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BUSY PERIOD AND STATIONARY DISTRIBUTION FOR THE QUEUEING SYSTEM $\mathrm{M}^{\theta}/\mathrm{G}/1/\infty$ WITH A THRESHOLD SWITCHING BETWEEN SERVICE MODES

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Abstract. We consider the queueing system $\mathrm{M}^{\theta}/\mathrm{G}/1/\infty$ with two service modes. The server switches between the main service mode and threshold mode at the beginning of the current customer service when the number of customers present in the system is larger than h. The mean duration of the busy time and formulas for the stationary distribution of the number of customers in the system are obtained.

1. Description of the model

The queueing system $\mathrm{M}^{\theta}/\mathrm{G}/1/\mathrm{m}$ with two service modes is studied in the paper [1]. The server switches between the modes at the beginning of the current customer service if the total number of customers in the system exceeds a given threshold level h. In the current paper, we use the results obtained in [1] to study a similar system without restrictions on the length of the queue, that is, we consider the case of $m = \infty$.

Let there be given three sequences of independent identically distributed random variables $\{\alpha_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$, $n \ge 1$, where α_n denotes the interarrival time between the $(n-1)^{\text{th}}$ and n^{th} batches of customers arriving to the system, θ_n denotes the number of customers in the n^{th} batch, and β_n denotes the service time for the n^{th} customer. Assume that

$$\mathsf{P}\{\alpha_n < x\} = 1 - e^{-\lambda x}, \qquad \lambda > 0,$$

and

$$\mathsf{P}\{\theta_n = i\} = a_i, \qquad i \ge 1.$$

If $\mathsf{P}\{\theta_n = 1\} = a_1 = 1$, then each batch consists of only one customer.

The customers are served one by one. After the service is completed, the customer leaves the system and the server starts the service of the next customer if there is at least one in the queue; otherwise, the server waits until a batch of customers arrives to the system. The FIFO discipline is used to maintain the queue in the system. The order of customers within a batch is arbitrary.

Denote by $\xi_\infty(t)$ the total number of customers in the system at time t. The phase space $X_\infty = \{0, 1, \dots\}$ corresponds to the stochastic process $(\xi_\infty(t), t \ge 0)$. For $\xi_\infty(t)$, we introduce a threshold level h, $h \ge 1$. Namely, if t is the moment when the service of the n^{th} customer starts and if $\xi_\infty(t) \le h$, then $\mathsf{P}\{\beta_n < x\} = F(x)$ for $x \ge 0$ (we agree that $F(0) = 0$). Otherwise, that is if $\xi_\infty(t) > h$, then $\mathsf{P}\{\beta_n < x\} = \widetilde{F}(x)$, $x \ge 0$, $\widetilde{F}(0) = 0$.

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The queueing system introduced above is denoted by $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\infty$. The stochastic process describing the functioning of this system belongs to the class of switching processes [2].

The stochastic process $(\xi_m(t), t \ge 0)$ with the phase space

$$X_m = \{0, 1, \dots, m+1\}$$

is used to denote the total number of customers at time t in the system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\mathrm{m}$. Recall that such a process is studied in [1].

2. Main notation and auxiliary results

We introduce the following notation: P_n is the conditional probability given that $n \ge 0$ customers are present in the system at the initial time; E (P) is the conditional expectation (conditional probability), given that the system starts functioning at the moment when the first batch of customers arrives to the system; a_i^{k*} is the k-fold convolution of the sequence a_i with itself; $\eta(x)$ is the total number of customers arriving to the system in the time interval $[0; x)$;

$$\rho_k(m) = \lim_{t\to\infty} \mathsf{P}\{\xi_m(t) = k\}; \qquad \rho_k(\infty) = \lim_{t\to\infty} \mathsf{P}\{\xi_\infty(t) = k\};$$

(1)
$$\begin{gathered} f(s) = \int_0^\infty e^{-sx}\, dF(x), \qquad m_1 = \int_0^\infty x\, dF(x) < \infty, \qquad \overline{F}(x) = 1 - F(x); \\ b_1 = \sum_{k=1}^\infty k a_k < \infty, \qquad \overline{a}_n = \sum_{k=n}^\infty a_k, \qquad \rho = \lambda m_1 b_1; \\ p_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!}\, dF(x), \qquad i \ge -1; \\ q_i = \sum_{k=0}^{i} a_i^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} \overline{F}(x)\, dx, \qquad i \ge 0. \end{gathered}$$

It is known that

$$q_0 = \frac{1 - f(\lambda)}{\lambda},$$
$$q_k = \sum_{i=1}^k a_i q_{k-i} - \frac{p_{k-1}}{\lambda}, \qquad k \ge 1$$

(see [1]).

Consider the sequence $\{R_n\}$ defined recursively as follows:

(2)
$$R_1 = \frac{1}{p_{-1}}, \qquad R_{n+1} = \frac{1}{p_{-1}} \left(R_n - \sum_{i=0}^{n-1} p_i R_{n-i} \right), \quad n \ge 1.$$

Similarly to (1)–(2), we define the functions and constants for the distribution function $\widetilde{F}(x)$ of the service time in the threshold mode arising when the total number of customers in the system exceeds h. We use the symbol "tilde" to denote these functions and constants, namely

$$\widetilde{f}(s) = \int_0^\infty e^{-sx}\, d\widetilde{F}(x), \qquad \widetilde{m}_1 = \int_0^\infty x\, d\widetilde{F}(x) < \infty, \qquad \overline{\widetilde{F}}(x) = 1 - \widetilde{F}(x);$$

$$\widetilde{\rho} = \lambda \widetilde{m}_1 b_1; \qquad \widetilde{p}_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!}\, d\widetilde{F}(x), \qquad i \geq -1;$$

(3)
$$\widetilde{q}_0 = \frac{1 - \widetilde{f}(\lambda)}{\lambda}, \qquad \widetilde{q}_k = \sum_{i=1}^{k} a_i \widetilde{q}_{k-i} - \frac{\widetilde{p}_{k-1}}{\lambda}, \qquad k \geq 1;$$

$$\widetilde{R}_1 = \frac{1}{\widetilde{p}_{-1}}, \qquad \widetilde{R}_{n+1} = \frac{1}{\widetilde{p}_{-1}} \left(\widetilde{R}_n - \sum_{i=0}^{n-1} \widetilde{p}_i \widetilde{R}_{n-i} \right), \qquad n \geq 1.$$

In what follows we use the sequences $\{p_n(s)\}$, $\{q_n(s)\}$, $\{R_n(s)\}$, $\{\widetilde{p}_n(s)\}$, $\{\widetilde{q}_n(s)\}$, and $\{\widetilde{R}_n(s)\}$ introduced in [1].

The following assertion for the sequence $\{\widetilde{R}_n\}$ follows explicitly from Theorem 1.5 of [3].

Lemma 2.1. *If $\widetilde{\rho} < 1$, then*

(4)
$$\lim_{n\to\infty} \widetilde{R}_n = \frac{1}{1 - \widetilde{\rho}}.$$

Otherwise, that is if $\widetilde{\rho} \geq 1$, then $\lim_{n\to\infty} \widetilde{R}_n = \infty$.

Lemma 2.2. *The sequence $\{R_n\}$ is increasing.*

Proof. Using relations (2), we get

$$R_2 = R_1^2(1 - p_0);$$
$$R_2 - R_1 = R_1\big(R_1(1 - p_0) - 1\big) = R_1^2(1 - p_0 - p_{-1}) = R_1^2 \overline{p}_1 > 0.$$

Assume that $R_k - R_{k-1} > 0$ for all $k = 1, \ldots, n$. We want to prove that $R_{n+1} - R_n > 0$. Indeed,

$$R_{n+1} = R_1\big(R_n(1 - p_0) - p_1 R_{n-1} - p_2 R_{n-2} - \cdots - p_{n-1} R_1\big);$$
$$R_{n+1} - R_n = R_1\big(R_n(1 - p_0 - p_{-1}) - p_1 R_{n-1} - p_2 R_{n-2} - \cdots - p_{n-1} R_1\big)$$
$$> R_1\big(R_n \overline{p}_1 - (p_1 + p_2 + \cdots + p_{n-1}) R_{n-1}\big) > R_1 \overline{p}_1 (R_n - R_{n-1}) > 0.$$

The lemma is proved. □

Lemma 2.3. *The limit relations*

(5)
$$\lim_{m\to\infty} m \overline{p}_m = \lim_{m\to\infty} m \overline{a}_m = 0$$

hold for the sequences $\{p_n\}$ and $\{a_n\}$.

Proof. It is known that $\sum_{k=0}^\infty (k+1) p_k = \rho$ and $\sum_{k=1}^\infty k a_k = b_1$. Since

$$m \overline{p}_m = \sum_{k=m}^\infty m p_k \leq \sum_{k=m}^\infty k p_k,$$

we get $\lim_{m\to\infty} m \overline{p}_m = 0$ in view of

$$\lim_{m\to\infty} m \overline{p}_m \leq \lim_{m\to\infty} \sum_{k=m}^\infty k p_k = 0.$$

The relation

$$\lim_{m\to\infty} m\overline{a}_m = 0$$

is proved similarly. The proof of the lemma is complete. □

Lemma 2.4. *Let*

$$r_k(h) = \sum_{i=1}^{h} R_i p_{k-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i p_{k-n-i}. \tag{6}$$

Then

$$\sum_{k=h+1}^{\infty} r_k(h) = p_{-1}R(h) - \overline{a}_h, \tag{7}$$

where

$$R(h) = R_{h+1} - \sum_{n=1}^{h-1} a_n R_{h+1-n}.$$

Proof. Using the equalities

$$\sum_{k=1}^{n} R_k \overline{p}_{n-k} = R_n - 1, \qquad n \ge 1 \tag{8}$$

(see [4]) and relation

$$\sum_{k=1}^{h} R_k p_{h-k} = R_h - p_{-1}R_{h+1} \tag{9}$$

that follows from (2), we prove for the sequence defined by (6) that

$$\begin{aligned}\sum_{k=h+1}^{\infty} r_k(h) &= \sum_{i=1}^{h} R_i \overline{p}_{h-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i \overline{p}_{h-n-i} - r_h(h)\\ &= R_h - 1 - \sum_{n=1}^{h-1} a_n(R_{h-n} - 1) - R_h + p_{-1}R_{h+1}\\ &\quad + \sum_{n=1}^{h-1} a_n(R_{h-n} - p_{-1}R_{h+1-n})\\ &= \sum_{n=1}^{h-1} a_n - 1 + p_{-1}R_{h+1} - p_{-1}\sum_{n=1}^{h-1} a_n R_{h+1-n} = p_{-1}R(h) - \overline{a}_h.\end{aligned}$$

The lemma is proved. □

Lemma 2.5. *The sequence $r_k(h)$ defined by* (6) *is such that*

$$\begin{aligned}\sum_{k=h+1}^{\infty} k r_k(h) &= (\rho - 1 + p_{-1}) \sum_{i=1}^{h} R_i \overline{a}_{h+1-i} + \sum_{i=1}^{h} R_i\left(i\overline{p}_{h+1-i} - \sum_{k=1}^{h-i} k p_k\right)\\ &\quad - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i\left((n+i)\overline{p}_{h+1-n-i} - \sum_{k=1}^{h-n-i} k p_k\right).\end{aligned} \tag{10}$$

If $a_1 = 1$, then

$$\sum_{k=h+1}^{\infty} k r_k(h) = 1 + (\rho - 1)R_h + h p_{-1}(R_{h+1} - R_h). \tag{11}$$

Proof. Since

$$\sum_{k=0}^{\infty}(k+1)p_k = \rho,$$

we deduce that

$$\begin{aligned}\sum_{k=h+1}^{\infty} kp_{k-i} &= \sum_{k=h+1}^{\infty}(k-i+1)p_{k-i} + (i-1)\sum_{k=h+1}^{\infty} p_{k-i}\\ &= \rho - \sum_{k=0}^{h-i}(k+1)p_k + (i-1)\overline{p}_{h+1-i} \qquad (12)\\ &= \rho - 1 + p_{-1} + i\overline{p}_{h+1-i} - \sum_{k=1}^{h-i} kp_k.\end{aligned}$$

Since

$$\sum_{k=h+1}^{\infty} kr_k(h) = \sum_{i=1}^{h} R_i \sum_{k=h+1}^{\infty} kp_{k-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i \sum_{k=h+1}^{\infty} kp_{k-n-i},$$

we derive equality (10) from (12). If $a_1 = 1$, then we again use equalities (8) and (9) and derive relation (11) from (10). The lemma is proved. □

3. Busy period and stationary distribution

Denote by

$$\tau(\infty) = \inf\{t \ge 0 \colon \xi_\infty(t) = 0\}$$

the first busy period in the queueing system $\mathrm{M}^\theta/\mathrm{G},\ \widetilde{\mathrm{G}}/1/\infty$. Let $\mathsf{E}\,\tau(\infty)$ denote the expectation of the first busy period. We denote by $\tau(m)$ the busy period of the queueing system $\mathrm{M}^\theta/\mathrm{G},\ \widetilde{\mathrm{G}}/1/\mathrm{m}$ studied in [1].

Theorem 3.1. *If* $\widetilde{\rho} < 1$, *then*

$$\begin{aligned}\lim_{m\to\infty} \mathsf{E}\,\tau(m) &= m_1 \sum_{i=1}^{h} R_i \overline{a}_{h+1-i}\\ &\quad + \frac{\widetilde{m}_1}{1-\widetilde{\rho}}\Bigg(b_1 - \sum_{k=1}^{h-1} ka_k - hp_{-1}R(h)\\ &\qquad + (\rho - 1 + p_{-1})\sum_{i=1}^{h} R_i \overline{a}_{h+1-i} \qquad (13)\\ &\qquad + \sum_{i=1}^{h} R_i\Bigg(i\overline{p}_{h+1-i} - \sum_{k=1}^{h-i} kp_k\Bigg)\\ &\qquad - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i\Bigg((n+i)\overline{p}_{h+1-n-i} - \sum_{k=1}^{h-n-i} kp_k\Bigg)\Bigg).\end{aligned}$$

If $\widetilde{\rho} < 1$ *and* $a_1 = 1$, *then*

$$\lim_{m\to\infty} \mathsf{E}\,\tau(m) = m_1 R_h + \frac{\widetilde{m}_1}{1-\widetilde{\rho}}\Big(1 - (1-\rho)R_h\Big). \qquad (14)$$

Proof. We use a result from [1] concerning the mean duration of the busy period in the queueing system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\mathrm{m}$. According to this result,

$$\begin{aligned}\mathsf{E}\,\tau(m) &= m_1 \sum_{i=1}^{h} R_i \overline{a}_{h+1-i} \\ &\quad + \widetilde{m}_1 \Bigg(p_{-1} R(h) \sum_{i=1}^{m-h} \widetilde{R}_i - \sum_{k=h+1}^{m-1} r_k(h) \sum_{i=1}^{m-k} \widetilde{R}_i \\ &\qquad - \sum_{n=h}^{m-1} a_n \sum_{i=1}^{m-n} \widetilde{R}_i + \overline{a}_{m+1} \Bigg).\end{aligned} \tag{15}$$

To pass to the limit in (15) as $m \to \infty$ we write

$$\begin{aligned} S(m) &= p_{-1} R(h) \sum_{i=1}^{m-h} \widetilde{R}_i - \sum_{k=h+1}^{m-1} r_k(h) \sum_{i=1}^{m-k} \widetilde{R}_i - \sum_{n=h}^{m-1} a_n \sum_{i=1}^{m-n} \widetilde{R}_i \\ &= \sum_{k=0}^{m-h-1} \widetilde{R}_{m-h-k} \Bigg(p_{-1} R(h) - \sum_{i=h+1}^{h+k} r_i(h) - \sum_{n=h}^{h+k} a_n \Bigg) \\ &= \widetilde{R}_{m-h} \Bigg((m-h) p_{-1} R(h) \\ &\qquad - \sum_{k=h+1}^{m-1} (m-k) r_k(h) - \sum_{k=h}^{m-1} (m-k) a_k \Bigg) \\ &\quad + \big(\widetilde{R}_{m-h} - \widetilde{R}_{m-h-1}\big)\big(r_{h+1}(h) + a_h + a_{h+1} - p_{-1} R(h)\big) \\ &\quad + \big(\widetilde{R}_{m-h} - \widetilde{R}_{m-h-2}\big) \\ &\qquad \times \big(r_{h+1}(h) + r_{h+2}(h) + a_h + a_{h+1} + a_{h+2} - p_{-1} R(h)\big) \\ &\quad + \cdots + \big(\widetilde{R}_{m-h} - \widetilde{R}_2\big) \Bigg(\sum_{k=h+1}^{m-2} r_k(h) + \sum_{k=h}^{m-2} a_k - p_{-1} R(h) \Bigg) \\ &\quad + \big(\widetilde{R}_{m-h} - \widetilde{R}_1\big) \Bigg(\sum_{k=h+1}^{m-1} r_k(h) + \sum_{k=h}^{m-1} a_k - p_{-1} R(h) \Bigg). \end{aligned} \tag{16}$$

Equalities (4) and (7) imply that all the terms in the sum on the right hand side of (16) except the first one approach zero as $m \to \infty$, that is,

$$\begin{aligned} \lim_{m\to\infty} S(m) &= \lim_{m\to\infty} \widetilde{R}_{m-h} \Bigg((m-h) p_{-1} R(h) \\ &\qquad - \sum_{k=h+1}^{m-1} (m-k) r_k(h) - \sum_{k=h}^{m-1} (m-k) a_k \Bigg) \\ &= \frac{1}{1-\widetilde{\rho}} \Bigg(b_1 - \sum_{k=1}^{h-1} k a_k - h p_{-1} R(h) + \sum_{k=h+1}^{\infty} k r_k(h) \Bigg) \\ &\quad + \lim_{m\to\infty} m \widetilde{R}_{m-h} \Bigg(p_{-1} R(h) - \sum_{k=h+1}^{m-1} r_k(h) - \sum_{k=h}^{m-1} a_k \Bigg). \end{aligned} \tag{17}$$

Since

$$p_{-1}R(h) - \sum_{k=h+1}^{m-1} r_k(h) - \sum_{k=h}^{m-1} a_k = p_{-1}R(h) - \sum_{k=h+1}^{\infty} r_k(h) - \overline{a}_h + \sum_{k=m}^{\infty} r_k(h) + \overline{a}_m$$
$$= \sum_{k=m}^{\infty} r_k(h) + \overline{a}_m,$$

we use relations (4)–(6) and conclude that

$$\lim_{m\to\infty} m\widetilde{R}_{m-h}\left(p_{-1}R(h) - \sum_{k=h+1}^{m-1} r_k(h) - \sum_{k=h}^{m-1} a_k\right) = \frac{1}{1-\widetilde{\rho}} \lim_{m\to\infty} m\left(\sum_{k=m}^{\infty} r_k(h) + \overline{a}_m\right)$$
$$= \frac{1}{1-\widetilde{\rho}}\left(\sum_{i=1}^{h} R_i \lim_{m\to\infty} m\overline{p}_{m-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i \lim_{m\to\infty} m\overline{p}_{m-n-i}\right) = 0.$$

Taking into account (17), (10), and (11), we obtain equalities (13) and (14) by passing to the limit in (15) as $m \to \infty$. The theorem is proved. □

Put

$$\varphi_n^{(m)}(t,k) = \mathsf{P}_n\{\xi_m(t) = k, \tau(m) > t\}, \qquad 1 \le n, k \le m+1;$$
$$\varphi_n^{\infty}(t,k) = \mathsf{P}_n\{\xi_\infty(t) = k, \tau(\infty) > t\}, \qquad n, k \ge 1;$$
$$\Phi_n^{(m)}(s,k) = \int_0^\infty e^{-st}\varphi_n^{(m)}(t,k)\,dt, \qquad \Phi_n^{\infty}(s,k) = \int_0^\infty e^{-st}\varphi_n^{\infty}(t,k)\,dt, \qquad \operatorname{Re} s > 0;$$
$$\Phi_n^{(m)}(k) = \lim_{s\to+0} \Phi_n^{(m)}(s,k), \qquad \Phi_n^{\infty}(k) = \lim_{s\to+0} \Phi_n^{\infty}(s,k);$$
$$\Phi_n^{(m)} = \sum_{k=1}^{m+1} \Phi_n^{(m)}(k), \qquad \Phi_n^{\infty} = \sum_{k=1}^{\infty} \Phi_n^{\infty}(k);$$
$$f_n(k) = q_{k-n} + I\{k = m+1\}\overline{q}_{m+2-n}, \qquad \widetilde{f}_n(k) = \widetilde{q}_{k-n} + I\{k = m+1\}\overline{\widetilde{q}}_{m+2-n}.$$

Lemma 3.1. *We have*

$$\lim_{m\to\infty} \Phi_n^{(m)}(k) = \Phi_n^{\infty}(k), \qquad n, k \ge 1. \tag{18}$$

Moreover,

$$\Phi_n^{(\infty)}(k) = \Phi_m^{(m)}(k) - D_n(k), \qquad 1 \le n \le h-1;$$
$$\Phi_n^{(\infty)}(k) = \Phi_m^{(m)}(k) - \sum_{i=1}^{k-n} \widetilde{R}_i \widetilde{q}_{k-n-i}, \qquad n \ge h, \tag{19}$$

where

$$\Phi_m^{(m)}(k) = \sum_{i=1}^{k} R_i q_{k-i}, \qquad 1 \le k \le h;$$
$$\Phi_m^{(m)}(k) = \sum_{i=1}^{h} R_i q_{k-i} + p_{-1}R_{h+1} \sum_{i=1}^{k-h} \widetilde{R}_i \widetilde{q}_{k-h-i} - \sum_{u=1}^{h} R_u \sum_{j=h+1}^{k-1} p_{j-u} \sum_{i=1}^{k-j} \widetilde{R}_i \widetilde{q}_{k-j-i}, \tag{20}$$
$$h+1 \le k \le m;$$

$$D_n(k) = \sum_{i=1}^{k-n} R_i q_{k-i}, \qquad 1 \le k \le h;$$

$$D_n(k) = \sum_{i=1}^{h-n} R_i q_{k-i} + p_{-1} R_{h+1-n} \sum_{i=1}^{k-n} \widetilde{R}_i \widetilde{q}_{k-h-i} - \sum_{u=1}^{h-n} R_u \sum_{j=h+1}^{k-1} p_{j-n-u} \sum_{i=1}^{k-j} \widetilde{R}_i \widetilde{q}_{k-j-i},$$
$$k \ge h+1.$$

Proof. The total probability formula implies that

$$(21)\qquad \begin{aligned} \varphi_n^\infty(t,k) &= \sum_{j=0}^{\infty} \int_0^t \mathsf{P}\{\eta(x) = j\} \varphi_{n+j-1}^\infty(t-x,k)\, dF(x) + \mathsf{P}\{\eta(t) = k-n\} \overline{F}(t), \\ &\qquad 1 \le n \le h; \\ \varphi_n^\infty(t,k) &= \sum_{j=0}^{\infty} \int_0^t \mathsf{P}\{\eta(x) = j\} \varphi_{n+j-1}^\infty(t-x,k)\, d\widetilde{F}(x) + \mathsf{P}\{\eta(t) = k-n\} \overline{\widetilde{F}}(t), \\ &\qquad n \ge h+1. \end{aligned}$$

Passing to the Laplace transforms in (21) we derive the following system of equations to determine the functions $\Phi_n^\infty(s,k)$:

$$(22)\qquad \begin{aligned} \Phi_n^\infty(s,k) &= f(s) \sum_{j=0}^{\infty} p_{j-1}(s) \Phi_{n+j-1}^\infty(s,k) + q_{k-n}(s), \qquad 1 \le n \le h, \\ \Phi_n^\infty(s,k) &= \widetilde{f}(s) \sum_{j=0}^{\infty} \widetilde{p}_{j-1}(s) \Phi_{n+j-1}^\infty(s,k) + \widetilde{q}_{k-n}(s), \qquad n \ge h+1, \end{aligned}$$

with the boundary condition $\Phi_0^\infty(s,k) = 0$ that follows from an obvious equality

$$\varphi_0^\infty(t,k) = 0.$$

Passing to the limit as $s \to +0$, equations (22) become of the form

$$(23)\qquad \begin{aligned} \Phi_n^\infty(k) &= \sum_{j=0}^{\infty} p_{j-1} \Phi_{n+j-1}^\infty(k) + q_{k-n}, \qquad 1 \le n \le h, \\ \Phi_n^\infty(k) &= \sum_{j=0}^{\infty} \widetilde{p}_{j-1} \Phi_{n+j-1}^\infty(k) + \widetilde{q}_{k-n}, \qquad n \ge h+1. \end{aligned}$$

Applying the equations obtained in [1] for functions $\Phi_n^{(m)}(s,k)$, we deduce that

$$(24)\qquad \begin{aligned} \Phi_n^{(m)}(k) &= \sum_{j=0}^{m-n} p_{j-1} \Phi_{n+j-1}^{(m)}(k) + \overline{p}_{m-n} \Phi_m^{(m)}(k) + f_n(k), \qquad 1 \le n \le h, \\ \Phi_n^{(m)}(k) &= \sum_{j=0}^{m-n} \widetilde{p}_{j-1} \Phi_{n+j-1}^{(m)}(k) + \overline{\widetilde{p}}_{m-n} \Phi_m^{(m)}(k) + \widetilde{f}_n(k), \qquad h+1 \le n \le m. \end{aligned}$$

It follows from the results of [1] that $\Phi_m^{(m)}(k)$ can be rewritten similarly to the right hand sides of (20). Passing to the limit as $s \to +0$ and then as $m \to \infty$ in the expressions for the functions $\Phi_n^{(m)}(s,k)$ obtained in [1], we prove that the limit $\lim_{m\to\infty} \Phi_n^{(m)}(k)$ is expressed in terms involved in the right hand sides of equalities (19). Since the functions $\Phi_m^{(m)}(k)$ are bounded (this follows from equality (20)), we see that the limits $\lim_{m\to\infty} \Phi_n^{(m)}(k)$ are finite. This allows one to pass to the limit in (24) as $m \to \infty$.

Thus

$$\lim_{m\to\infty}\Phi_n^{(m)}(k)=\sum_{j=0}^{\infty}p_{j-1}\lim_{m\to\infty}\Phi_{n+j-1}^{(m)}(k)+q_{k-n},\qquad 1\le n\le h,$$

(25)

$$\lim_{m\to\infty}\Phi_n^{(m)}(k)=\sum_{j=0}^{\infty}\widetilde{p}_{j-1}\lim_{m\to\infty}\Phi_{n+j-1}^{(m)}(k)+\widetilde{q}_{k-n},\qquad n\ge h+1.$$

Comparing equalities (23) and (25), we conclude that the limit relation (18) holds. The lemma is proved. □

Put

$$r_{kn}(h)=\widetilde{r}_k(h)-\sum_{i=1}^{h-n}R_ip_{k-n-i},\qquad \widetilde{r}_k(h)=\sum_{i=1}^{h}R_ip_{k-i},\qquad n,k\ge 1.$$

Lemma 3.2. *If $\widetilde{\rho}<1$, then*

(26)
$$\lim_{m\to\infty}\Phi_n^{(m)}=\Phi_n^{\infty},\qquad n\ge 1,$$

where

$$\Phi_n^{\infty}=m_1\sum_{i=h+1-n}^{h}R_i+\frac{\widetilde{m}_1}{1-\widetilde{\rho}}\bigg(\sum_{k=h+1}^{\infty}kr_{kn}(h)-hp_{-1}(R_{h+1}-R_{h+1-n})\bigg),$$
$$1\le n\le h-1;$$

(27)

$$\Phi_n^{\infty}=m_1\sum_{i=1}^{h}R_i+\frac{\widetilde{m}_1}{1-\widetilde{\rho}}\bigg(n-hp_{-1}R_{h+1}+\sum_{k=h+1}^{\infty}k\widetilde{r}_k(h)\bigg),\qquad n\ge h;$$

$$\sum_{k=h+1}^{\infty}kr_{kn}(h)=\sum_{k=h+1}^{\infty}k\widetilde{r}_k(h)$$
$$-\sum_{i=1}^{h-n}R_i\bigg(\rho-1+p_{-1}+(n+i)\overline{p}_{h+1-n-i}-\sum_{k=1}^{h-n-i}kp_k\bigg),$$
$$n\ge 1;$$

(28)

$$\sum_{k=h+1}^{\infty}k\widetilde{r}_k(h)=\sum_{i=1}^{h}R_i\bigg(\rho-1+p_{-1}+i\overline{p}_{h+1-i}-\sum_{k=1}^{h-i}kp_k\bigg).$$

Proof. Passing to the limit as $s\to+0$ and then evaluating the sum over k running from 1 to $m+1$ in the expressions for the functions $\Phi_n^{(m)}(s,k)$ obtained in [1], we get

$$\Phi_n^{(m)}=m_1\sum_{i=h+1-n}^{h}R_i$$

(29)

$$+\widetilde{m}_1\bigg(p_{-1}(R_{h+1}-R_{h+1-n})\sum_{i=1}^{m-h}\widetilde{R}_i-\sum_{k=h+1}^{m-1}r_{kn}(h)\sum_{i=1}^{m-k}\widetilde{R}_i\bigg),$$
$$1\le n\le h-1;$$

(30)
$$\Phi_n^{(m)}=m_1\sum_{i=1}^{h}R_i+\widetilde{m}_1\bigg(p_{-1}R_{h+1}\sum_{i=1}^{m-h}\widetilde{R}_i-\sum_{k=h+1}^{m-1}\widetilde{r}_k(h)\sum_{i=1}^{m-k}\widetilde{R}_i-\sum_{i=1}^{m-n}\widetilde{R}_i\bigg),$$
$$h\le n\le m.$$

If $\widetilde{\rho} < 1$, then the limits as $m \to \infty$ on the right hand sides of equalities (29) and (30) are evaluated similarly to the limit $\lim_{m\to\infty} S(m)$ in the proof of Theorem 3.1. As a result, the limits $\lim_{m\to\infty} \Phi_n^{(m)}$ equal the right hand sides of (27). The proof of relations (28) is similar to the proof of Lemma 2.5. By the definition of $\Phi_n^{(m)}$ and Φ_n^{∞} and since the limits $\lim_{m\to\infty} \Phi_n^{(m)}(k)$ and $\lim_{m\to\infty} \Phi_n^{(m)}$ are finite, the limit relations (18) imply that

$$\lim_{m\to\infty} \Phi_n^{(m)} = \lim_{m\to\infty} \sum_{k=1}^{m+1} \Phi_n^{(m)}(k) = \sum_{k=1}^{\infty} \lim_{m\to\infty} \Phi_n^{(m)}(k) = \sum_{k=1}^{\infty} \Phi_n^{\infty}(k) = \Phi_n^{\infty}.$$

The lemma is proved. □

Theorem 3.2. *If $\widetilde{\rho} < 1$, then the mean duration of the busy period of the queueing system* $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\infty$ *is finite. Moreover,*

$$\mathsf{E}\,\tau(\infty) = \lim_{m\to\infty} \mathsf{E}\,\tau(m),$$

that is, the expectation $\mathsf{E}\,\tau(\infty)$ equals the right hand side of (14).

Proof. The equality

$$\mathsf{E}\,\tau(m) = \sum_{n=1}^{m} a_n \Phi_n^{(m)} + \overline{a}_{m+1} \Phi_{m+1}^{(m)} \tag{31}$$

is proved in [1], where $\Phi_{m+1}^{(m)} = \Phi_m^{(m)}$. A similar reasoning for the system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\infty$ yields

$$\mathsf{E}\,\tau(\infty) = \sum_{n=1}^{\infty} a_n \Phi_n^{\infty}.$$

If $\widetilde{\rho} < 1$, then the limit $\lim_{m\to\infty} \mathsf{E}\,\tau(m)$ is finite by Theorem 3.1, whence, by relations (26) and (31),

$$\lim_{m\to\infty} \mathsf{E}\,\tau(m) = \sum_{n=1}^{\infty} a_n \lim_{m\to\infty} \Phi_n^{(m)} = \sum_{n=1}^{\infty} a_n \Phi_n^{\infty} = \mathsf{E}\,\tau(\infty).$$

The theorem is proved. □

Theorem 3.3. *If $\widetilde{\rho} < 1$, then the stationary distribution of the total number of customers in the queueing system* $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\infty$ *is given by*

$$\rho_0(\infty) = \frac{1}{1 + \lambda\,\mathsf{E}\,\tau(\infty)};$$

$$\rho_k(\infty) = \frac{\lambda}{1 + \lambda\,\mathsf{E}\,\tau(\infty)} \left(\sum_{i=1}^{k} R_i q_{k-i} - \sum_{n=1}^{k-1} a_n \sum_{i=1}^{k-n} R_i q_{k-n-i} \right), \qquad k = 1, \dots, h;$$

$$\begin{aligned} \rho_k(\infty) = {} & \frac{\lambda}{1 + \lambda\,\mathsf{E}\,\tau(\infty)} \\ & \times \left(\sum_{i=1}^{h} R_i q_{k-i} - \sum_{n=1}^{h-1} a_n \sum_{i=1}^{h-n} R_i q_{k-n-i} + p_{-1} R(h) \sum_{i=1}^{k-h} \widetilde{R}_i \widetilde{q}_{k-h-i} \right. \\ & \left. - \sum_{j=h+1}^{k-1} r_j(h) \sum_{i=1}^{k-j} \widetilde{R}_i \widetilde{q}_{k-j-i} - \sum_{n=h}^{k-1} a_n \sum_{i=1}^{k-n} \widetilde{R}_i \widetilde{q}_{k-n-i} \right), \\ & k \ge h+1. \end{aligned} \tag{32}$$

Proof. According to [1],

$$\rho_0(m) = \frac{1}{1+\lambda\,\mathsf{E}\,\tau(m)};$$
(33)
$$\rho_k(m) = \lambda\rho_0(m)\left(\sum_{n=1}^{m} a_n\Phi_n^{(m)}(k) + \overline{a}_{m+1}\Phi_{m+1}^{(m)}(k)\right), \qquad 1\le k\le m+1,$$

where

$$\Phi_{m+1}^{(m)}(k) = \Phi_m^{(m)}(k).$$

A similar reasoning for the queueing system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\infty$ yields

$$\rho_0(\infty) = \frac{1}{1+\lambda\,\mathsf{E}\,\tau(\infty)};$$
$$\rho_k(\infty) = \lambda\rho_0(\infty)\sum_{n=1}^{\infty} a_n\Phi_n^{\infty}(k), \qquad k\ge 1.$$

If $\widetilde{\rho}<1$, then the limits $\lim_{m\to\infty}\rho_k(m)$ are finite. The latter result follows from equalities (33) and from the finiteness of the limits

$$\lim_{m\to\infty}\Phi_n^{(m)}(k)$$

and

$$\lim_{m\to\infty}\mathsf{E}\,\tau(m)$$

(see Theorem 3.1). Using the limit relations (18) and Theorem 3.2 we derive from equalities (33) that

$$\lim_{m\to\infty}\rho_0(m) = \frac{1}{1+\lambda\,\mathsf{E}\,\tau(\infty)} = \rho_0(\infty);$$
$$\lim_{m\to\infty}\rho_k(m) = \lambda\rho_0(\infty)\sum_{n=1}^{\infty} a_n \lim_{m\to\infty}\Phi_n^{(m)}(k)$$
$$= \lambda\rho_0(\infty)\sum_{n=1}^{\infty} a_n\Phi_n^{\infty}(k) = \rho_k(\infty), \qquad k\ge 1.$$

Passing to the limit as $m\to\infty$ in formulas for the stationary distributions $\rho_k(m)$ of the queueing system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\mathrm{m}$ obtained in [1], we prove equalities (32). The theorem is proved. □

4. An example of the evaluation of the stationary distribution

Assume that customers arrive to the system in batches containing only one or two customers (that is, $a_1+a_2=1$) and that the service time in the main mode is uniformly distributed in the interval $[a,b]$, while the service time of the threshold mode is distributed according to the Erlang law of the second order with parameter $\widetilde{\mu}$. Then the expectations of the service time are equal to $m_1=(a+b)/2$ and $\widetilde{m}_1=2/\widetilde{\mu}$, respectively.

Denote

$$S_n(a,b) = \frac{1}{\lambda(b-a)}\sum_{k=0}^{n}\left(\frac{(a\lambda)^k}{k!}e^{-a\lambda} - \frac{(b\lambda)^k}{k!}e^{-b\lambda}\right), \qquad n\ge 0.$$

By formulas (1) and (3) we obtain

$$p_{-1} = S_0(a,b), \qquad p_0 = a_1 S_1(a,b), \qquad p_1 = a_1^2 S_2(a,b) + a_2 S_1(a,b);$$
$$p_2 = a_1^3 S_3(a,b) + 2a_1 a_2 S_2(a,b), \qquad p_3 = a_1^4 S_4(a,b) + 3a_1^2 a_2 S_3(a,b) + a_2^2 S_2(a,b);$$
$$p_4 = a_1^5 S_5(a,b) + 4a_1^3 a_2 S_4(a,b) + 3a_1 a_2^2 S_3(a,b);$$
$$p_5 = a_1^6 S_6(a,b) + 5a_1^4 a_2 S_5(a,b) + 6a_1^2 a_2^2 S_4(a,b) + a_2^3 S_3(a,b);$$
$$p_6 = a_1^7 S_7(a,b) + 6a_1^5 a_2 S_6(a,b) + 10a_1^3 a_2^2 S_5(a,b) + 4a_1 a_2^3 S_4(a,b);$$
$$p_7 = a_1^8 S_8(a,b) + 7a_1^6 a_2 S_7(a,b) + 15a_1^4 a_2^2 S_6(a,b) + 10a_1^2 a_2^3 S_5(a,b) + a_2^4 S_4(a,b);$$
$$\widetilde{p}_{-1} = \frac{\widetilde{\mu}^2}{(\lambda+\widetilde{\mu})^2}; \qquad \widetilde{p}_0 = \frac{2a_1\widetilde{\mu}^2\lambda}{(\lambda+\widetilde{\mu})^3}; \qquad \widetilde{p}_1 = \frac{3a_1^2\widetilde{\mu}^2\lambda^2}{(\lambda+\widetilde{\mu})^4} + \frac{2a_2\widetilde{\mu}^2\lambda}{(\lambda+\widetilde{\mu})^3};$$
$$\widetilde{p}_2 = \frac{4a_1^3\widetilde{\mu}^2\lambda^3}{(\lambda+\widetilde{\mu})^5} + \frac{6a_1a_2\widetilde{\mu}^2\lambda^2}{(\lambda+\widetilde{\mu})^4}; \qquad \widetilde{p}_3 = \frac{5a_1^4\widetilde{\mu}^2\lambda^4}{(\lambda+\widetilde{\mu})^6} + \frac{12a_1^2a_2\widetilde{\mu}^2\lambda^3}{(\lambda+\widetilde{\mu})^5} + \frac{3a_2^2\widetilde{\mu}^2\lambda^2}{(\lambda+\widetilde{\mu})^4};$$
$$\widetilde{p}_4 = \frac{6a_1^5\widetilde{\mu}^2\lambda^5}{(\lambda+\widetilde{\mu})^7} + \frac{20a_1^3a_2\widetilde{\mu}^2\lambda^4}{(\lambda+\widetilde{\mu})^6} + \frac{12a_1a_2^2\widetilde{\mu}^2\lambda^3}{(\lambda+\widetilde{\mu})^5}.$$

Consider an example corresponding to the case of

$$a_1 = 0.75; \quad a_2 = 0.25; \quad h = 5; \quad \lambda = 1; \quad a = \frac{1}{3}; \quad b = 1; \quad \widetilde{\mu} = 6.$$

Then

$$m_1 = \frac{2}{3}; \quad \widetilde{m}_1 = \frac{1}{3}; \quad b_1 = 1.25; \quad \rho = \frac{5}{6}; \quad \widetilde{\rho} = \frac{5}{12}$$

and the mean duration of the busy period $\mathsf{E}\,\tau(\infty)$ found by (13) is equal to 3.79439.

The row "$\rho_k(\infty)$" in Table 1 contains the probabilities $\rho_k(\infty)$ evaluated by formulas (32). For the sake of comparison, this table also contains the corresponding probabilities obtained with the help of GPSS World [5, 6] for $t = 100\,000$.

Table 1. Stationary distribution of the number of customers in the queueing system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\infty$ $(h = 5)$

Number of customers (k)	0	1	2	3	4
$\rho_k(\infty)$	0.2086	0.1902	0.1754	0.1411	0.1128
$\rho_k(\infty)$ (GPSS World, $t = 10^5$)	0.2093	0.1900	0.1756	0.1413	0.1126
Number of customers (k)	5	6	7	8	...
$\rho_k(\infty)$	0.0891	0.0437	0.0218	0.0097	...
$\rho_k(\infty)$ (GPSS World, $t = 10^5$)	0.0898	0.0429	0.0218	0.0093	...

5. Concluding remarks

Based on the results obtained in [1] for the queueing system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\mathrm{m}$, we find the mean duration of the busy period and stationary distribution of the total number of customers in the queueing system $\mathrm{M}^\theta/\mathrm{G}, \widetilde{\mathrm{G}}/1/\infty$ by studying the properties of the sequences $\{R_n\}$ and $\{\widetilde{R}_n\}$ related to the resolvents of two lower continuous random walks corresponding to the main and threshold service modes, respectively.

The analytical results obtained in the paper are verified with the help of a GPSS World simulation model.

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