{2\alpha} ds \\
 &= \frac{1}{2\alpha + 1} \sum_{i=1}^N (Q(v(s_i^{N*})) - Q(u(s_i^{N*})))^2 \\
 &\quad \times ((s_i^N)^{2\alpha+1} - (s_{i-1}^N)^{2\alpha+1}).
 \end{aligned}$$

By definition  $D_\alpha$ , we obtain

$$(17) \quad D_\alpha = \max_{0 \leq t \leq 1} g(t) = \max_{0 \leq t \leq 1} I(0, t, u_t, v),$$

where  $u_t(y) = t/y, v = 1/y, y \in (0, 1]$ .

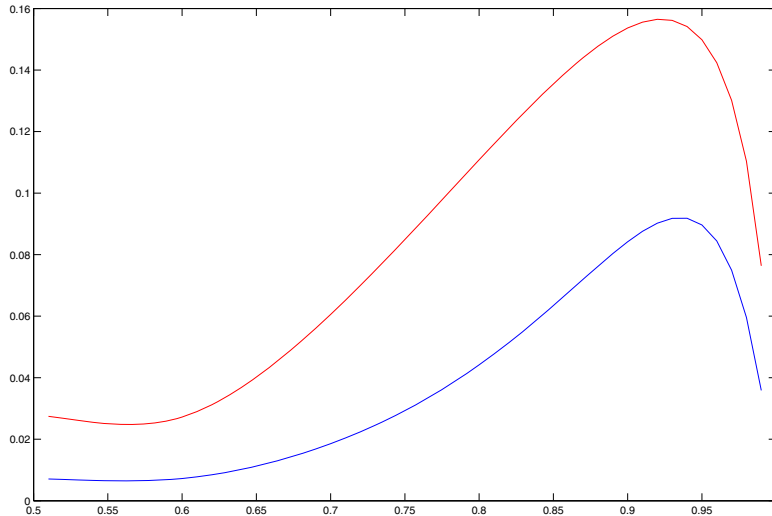


FIGURE 1.  $\check{\varrho}_H^2$  (lower graph),  $\hat{\varrho}_H^2$  (upper graph)

**The algorithm for the evaluation of  $D_\alpha$ .** We find the maximum of the function  $g(t)$ ,  $t \in [0, 1]$ , by using standard methods of optimization of unimodal functions (note that  $g$  is unimodal in view of Lemma 3.1). We use formula (17) to evaluate  $g$ .

The following table contains the results of evaluation for some  $H$ :

H	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
$D_\alpha$	0.0260	0.0289	0.0451	0.0742	0.1171	0.1767	0.2537	0.3368	0.3586
$t_0^*$	0.9650	0.9504	0.9156	0.8651	0.8069	0.7471	0.6914	0.6467	0.6317
$\check{\varrho}_H^2$	0.0065	0.0072	0.0113	0.0185	0.0293	0.0442	0.0634	0.0842	0.0897
$\hat{\varrho}_H^2$	0.0250	0.0272	0.0402	0.0606	0.0849	0.1108	0.1355	0.1537	0.1499

The values of  $\check{\varrho}_H^2$  and  $\hat{\varrho}_H^2$  are depicted in Figure 1 for  $H$  running from 0.51 to 0.99 with the step 0.01.

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