Теорія Ймовір. та Матем. Статист.
Вип. 87, 2012

Theor. Probability and Math. Statist.
No. 87, 2013, Pages 65–78
S 0094-9000(2014)00905-9
Article electronically published on March 21, 2014

MAXIMAL COUPLING PROCEDURE AND STABILITY OF DISCRETE MARKOV CHAINS. II

UDC 519.21

M. V. KARTASHOV AND V. V. GOLOMOZYĬ

Abstract. Two discrete Markov chains whose one-step transition probabilities are close to each other in the uniform total variation norm or in the V-norm are considered. The problem of stability of the transition probabilities over an arbitrary number of steps is investigated. The main assumption is that either the uniform mixing or V-mixing condition holds. In particular, we prove that the uniform distance between the distributions of the chains after an arbitrary number of steps does not exceed $\varepsilon/(1-\rho)$, where ε is the uniform distance between the transition matrices and where ρ is the uniform mixing coefficient. A number of general examples are considered. The proofs are based on the maximal coupling procedure that maximizes the one-step coupling probabilities.

This is the second part of our paper, and it contains the proofs of the results stated in its first part [29]. We continue the numbering of the formulas after the first part of the paper.

4. Proofs

Recall that if $Q=(Q_{ij})$ is a matrix, $f=(f_j)$ is a function, and $\mu=(\mu_i)$ is a measure, then Qf denotes the function with the coordinates

$$Qf_i=\sum_j Q_{ij}f_j,$$

while the symbol μQ stands for the measure with the coordinates $\mu Q_j=\sum_i \mu_i Q_{ij}$. The product PQ of matrices P and Q and a power $Q^{(n)}$ of a matrix Q are defined accordingly. The symbol $Q_i.=(Q_{ij}, j\in E)$ denotes the i-th row of a matrix Q. If f is a function, then we write either f_i or $=f(i)$ for its coordinates. The unit matrix is denoted by I, while 1 means the function assuming only one value 1 and the corresponding set of indices is determined by the co-dimension of a matrix by which a function is multiplied.

If the set for an index of summation is not specified, then we assume that the index runs through the whole set E. The same rule applies when evaluating the limit superior. Finally, $x^{\pm}$ stands for the positive and negative parts of a number x and $\delta_{ij}=1_{i=j}$ means the Kronecker symbol.

2010 *Mathematics Subject Classification.* Primary 60J45; Secondary 60A05, 60K05.

Key words and phrases. Coupling theory, coupling method, maximal coupling, discrete Markov chains, stability of distributions.

Lemma 1. *Using the notation introduced in* [29],

$$R1_i = R'1_i = h1_i = 1 - q_i, \qquad r(P,P') = \sup_i (1 - q_i), \tag{45}$$

$$S1_{ik} = S'1_{ik} = T1_{ik} = 1 - g1_{ik} = 1 - q_{ik}, \qquad \rho(P,P') = \sup_{i \neq k} (1 - q_{ik}). \tag{46}$$

In particular, if assumptions (1) *and* (2) *hold, then*

$$R1_i = R'1_i = h1_i \le r(P,P') \le \varepsilon,$$

$$S1_{ik} = S'1_{ik} = T1_{ik} = 1 - g1_{ik} \le \rho(P,P') \le \rho,$$

and

$$T^{(s)}1_{ik} \le \rho^s \to 0, \qquad s \to \infty. \tag{47}$$

Proof. We derive from (29) that

$$R1_i = (P - Q)1_i = 1 - \sum_j Q_{ij} = 1 - q_i, \qquad R'1_i = 1 - q_i,$$

and

$$h1_i = \sum_{j,l} R_{ij} R'_{il}/(1 - q_i) = \sum_j R_{ij} \sum_l R'_{il}/(1 - q_i) = 1 - q_i.$$

Then

$$r(P,P') = \sup_i \sum_j \left| P_{ij} - P'_{ij} \right| /2 = \sup_i \sum_j \left(R_{ij} + R'_{ij} \right)/2 = \sup_i (1 - q_i),$$

which proves (45).

Similarly, we obtain from (34) and (35) that (46) holds and

$$S1_{ik} = (P - g)1_{ik} = 1 - g1_{ik} = 1 - q_{ik}, S'1_{ik} = 1 - q_{ik},$$

$$T1_{ik} = \sum_{j,l} S_{ik,j} S'_{ik,l}/(1 - q_{ik}) = \sum_j S_{ik,j} \sum_l S'_{ik,l}/(1 - q_{ik}) = 1 - q_{ik},$$

$$\rho(P,P') = \sup_{i \neq k} \sum_j \left| P_{ij} - P'_{kj} \right| /2 = \sup_{i \neq k} \sum_j \left(S_{ik,j} + S'_{ik,j} \right)/2 = \sup_{i \neq k} (1 - q_{ik}).$$

Applying assumptions (1) and (2) to equalities (45) and (46) we prove the first two inequalities in (47).

Using the induction, the second inequality in (47) implies that

$$T^{(s)}1_{ik} = \left(T^{(s-1)}(T1) \right)_{ik} \le \left(T^{(s-1)}(\rho 1) \right)_{ik} = \rho T^{(s-1)} 1_{ik} \le \cdots \le \rho^s 1,$$

since the matrix T is nonnegative. □

We agree that $\inf(\varnothing) = \infty$ in what follows.

Definition. *The switching moments for the coordinate* d_n *of the coupling process* $\overline{X}$ *are defined by the following recurrence relation*:

$$\tau_0^{d_0} = 0, \qquad \tau_m^d = \inf\left(n > \tau_{m-1}^{1-d} \colon d_n = d \right), \qquad d = \mathrm{mod}(m - d_0, 2), \qquad m \ge 1. \tag{48}$$

If $d_0 = 1$, then we deal with a sequence of a Markov moment,

$$0 = \tau_0^1 < \tau_1^0 < \tau_2^1 < \tau_3^0 < \cdots \le \infty.$$

If $d_0 = 0$, then

$$0 = \tau_0^0 < \tau_1^1 < \tau_2^0 < \tau_3^1 < \cdots \le \infty.$$

Lemma 2. *Using the notation introduced in* [29],

$$\begin{aligned} \mathsf{P}_{ii1}\left(\tau_1^0 > t, \overline{X}_t = (k,k,1)\right) &= Q_{ik}^{(t)}, \qquad t \ge 1, \\ \mathsf{P}_{ii1}\left(\tau_1^0 = t, \overline{X}_t = (j,l,0)\right) &= Q^{(t-1)} h_{i,jl}, \qquad t \ge 1, \\ \mathsf{P}_{ik0}\left(\tau_1^1 > t, \overline{X}_t = (j,l,0)\right) &= T_{ik,jl}^{(t)}, \qquad t \ge 1, \end{aligned} \tag{49}$$

and

$$\mathsf{P}_{ik0}\left(\tau_1^1 = t, \overline{X}_t = (j,j,1)\right) = T^{(t-1)} g_{ik,j}, \qquad t \ge 1. \tag{50}$$

Proof. The proof follows from the Markov property of the chain $\overline{X}$, since the transition probability over several steps is equal to the product of one-step transition probabilities. According to (31), (32), (36), and (37) the latter probabilities for the coordinate d_n are given by the matrix Q for the transition from 1 to 1, by matrix h for the transition from 1 to 0, by matrix T for the transition from 0 to 0, and by matrix g for the transition from 0 to 1. $\square$

Lemma 3. *Assume that condition* (7) *holds. Then* $\mathsf{P}_{ik0}(\tau_1^1 < \infty) = 1$ *for all* $i \ne k$.

Proof. We deduce from Lemma 2 that the function

$$\varphi_{ik} = \mathsf{P}_{ik0}\left(\tau_1^1 < \infty\right) = \sum_{t\ge1} \mathsf{P}_{ik0}\left(\tau_1^1 = t\right) = \sum_{t\ge1}\sum_j T^{(t-1)} g_{ik,j} = \sum_{s\ge0} T^{(s)} g 1_{ik}$$

is the minimal solution of the equation $\varphi_{ik} = g1_{ik} + T\varphi_{ik}$. By Lemma 1, the function 1 satisfies this equation. Then (7) together with Lemma 1 of [27, 2009] implies that the above function is the minimal solution. $\square$

Definition. The number of complete coupling–decoupling cycles for the initial condition $d_0 = 1$ is defined by

$$\begin{aligned} \nu_n = \sup\left(m \ge 0\colon \tau_{2m}^1 \le n\right) &< \infty, \qquad n \ge 1, \\ \{\nu_n = m\} &= \left\{\tau_{2m}^1 \le n < \tau_{2m+2}^1\right\}. \end{aligned} \tag{51}$$

This definition is correct, since $\tau_0^1 = 0$ and $\tau_m^d \ge m$.

Remark 8. In the general case, the random events $\{\tau_{2m}^1 = \infty\}$ can have positive probabilities. Note however that if $d_0 = 1$, then condition (7) implies that

$$\tau_{2m}^1 - \tau_{2m-1}^0 < \infty$$

almost surely. Hence if τ_{2m-1}^0 is finite almost surely, then τ_{2m}^1 is finite almost surely as well. Therefore only one sequence may have a positive probability, namely the sequence whose members d_n equal each other, that is, $d_n = 1$ for $n \ge \tau_{2\mu}^1$ on the event $\{\mu < \infty\}$, where $\mu = \sup_{n\ge0} \nu_n$.

Lemma 4. *For all* $B \subset E$ *and* $n \ge 1$, *we have*

$$\mathsf{P}_{ii1}(X_n \in B, d_n = 1) = \mathsf{P}_{ii1}(X_n' \in B, d_n = 1). \tag{52}$$

Proof. Consider the random events

$$C_{sk}^{nm} = \left\{\nu_n = m, \tau_{2m}^1 = s, \overline{X}_s = (k,k,1)\right\}$$

being disjoint for different $m \ge 0$, $s \ge 0$, and $k \in E$. Since

$$\{d_n = 1\} = \bigcup_{m\ge0} \left\{\nu_n = m, \tau_{2m}^1 < \infty, d_n = 1\right\},$$

we get

$$\begin{aligned}\mathsf{P}_{ii1}(X_n\in B, d_n=1) &= \sum_{m\ge 0}\sum_{s\le n}\sum_k \mathsf{P}_{ii1}(C_{sk}^{nm}, X_n\in B, d_n=1)\\ &= \sum_{m\ge 0}\sum_{s\le n}\sum_k \mathsf{P}_{ii1}\left(C_{sk}^{sm}, X_n\in B, \tau_{2m}^1\le n<\tau_{2m+1}^0\right)\end{aligned}$$

because $\{\nu_n=m\}=\{\nu_s=m\}$ on the random event $\{d_n=1,\tau_{2m}^1=s\}$ for $s\le n$ and because

$$C_{sk}^{nm}\cap\{d_n=1\}=C_{sk}^{nm}\cap\left\{\tau_{2m}^1\le n<\tau_{2m+1}^0\right\}.$$

Taking into account the strong Markov property of the chain $\overline{X}$, we conclude that

$$\begin{aligned}\mathsf{P}_{ii1}(X_n\in B, d_n=1) &= \sum_{m\ge 0}\sum_{s\le n}\sum_k \mathsf{P}_{ii1}(C_{sk}^{sm})\,\mathsf{P}\left(X_n\in B, s\le n<\tau_{2m+1}^0 \,\middle|\, C_{sk}^{sm}\right)\\ &= \sum_{m\ge 0}\sum_{s\le n}\sum_k \mathsf{P}_{ii1}(C_{sk}^{sm})\,\mathsf{P}_{kk1}\left(X_{n-s}\in B, n-s<\tau_1^0\right)\\ &= \sum_{m\ge 0}\sum_{s\le n}\sum_k \mathsf{P}_{ii1}(C_{sk}^{sm})\,\mathsf{P}_{kk1}\left(X'_{n-s}\in B, n-s<\tau_1^0\right),\end{aligned}$$

since $X_n=X'_n$ on the random event $\{n<\tau_1^0\}$ if $d_0=1$. Hence

$$\mathsf{P}_{ii1}(X_n\in B, d_n=1)=\mathsf{P}_{ii1}(X'_n\in B, d_n=1).$$

The latter equality is proved by a similar reasoning if X_n is changed by X'_n. □

Lemma 5. *Let* $A_n=\{X_n\in B\}\cap\{X'_n\in B'\}$, *where* $B,B'\subset E$. *Then*

$$\mathsf{P}_{ii1}(A_n, d_n=0)=\sum_{1\le t\le n}\sum_k \mathsf{P}_{ii1}(\overline{X}_{t-1}=(k,k,1))r_k^{(n-t)}, \tag{53}$$

where

$$\begin{gathered} r_k^{(s)}\equiv \mathsf{P}_{kk1}\left(d_1=0, s+1<\tau_2^1, A_{s+1}\right)=\left(hT^{(s)}1_{B\times B'}\right)_k,\\ s\ge 0,\qquad k\in E,\qquad (1_{B\times B'})_{jl}\equiv 1_{j\in B, l\in B'}.\end{gathered} \tag{54}$$

Proof. Since $\{\nu_n=m, d_n=0\}\subset\{\tau_{2m+1}^0\le n\}$, the full probability formula implies that

$$\begin{aligned}\mathsf{P}_{ii1}(A_n, d_n=0) &= \sum_{m\ge 0}\sum_{0\le s<t\le n}\mathsf{P}_{ii1}\left(\nu_n=m, \tau_{2m}^1=s, \tau_{2m+1}^0=t, d_t=0, d_n=0, A_n\right)\\ &= \sum_{m\ge 0}\sum_{0\le s<t\le n}\sum_k \mathsf{P}_{ii1}\big(\nu_{t-1}=m, \tau_{2m}^1=s, \overline{X}_{t-1}=(k,k,1),\\ &\qquad\qquad d_t=0, n<\tau_{2m+1}^1, A_n\big).\end{aligned}$$

The latter equality holds since $\{\nu_s=\nu_{t-1}=\nu_n=m\}$ on the random event

$$\left\{\nu_n=m, \tau_{2m}^0=s<t\le n, d_n=0\right\}$$

and since

$$\left\{\tau_{2m+1}^0=t\right\}=\bigcup_k\left\{d_t=0, \overline{X}_{t-1}=(k,k,1)\right\}.$$

The random events

$$D_{sk}^{tm}\equiv\{\nu_{t-1}=m, \tau_{2m}^1=s, \overline{X}_{t-1}=(k,k,1)\}$$

are disjoint. Hence the above equality yields

$$\begin{aligned}
\mathsf{P}_{ii1}(A_n, d_n = 0) &= \sum_{m\ge 0}\sum_{0\le s<t\le n}\sum_k \mathsf{P}_{ii1}\left(D^{tm}_{sk}, d_t = 0, n < \tau^1_{2m+1}, A_n\right)\\
&= \sum_{m\ge 0}\sum_{0\le s<t\le n}\sum_k \mathsf{P}_{ii1}\left(D^{tm}_{sk}\right)\mathsf{P}\left(\nu_{t-1} = m, d_t = 0, n < \tau^1_{2m+1}, A_n \;\middle|\; D^{tm}_{sk}\right)\\
&= \sum_{m\ge 0}\sum_{0\le s<t\le n}\sum_k \mathsf{P}_{ii1}\left(D^{tm}_{sk}\right)\mathsf{P}_{kk1}\left(d_1 = 0, n-t+1 < \tau^1_2, A_{n-t+1}\right)\\
&= \sum_{t\le n}\sum_k \mathsf{P}_{ii1}\left(\overline{X}_{t-1} = (k,k,1)\right) r_k^{(n-t)}
\end{aligned}$$

in view of the Markov property of the chain $\overline{X}$ at the moment $t-1$. Here we used the definition of (54) and the property that the events $\{\nu_{t-1} = m, \tau^1_{2m} = s\}$ are disjoint for different m and s.

To prove the equality in (54), we use Lemmas 1 and 2 together with the Markov property of the chain $\overline{X}$ at the moment 1. Then we get

$$\begin{aligned}
r_k^{(s)} &= \sum_{j\ne l}\mathsf{P}_{kk1}\left(\overline{X}_1 = (j,l,0)\right)\mathsf{P}_{jl0}\left(s < \tau^1_1, (X_s, X'_s)\in B\times B', d_s = 0\right)\\
&= \sum_{j\ne l} h_{k,jl}\left(T^{(s)}1_{B\times B'}\right)_{jl} = \left(hT^{(s)}1_{B\times B'}\right)_k . \qquad \square
\end{aligned}$$

Proof of Remark 1. The inequality for the increment of the distance ρ follows from the triangle inequality for the norm $\|\cdot\|$. The contraction inequality is equivalent to the same inequality for the difference of point measures $\mu_j = \delta_{ij} - \delta_{kj}$, since these measures generate the space of measures with $\mu(E) = 0$ under the closure of the linear span. $\square$

Proof of Theorem 1. Using the marginal distributions (43) of the chain $\overline{X}$ together with Lemma 4 we obtain

$$\begin{aligned}
\left|\mathsf{P}_i(X_n\in B) - \mathsf{P}'_i(X'_n\in B)\right| &= |\mathsf{P}_{ii1}(X_n\in B) - \mathsf{P}_{ii1}(X'_n\in B)|\\
&= |\mathsf{P}_{ii1}(X_n\in B, d_n = 0) - \mathsf{P}_{ii1}(X'_n\in B, d_n = 0)|\\
&\le \mathsf{P}_{ii1}(\{X_n\in B\}\Delta\{X'_n\in B\}, d_n = 0)\\
&\le \mathsf{P}_{ii1}(d_n = 0).
\end{aligned}\tag{55}$$

Then we choose $B = B' = E$ and $A_n = \Omega$ in Lemma 5 to prove bound (4):

$$\begin{aligned}
\mathsf{P}_{ii1}(d_n = 0) &= \sum_{1\le t\le n}\sum_k \mathsf{P}_{ii1}\left(\overline{X}_{t-1} = (k,k,1)\right) r_k^{(n-t)}\\
&\le \sum_{1\le t\le n}\sup_k r_k^{(n-t)}\sum_k \mathsf{P}_{ii1}\left(\overline{X}_{t-1} = (k,k,1)\right)\\
&\le \sum_{1\le t\le n}\sup_k r_k^{(n-t)}\\
&\le \sum_{1\le t\le n}\varepsilon\rho^{n-t} = \varepsilon(1-\rho^n)/(1-\rho).
\end{aligned}\tag{56}$$

Here we used Lemmas 3 and 5 together with the relations

$$r_k^{(s)} = \sum_{j\ne l} h_{k,jl}(T^{(s)}1)_{jl} \le \left(\sum_{j\ne l} h_{k,jl}\right)\sup_{j\ne l}(T^{(s)}1)_{jl} \le (1-q_k)\rho^s \le \varepsilon\rho^s.$$

Substituting (56) to (55) we prove the first inequality in (4). The second inequality is obvious. $\square$

Proof of Corollary 1. Using the strong Markov property of the chain $\overline{X}$ we see that

$$\begin{aligned}\mathsf{P}_{ik0}(X_n \in B) &= \mathsf{P}_{ik0}\left(\tau_1^1 > n, X_n \in B\right)\\ &\quad + \sum_{1\le t\le n}\sum_j \mathsf{P}_{ik0}\left(\tau_1^1 = t, \overline{X}_t = (j,j,1), X_n \in B\right)\\ &= \mathsf{P}_{ik0}\left(\tau_1^1 > n, X_n \in B\right)\\ &\quad + \sum_{1\le t\le n}\sum_j \mathsf{P}_{ik0}\left(\tau_1^1 = t, \overline{X}_t = (j,j,1)\right)\mathsf{P}_{jj1}(X_{n-t} \in B).\end{aligned}$$

Subtracting a similar equality for X'_n and using (43) and (4), we obtain for $i \ne k$ that

$$\begin{aligned}&\left|\mathsf{P}_i(X_n \in B) - \mathsf{P}'_k(X'_n \in B)\right| = \left|\mathsf{P}_{ik0}(X_n \in B) - \mathsf{P}_{ik0}(X'_n \in B)\right|\\ &\quad\le \mathsf{P}_{ik0}(\tau_1^1 > n, \{X_n \in B\}\Delta\{X'_n \in B\})\\ &\qquad + \sum_{t\le n}\sum_j \mathsf{P}_{ik0}\left(\tau_1^1 = t, \overline{X}_t = (j,j,1)\right)\left|\mathsf{P}_{jj1}(X_{n-t} \in B) - \mathsf{P}_{jj1}(X'_{n-t} \in B)\right|\\ &\quad\le \mathsf{P}_{ik0}\left(\tau_1^1 > n\right) + \sum_{t\le n}\mathsf{P}_{ik0}(\tau_1^1 = t)\varepsilon/(1-\rho)\\ &\quad= \mathsf{P}_{ik0}\left(\tau_1^1 > n\right) + \mathsf{P}_{ik0}\left(\tau_1^1 \le n\right)\varepsilon/(1-\rho)\\ &\quad= \varepsilon/(1-\rho) + (1-\varepsilon/(1-\rho))\,\mathsf{P}_{ik0}\left(\tau_1^1 > n\right)\\ &\quad= \varepsilon/(1-\rho) + (1-\varepsilon/(1-\rho))T^{(n)}1_{ik}\\ &\quad\le \varepsilon/(1-\rho) + (1-\varepsilon/(1-\rho))\rho^n\end{aligned}$$

by Lemmas 1 and 2. The bound (5) for $i = k$ follows from (4). □

Proof of Corollary 2. Let μ be a probability measure on E. Consider the measure

$$\pi = \mu - \sum_{n\ge 0}\mu(I-P)P^{(n)}. \tag{57}$$

Since $\mu(I-P) \in l_1^0(E)$, relation (3) implies that the series in (57) converges in the full variation norm in view of

$$\left\|\mu(I-P)P^{(n)}\right\| \le \rho^n\left\|\mu(I-P)\right\|.$$

In particular, the partial sums converge, namely

$$\mu - \sum_{t<n}\mu(I-P)P^{(t)} = \mu P^{(n)} \to \pi, \qquad n\to\infty.$$

This implies the invariance $\pi = \pi P$. The contracting mapping theorem together with (3) implies that the measure π is unique. Taking into account $\varepsilon + \rho < 1$ and Remark 1, we similarly deduce that the chain X' is ergodic.

Finally, inequality (6) is proved by passing to the limit as $n\to\infty$ in the bound (4) of Theorem 1. □

Proof of Theorem 2. We use inequality (55) and equalities of Lemma 5 for $A_n = \Omega$ together with the inclusion $\{\overline{X}_s = (k,k,0)\} \subset \{X_s = k\}$:

$$\begin{aligned}
\left|\mathsf{P}'_\pi(X'_n \in B) - \pi(B)\right| &= \left|\sum_i \pi_i \big(\mathsf{P}'_i(X'_n \in B) - \mathsf{P}_i(X_n \in B)\big)\right| \\
&\le \sum_i \pi_i \left|\mathsf{P}'_i(X'_n \in B) - \mathsf{P}_i(X_n \in B)\right| \le \sum_i \pi_i\, \mathsf{P}_{ii1}(d_n = 0) \\
&= \sum_i \pi_i \sum_{1\le t\le n} \sum_k \mathsf{P}_{ii1}\left(\overline{X}_{t-1} = (k,k,0)\right) r_k^{(n-t)} \\
&\le \sum_i \pi_i \sum_{1\le t\le n} \sum_k \mathsf{P}_i(X_{t-1} = k) r_k^{(n-t)} = \sum_{1\le t\le n} \sum_k \pi_k r_k^{(n-t)} \\
&= \sum_{0\le s<n} \sum_k \pi_k \left(hT^{(s)}\right)_k \le \sum_{0\le s} \pi h T^{(s)} 1 = \varepsilon_T.
\end{aligned}$$

The first inequality in (10) follows in view of the ergodicity by passing to the limit in (8) as $n \to \infty$. The second inequality follows from (56). □

Proof of Remark 3. Let n be fixed. Consider the functions

$$f_i = \mathsf{P}_i(X_n \in B) = \left(P^{(n)} 1_B\right)_i, \qquad f'_i = \mathsf{P}'_i(X'_n \in B) = \left(P'^{(n)} 1_B\right)_i.$$

By definition,

$$\mathsf{P}_i(X_{n+t} \in B) = P^{(t)} f_i \quad \text{and} \quad \mathsf{P}'_i(X'_{n+t} \in B) = P'^{(t)} f'_i.$$

Note that

$$|Pg_i - P'g_i| \le \varepsilon \sup g$$

by condition (1) for nonnegative bounded functions g. Thus

$$P^{(t)} f - P'^{(t)} f' = P^{(t)}(f - f') + \sum_{0\le s<t} P^{(t-1-s)}(P - P')P'^{(s)} f'$$

implies that

$$\left|P^{(t)} f - P'^{(t)} f'\right| \le \sup_i |f_i - f'_i| + \sum_{0\le s<t} \varepsilon \le \sup_i |f_i - f'_i| + (m-1)\varepsilon$$

for $t < m$, $t + n = 0 \mod m$, since the matrices P and P' are stochastic.

Inequalities (4) and (6) hold for $t + n = 0 \mod m$, whence we derive the result.

The inequality $r(P^{(m)}, P'^{(m)}) \le m\, r(P, P')$ follows from the above equality after the substitution $f = f' = 1_B$ and $t = m$. □

Proof of Example 2. We evaluate the number defined by (2):

$$\begin{aligned}
\rho(P, P') = \sup_{i\ne k} \sum_j \left|P_{ij} - P'_{kj}\right|/2 &= \sup_{i\ne k}\left(1 - \sum_j \min(P_{ij}, P'_{kj})\right) \\
&\le 1 - \inf_{i\ne k} \min(P_{io}, P'_{ko}) \le 1 - d = \rho < 1.
\end{aligned}$$

The rest of the proof follows from (4), (5), and Remark 3. □

In what follows, the symbol $\mathsf{E}(\xi, A)$ stands for $\mathsf{E}(\xi 1_A)$.

Proof of Theorem 3. Put

$$W(j,l) \equiv V_j + V_l, \qquad j,l \in E,$$

and let $|f_i| \le V_i$, $i \in E$.

Consider the sum of the values of the function $f = (f_i)$ with respect to the measures of the sets $B \subset E$ in equality (52) of Lemma 4. We obtain

$$\mathsf{E}_{ii1}(f(X_n), d_n = 1) = \mathsf{E}_{ii1}(f(X'_n), d_n = 1), \tag{58}$$

whence

$$\begin{aligned}\left|\mathsf{E}_i\, f(X_n) - \mathsf{E}'_i\, f(X'_n)\right| &= \left|\mathsf{E}_{ii1}\, f(X_n) - \mathsf{E}_{ii1}\, f(X'_n)\right| \\ &= \left|\mathsf{E}_{ii1}(f(X_n), d_n = 0) - \mathsf{E}_{ii1}(f(X'_n), d_n = 0)\right| \\ &\le \mathsf{E}_{ii1}(W(X_n, X'_n), d_n = 0).\end{aligned} \tag{59}$$

The finiteness of the right hand side proved below implies the same property for the left hand side.

Consider the sum of the values of the function $W(j,l)$ with respect to the measures of the sets $B \times B' \subset E \times E$ in equality (53) of Lemma 5. Taking into account the definition of (54) we obtain

$$\begin{aligned}\mathsf{E}_{ii1}(W(X_n, X'_n), d_n = 0) &= \sum_{1\le t\le n} \sum_k \mathsf{P}_{ii1}\left(\overline{X}_{t-1} = (k,k,1)\right) \left(hT^{(n-t)}W\right)_k \\ &\le \sum_{1\le t\le n} \sum_k \mathsf{P}_{ii1}(X_{t-1} = k) \left(hT^{(n-t)}W\right)_k \\ &\le \sum_{1\le t\le n} \sum_k \mathsf{P}_i(X_{t-1} = k) V_k C^{(n-t)} \\ &= \sum_{1\le t\le n} \mathsf{E}_i\, V(X_{t-1}) C^{(n-t)} \le K_i^{(n)} \sum_{s<n} C^{(s)},\end{aligned} \tag{60}$$

where

$$\begin{aligned}C^{(s)} &\equiv \sup_k V_k^{-1} \left(hT^{(s)}W\right)_k \\ &= \sup_k V_k^{-1} \sum_{j\ne l} h_{k,jl} \left(T^{(s)}W\right)_{jl} = \sup_k V_k^{-1} \sum_{j\ne l} h_{k,jl} W_{jl} W_{jl}^{-1} \left(T^{(s)}W\right)_{jl} \\ &\le \sup_k V_k^{-1} (hW)_k \sup_{j\ne l} W_{jl}^{-1} \left(T^{(s)}W\right)_{jl}.\end{aligned} \tag{61}$$

In view of the definition of (30) and Lemma 1,

$$\begin{aligned}V_k^{-1}(hW)_k &= V_k^{-1} \sum_{j\ne l} h_{k,jl}(V_j + V_l) = V_k^{-1} \sum_{j\ne l} R_{kj} R'_{kl} (V_j + V_l)/(1 - q_k) \\ &= V_k^{-1} \left(\sum_j R_{kj} V_j + \sum_l R'_{kl} V_l\right) = V_k^{-1} \sum_j \left|P_{kj} - P'_{kj}\right| V_j \le \varepsilon_V\end{aligned} \tag{62}$$

by condition (12).

Further, Lemma 1 and the definitions of (34) and (35) imply that

$$\begin{aligned}TW_{ik} &= \sum_{j\ne l} T_{ik,jl}(V_j + V_l) = \sum_j \sum_l T_{ik,jl} V_j + \sum_l \sum_j T_{ik,jl} V_l \\ &= \sum_j (P_{ij} - g_{ik,j}) V_j + \sum_l (P'_{kl} - g_{ik,l}) V_l \\ &= \sum_j (P_{ij} - P'_{kj})^+ V_j + \sum_j (P'_{kj} - P_{ij})^+ V_j = \sum_j \left|P_{ij} - P'_{kj}\right| V_j \\ &\le \rho_V (V_i + V_k) = \rho_V W_{ik}\end{aligned} \tag{63}$$

if condition (13) holds. Using the induction, we derive from here that

$$T^{(s)}W_{ik} \le \rho_V^s W_{ik}, \qquad s \ge 1,\ i \ne k \in E. \tag{64}$$

Substituting this inequality into (60) and (61) and applying (62) we conclude that

$$\mathsf{E}_{ii1}(W(X_n, X_n'), d_n = 0) \le K_i^{(n)} \sum_{0 \le s < n} \varepsilon_V \rho_V^s,$$

which proves (14). □

Proof of Example 3. The uniform mixing condition fails in this case, since both the first and last rows of the matrix P are singular. Then system (13) can be rewritten in the following form:

$$\begin{gathered} \alpha + 1 - \beta + \gamma v < 1 + 1, \\ 1 + 1 < 1 + v, \\ 1 - \alpha + \beta + \gamma v < 1 + v. \end{gathered}$$

This system has a solution $v \in (1, 1 + 2\beta/\gamma)$. □

Proof of Corollary 3. We check the strong mixing condition (13) for the case of

$$\rho_V = \rho_O + \varepsilon_V.$$

Let $i \notin O$ and $k \notin O$, $i \ne k$. Then (16) implies that

$$\begin{aligned} \sum_j |P_{ij} - P_{kj}|\, V_j &\le \sum_{j \in O} |P_{ij} - P_{kj}|\, V_j + \sum_{j \notin O} (P_{ij} + P_{kj}) V_j \\ &\le \sum_{j \in O} |P_{ij} - P_{kj}|\, V_j + \rho_O V_i - \sum_{j \in O} P_{ij} V_j + \rho_O V_k - \sum_{j \in O} P_{kj} V_j \\ &\le \rho_O (V_i + V_k). \end{aligned} \tag{65}$$

Assume that $i \ne k$ for $k \in O$. Then inequality (65) is equivalent to condition (17). The case $i \ne k$ for $i \in O$ is similar.

Thus inequalities of the form (65) and (12) proved above imply that, for all $i \ne k$,

$$\begin{aligned} \sum_j \left| P_{ij} - P'_{kj} \right| V_j &\le \sum_j |P_{ij} - P_{kj}|\, V_j + \varepsilon_V V_k \\ &\le \rho_O (V_i + V_k) + \varepsilon_V V_k < (\rho_O + \varepsilon_V)(V_i + V_k). \end{aligned}$$

Therefore (18) follows from (14).

To derive (19) from (15) put

$$k_i^{(n)} \equiv \mathsf{E}_i\, V(X_n), \qquad K_O \equiv \sup_{i \in O} PV_i.$$

Note that $\mathsf{E}_i\, V(X_1) \le \rho_O V_i$ for $i \notin O$ in view of (16).

Using the Markov property of the chain X,

$$\begin{aligned} k_i^{(n)} &= \mathsf{E}_i \left(1_{\{X_{n-1} \in O\}}\, \mathsf{E}_{X_{n-1}} V(X_1) + 1_{\{X_{n-1} \notin O\}}\, \mathsf{E}_{X_{n-1}} V(X_1) \right) \\ &\le \mathsf{E}_i (K_O + \rho_O V(X_{n-1})) = K_O + \rho_O k_i^{(n-1)}, \qquad n \ge 1. \end{aligned}$$

By induction, this implies that

$$k_i^{(n)} \le K_O \left(1 - \rho_O^n\right) / (1 - \rho_O) + \rho_O^n V_i, \qquad n \ge 0.$$

Since the right hand side is monotone with respect to n,

$$K_i^{(n)} \le \sup_{m\ge 0} k_i^{(m)} \le \max\left(k_i^{(0)}, k_i^{(\infty)}\right) = \max(V_i, K_O/(1-\rho_O)),$$

and this completes the proof of (19). □

Proof of Example 4. The equivalence of (21) and the condition imposed on the mean increment (22) is proved in [4, 1977] under the assumption that the exponential moment of a jump is bounded and by using the method based on finding power test functions. We apply this method below to check (16).

To prove condition (16) of Corollary 3 note that the function in (21) is convex in the interval $(0, \ln\beta_0)$ and equals 1 if $\beta_0 = 1$. Thus the inequality

$$\sum_j P_{ij} v^{j-i} \le \rho < 1, \qquad i \ne 0,$$

holds for all $v \in (1, \beta_0]$ and some $\rho < 1$. Thus

$$\rho v^i \ge \sum_j P_{ij} v^j = P_{i0} + \sum_{j\ne 0} P_{ij} v^j$$

and this proves (16). The value $V_0 = 1$ is chosen according to the definition.

Now we prove that bounds (24) hold for all sufficiently small numbers $v - 1 > 0$ and for some $\rho_V < 1$.

Define the functions

$$R_i(v) \equiv \left(\left(1 + v^i\right)^{-1} \sum (P_{ij} + P_{0j}) v^j - 1\right) \Big/ (v-1)$$

and

$$S_i(v) \equiv \sum_j P_{ij}\left(v^{j-i} - 1\right)/(v-1) \tag{66}$$

for $i \in E$ and $v \in (1, \beta_0]$.

Note that

$$S_i(v) - \Delta_i = o(1), \qquad v \downarrow 1, \tag{67}$$

uniformly with respect to i under the majorization condition (20), where Δ_i is defined by equality (22).

Further, the definition of (66) implies that

$$\begin{aligned}\left(1+v^i\right) R_i(v) - \left(v^i \Delta_i + \Delta_0\right) &= \sum_j \left(P_{ij}\left(v^j - v^i\right) + P_{0j}\left(v^j - 1\right)\right) - \left(v^i \Delta_i + \Delta_0\right)\\ &= v^i (S_i - \Delta_i) + S_0 - \Delta_0,\end{aligned}$$

whence

$$R_i(v) - (\Delta_i + \Delta_0)/2 = o(1), \qquad v \downarrow 1, \tag{68}$$

uniformly with respect to $i \ne 0$ in view of (67).

Denote by $-2d < 0$ the left hand side of (23).

(a) Let $i \ne 0$ be a state such that the minimum in (23) is attained at the first index: $\Delta_i + \Delta_0 \le -2d$. According to (66),

$$\begin{aligned}\rho_i(v) &\le (1+v^i)^{-1} \sum_j (P_{ij} + P_{0j}) v^j = 1 + (v-1) R_i(v)\\ &= 1 + (v-1)((\Delta_i + \Delta_0)/2 + o(1))\\ &\le 1 - (v-1)d + o(v-1) \le 1 - (v-1)d/2 = \rho_v < 1,\end{aligned}$$

starting with a certain $v-1>0$ and for all $i \neq 0$, since $o(1)$ in (68) is uniform with respect to i.

(b) Let $i \neq 0$ and let the inequality in (23) hold due to the second index under the sign of minimum, that is, $\sum_j |P_{ij}-P_{0j}|-2 \leq -2d$. The latter condition is equivalent to the inequality

$$\sum_j \min(P_{ij}, P_{0j}) \geq d.$$

If the constant N is such that $\sum_{j>N} P_{0j} < d/2$, then

$$\sum_{j\leq N} P_{ij} \geq \sum_{j\leq N} \min(P_{ij}, P_{0j}) \geq d/2.$$

This inequality for a fixed N contradicts condition (20) that the sums

$$\sum_{j<i} (i-j)P_{ij} \geq (i-N)d/2 \to \infty$$

are bounded as $i \to \infty$ along a subsequence of states i. Hence $i \leq m$ for some fixed m if the second inequality in (23) holds.

Therefore

$$\begin{aligned}
\rho_i(v) &= \left(1+v^i\right)^{-1} \left(\sum_j |P_{ij}-P_{0j}| + \sum_j |P_{ij}-P_{0j}| \left(v^j-1\right)\right) \\
&\leq \left(1+v^i\right)^{-1} \left(2-2d+\sum_j (P_{ij}+P_{0j}) \left(v^j-1\right)\right) \\
&= \left(1+v^i\right)^{-1} \left(2-2d-2+\left(1+v^i\right)(1+(v-1)R_i(v))\right) \\
&= 1-2d\left(1+v^i\right)^{-1} + (v-1)R_i(v) = 1-d+o(1), \qquad v \downarrow 1,
\end{aligned}$$

uniformly with respect to $i \leq m$ by relations (66) and (68). This means that $\rho_i(v) \leq 1-d/2 = \rho_v$, starting with some sufficiently small $v-1>0$ and for all $i \leq m$. □

Proof of Theorem 4. Consider the stopping time $\theta_1 \equiv \min(\theta, \tau_1^0)$ for the chain $\overline{X}$. Let

$$m \equiv \sup(n \geq 1\colon\ \mathsf{P}_{ii1}(\theta_1 > n) > 0) + 1 \leq \infty.$$

Taking into account the extended Markov property of $\overline{X}$ and the inequality $h1_i \leq \varepsilon 1_i$ of Lemma 1 we obtain

$$\begin{aligned}
\mathsf{P}_{ii1}\left(\theta \geq \tau_1^0 = n\right) &= \mathsf{P}_{ii1}\left(\theta_1 > n-1, \tau_1^0 = n\right) \\
&= \sum_j \mathsf{P}_{ii1}\left(\theta_1 > n-1, \tau_1^0 = n, \overline{X}_{n-1} = (j,j,1)\right) \\
&= \sum_j \mathsf{P}_{ii1}\left(\theta_1 > n-1, \overline{X}_{n-1} = (j,j,1)\right) \\
&\qquad \times \mathsf{P}\left(\tau_1^0 = n \;\middle|\; \theta_1 > n-1, \overline{X}_{n-1} = (j,j,1)\right) \\
&= \sum_j \mathsf{P}_{ii1}\left(\theta_1 > n-1, \overline{X}_{n-1} = (j,j,1)\right) \mathsf{P}_{jj1}\left(\tau_1^0 = 1\right) \\
&\leq \sum_j \mathsf{P}_{ii1}\left(\theta_1 > n-1, \overline{X}_{n-1} = (j,j,1)\right) \varepsilon = \varepsilon\, \mathsf{P}_{ii1}(\theta_1 > n-1)
\end{aligned} \tag{69}$$

for $n \geq 1$, whence

$$\mathsf{P}_{ii1}\left(\theta \geq \tau_1^0\right) = \sum_{1\leq n<m} \mathsf{P}_{ii1}\left(\theta \geq \tau_1^0 = n\right) \leq \varepsilon \sum_{1\leq n<m} \mathsf{P}_{ii1}(\theta_1 > n-1) = \varepsilon\, \mathsf{E}_{ii1}\, \theta_1. \tag{70}$$

Since $\{\theta' \geq \tau_1^0\} = \{\theta \geq \tau_1^0\}$ in view of the equality $\theta = \theta'$ being true in the set

$$\{\theta' < \tau_1^0\} \cap \{\theta < \tau_1^0\},$$

we get

$$\mathsf{P}_{ii1}\left(\theta' \geq \tau_1^0\right) = \mathsf{P}_{ii1}\left(\theta \geq \tau_1^0\right) \leq \varepsilon\, \mathsf{E}_{ii1}\, \theta_1. \tag{71}$$

If φ is a nonnegative and $(\mathfrak{F}_t)$-measurable random variable, then we use definition (25) and prove that the expectations are the same:

$$\begin{aligned}\mathsf{E}_{ii1}\left(\varphi 1_{\{\theta=n\}}, n < \tau_1^0\right) &= \mathsf{E}_{ii1}\left(\varphi_n(X_1,\dots,X_n), n < \tau_1^0\right) = \mathsf{E}_{ii1}\left(\varphi_n(X_1',\dots,X_n'), n < \tau_1^0\right) \\ &= \mathsf{E}_{ii1}\left(\varphi' 1_{\{\theta'=n\}}, n < \tau_1^0\right),\end{aligned}$$

since $(X_1,\dots,X_n) = (X_1',\dots,X_n')$ for $n < \tau_1^0$.

Then

$$\begin{aligned}\mathsf{E}_{ii1}\left(\varphi, \theta < \tau_1^0\right) &= \sum_{n\geq 1} \mathsf{E}_{ii1}\left(\varphi 1_{\{\theta=n\}}, n < \tau_1^0\right) = \sum_{n\geq 1} \mathsf{E}_{ii1}\left(\varphi' 1_{\{\theta'=n\}}, n < \tau_1^0\right) \\ &= \mathsf{E}_{ii1}\left(\varphi', \theta' < \tau_1^0\right).\end{aligned} \tag{72}$$

By Hölder inequality,

$$\begin{aligned}\mathsf{E}_i\, \varphi - \mathsf{E}_i'\, \varphi' &= \mathsf{E}_{ii1}\, \varphi - \mathsf{E}_{ii1}\, \varphi' = \mathsf{E}_{ii1}\left(\varphi, \theta \geq \tau_1^0\right) - \mathsf{E}_{ii1}\left(\varphi', \theta' \geq \tau_1^0\right) \\ &\leq \mathsf{E}_{ii1}\left(\varphi 1_{\{\theta \geq \tau_1^0\}}\right) \leq (\mathsf{E}_{ii1}\, \varphi^\alpha)^{1/\alpha} \left(\mathsf{E}_{ii1}\, 1^\beta_{\{\theta\geq\tau_1^0\}}\right)^{1/\beta} \\ &\leq K_i^\alpha(\varphi)\left(\mathsf{P}_{ii1}\left(\theta \geq \tau_1^0\right)\right)^{1/\beta}.\end{aligned} \tag{73}$$

A lower bound is derived similarly:

$$\mathsf{E}_i\, \varphi - \mathsf{E}_i'\, \varphi' \geq -K_i^\alpha(\varphi)\left(\mathsf{P}_{ii1}\left(\theta' \geq \tau_1^0\right)\right)^{1/\beta}. \tag{74}$$

Finally, we apply bound (70) in (73) and bound (71) in (74) and derive the two sided inequalities (26) in view of the inequality $\mathsf{E}_{ii1}\, \theta_1 \leq \mathsf{E}_{ii1}\, \theta = \mathsf{E}_i\, \theta$. □

Proof of Corollary 4. Substituting $\varphi = 1_A$ and $\varphi' = 1_{A'}$ to (26), we get $K_i^\alpha(\varphi) \leq 1$; thus (27) follows from (4) by passing to the limit as $\beta \downarrow 1$. □

Proof of Example 5. Let $A \in \mathfrak{F}_\infty$, that is, $A = \{X_\infty \in B\}$, where

$$X_\infty = (X_n, n \geq 1) \in E^\infty$$

is a trajectory of the chain X and $B \subset E^\infty$.

Let $\pi_n(x) = (x_1,\dots,x_n)$ be the projection for a sequence $x = (x_n, n \geq 1)$.

Consider the sets $B_n = \bigcup_{x\in B}\{\pi_n(x)\}$ and

$$B^n = B_n \times \{(o,o,\dots)\} = \{(x_1,\dots,x_n,o,o,\dots), (x_1,\dots,x_n) \in B_n\}.$$

Put $C_n = \bigcap_{t>n}\{X_t = o\}$.

We also introduce the random event

$$\widetilde{A} = \bigcup_{n\geq 1}\{\theta = n\} \cap \{(X_1,\dots,X_n) \in B_n\}.$$

By definition, $\widetilde{A} \in \mathfrak{F}_\theta$. Since $\mathsf{P}_i(\theta = n, \overline{C}_n) = 0$ by the almost sure absorption property, we obtain

$$\begin{aligned}\mathsf{P}_i(\theta = n, A\Delta\widetilde{A}) &= \mathsf{P}_i(\theta = n, C_n, A\Delta\widetilde{A}) \\ &= \mathsf{P}_i(\theta = n, C_n, \{X_\infty \in B\}\Delta\{(X_1,\dots,X_n) \in B_n\}) \\ &= \mathsf{P}_i(\theta = n, C_n, X_\infty \in B\Delta B^n) = 0,\end{aligned}$$

since

$$\mathsf{P}_i(\theta = n, C_n, X_\infty \in B) = \mathsf{P}_i(\theta = n, C_n, X_\infty \in B^n)$$

by the definition of the sets B_n, B^n, and C_n.

Since $\theta < \infty$ almost surely, we conclude that $\mathsf{P}_i(A\Delta\widetilde{A}) = 0$.

Applying Theorem 4 to the random event $\widetilde{A} \in \mathfrak{F}_\theta$, we complete the proof of the statement of Example 5. □

Proof of Remark 7. The Markov property of $\widetilde{X}$ follows from (43), since

$$\overline{X}_n = (X_n, X_n', d_n)$$

is a Markov chain and $d_n = 1_{\{X_n = X_n'\}}$ almost surely. Inequality (44) follows from the Monge theorem [12], [13] by using the definitions of (28) and (34) for transition probabilities of the chain $\widetilde{X}$ that include the minimum of two possible marginal distributions in each of the only two possible cases, namely $i = k$ and $i \neq k$. □

Bibliography

1. W. Doeblin, *Expose de la theorie des chaines simples constantes de Markov a un nomber fini d'estats*, Mathematique de l'Union Interbalkanique **2** (1938), 77–105.
2. W. Feller, *An Intoduction to Probability Theory and its Applications*, vol. 1, John Wiley & Sons, New York, 1966. MR0210154 (35:1048)
3. N. V. Kartashov, *Strong Stable Markov Chains*, VSP/TViMS, Utrecht/Kiev, The Netherlands/Ukraine, 1996. MR1451375 (99e:60150)
4. N. V. Kartashov, *Exponential asymptotics of the matrix of a Markov renewal*, Asymptotic Problems for Stochastic Processes, Preprint # 77-24, Institute of Mathematics, Academy of Science of Ukraine, Kiev, 1977, pp. 2–43. (Russian)
5. E. Nummelin, *A splitting technique for Harris recurrent chains*, Z. Wahrscheinlichkeitstheorie verw. Geb. **43** (1978), 309–318. MR0501353 (58:18732)
6. E. Nummelin and R. L. Tweedie, *Geometric ergodicity and R-positivity for general Markov chains*, Ann. Probab. **6** (1978), 404–420. MR0474504 (57:14143)
7. T. Lindvall, *On coupling of discrete renewal sequences*, Z. Wahrscheinlichkeitstheorie verw. Geb. **48** (1979), 57–70. MR533006 (80g:60091)
8. I. N. Kovalenko and N. Yu. Kuznetsov, *The construction of the embedded renewal process for essentially multidimensional queueing processes and its application for deriving the limit theorems*, Preprint # 80–12, Institute of Cybernetics, Academy of Science of Ukraine, Kiev, 1980. (Russian) MR612478 (82i:60142)
9. P. Ney, *A refinement of the coupling method in renewal theory*, Stochastic Processes Appl. **11** (1981), 11–26. MR608004 (82d:60169)
10. E. Numemelin and P. Tuominen, *Geometric ergodicity of Harris recurrent Markov chains with applicatoins to renewal theory*, Stoch. Proc. Appls. **12** (1982), 187–202. MR651903 (83f:60089)
11. E. Nummelin, *General Irreducible Markov Chains and Nonnegative Operators*, Cambridge University Press, Cambridge, 1984. MR776608 (87a:60074)
12. V. M. Zolotarev, *Modern Theory of Summation of Independent Random Variables*, "Nauka", Moscow, 1986; English transl., VSP, Utrecht, the Netherlands–Tokyo, Japan, 1997.
13. S. T. Rachev, *The Monge–Kantorovich mass transference problem and its stochastic applications*, Teor. Veroyatnost. Primenen, **29** (1984), no. 4, 625–653; English transl. in Theory Probab. Appl. **29** (1984), no. 4, 647–676. MR773434 (86m:60026)
14. T. Lindvall, *Lectures on the Coupling Method*, John Wiley & Sons, New York, 1992. MR1180522 (94c:60002)
15. S. P. Mayn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag London, Ltd., London, 1993. MR1287609 (95j:60103)
16. P. Tuominen and R. Tweedie, *Subgeometric rates of convergence of f-ergodic Markov chains*, Adv. Appl. Probab. **26** (1994), 775–798. MR1285459 (95m:60097)
17. R. L. Tweedie and J. N. Corcoran, *Perfect sampling of ergodic Harris chains*, Ann. Appl. Probab. **11** (2001), no. 2, 438–451. MR1843053 (2002g:60111)
18. H. Thorisson, *Coupling, Stationarity, and Regeneration*, Springer, New York, 2000. MR1741181 (2001b:60003)

19. S. F. Jarner and G. O. Roberts, *Polynomial convergence rates of Markov chains*, Ann. Appl. Probab. **12** (2001), 224–247. MR1890063 (2003c:60117)
20. R. Douc, E. Moulines, and J. S. Rosenthal, *Quantitative bounds for geometric convergence rates of Markov chains*, Ann. Appl. Probab. **14** (2004), 1643–1664. MR2099647 (2005i:60146)
21. R. Douc, E. Moulines, and J. S. Rothenthal, *Quantitative bounds on convergence of time-inhomogeneous Markov chains*, Ann. Appl. Probab. **14** (2004), no. 4, 1643–1665. MR2099647 (2005i:60146)
22. R. Douc, E. Moulines, and P. Soulier, *Practical drift conditions for subgeometric rates of convergence*, Ann. Appl. Probab. **14** (2004), no. 4, 1353–1377. MR2071426 (2005e:60156)
23. R. Douc, E. Moulines, and P. Soulier, *Computable convergence rates for subgeometrically ergodic Markov Chains*, Bernoulli **13** (2007), no. 3, 831–848. MR2348753 (2008j:60172)
24. R. Douc, G. Fort, and A. Guillin, *Subgeometric rates of convergence of f-ergodic strong Markov processes*, Stoch. Process. Appl. **119** (2009), no. 3, 897–923. MR2499863 (2010j:60184)
25. V. V. Golomozyĭ, *Stability of non-homogeneous Markov chains*, Visnyk Kyiv Univ., ser. fiz. mat. nauk **4** (2009), 10–15. (Ukrainian)
26. V. V. Golomozyĭ, *A subgeometric estimate for the stability of time-homogeneous Markov chains*, Teor. Imovir. Mat. Stat. **81** (2010), 31–45; English transl. in Theory Probab. Math. Statist. **81** (2010), 35–50. MR2667308 (2011c:60232)
27. M. V. Kartashov, *Boundedness, limits, and stability of solutions of an inhomogeneous perturbation of a renewal equation on a half-line*, Teor. Imovir. Mat. Stat. **81** (2009), 65–75; English transl. in Theory Probab. Math. Statist. **81** (2010), 71–83. MR2667311 (2011f:60154)
28. M. V. Kartashov and V. V. Golomozyĭ, *The mean coupling time of independent discrete renewal processes*, Teor. Imovir. Mat. Stat. **84** (2011), 77–83; English transl. in Theory Probab. Math. Statist. **84** (2012), 79–86. MR2857418 (2012f:60306)
29. M. V. Kartashov and V. V. Golomozyĭ, *Maximal coupling and stability of discrete Markov chains. I*, Teor. Imovir. Mat. Stat. **86** (2011), 81–91. MR2986452

Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue, 4E, Kiev 03127, Ukraine

E-mail address: nkartashov@skif.com.ua

Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue, 4E, Kiev 03127, Ukraine

Received 07/OCT/2011

Translated by S. KVASKO