

MAXIMAL COUPLING PROCEDURE AND STABILITY OF DISCRETE MARKOV CHAINS. II

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ABSTRACT. Two discrete Markov chains whose one-step transition probabilities are close to each other in the uniform total variation norm or in the V -norm are considered. The problem of stability of the transition probabilities over an arbitrary number of steps is investigated. The main assumption is that either the uniform mixing or V -mixing condition holds. In particular, we prove that the uniform distance between the distributions of the chains after an arbitrary number of steps does not exceed $\varepsilon/(1-\rho)$, where ε is the uniform distance between the transition matrices and where ρ is the uniform mixing coefficient. A number of general examples are considered. The proofs are based on the maximal coupling procedure that maximizes the one-step coupling probabilities.

This is the second part of our paper, and it contains the proofs of the results stated in its first part [29]. We continue the numbering of the formulas after the first part of the paper.

4. PROOFS

Recall that if $Q = (Q_{ij})$ is a matrix, $f = (f_j)$ is a function, and $\mu = (\mu_i)$ is a measure, then Qf denotes the function with the coordinates

$$Qf_i = \sum_j Q_{ij} f_j,$$

while the symbol μQ stands for the measure with the coordinates $\mu Q_j = \sum_i \mu_i Q_{ij}$. The product PQ of matrices P and Q and a power $Q^{(n)}$ of a matrix Q are defined accordingly. The symbol $Q_i = (Q_{ij}, j \in E)$ denotes the i -th row of a matrix Q . If f is a function, then we write either f_i or $f(i)$ for its coordinates. The unit matrix is denoted by I , while 1 means the function assuming only one value 1 and the corresponding set of indices is determined by the co-dimension of a matrix by which a function is multiplied.

If the set for an index of summation is not specified, then we assume that the index runs through the whole set E . The same rule applies when evaluating the limit superior. Finally, x^\pm stands for the positive and negative parts of a number x and $\delta_{ij} = 1_{i=j}$ means the Kronecker symbol.

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Lemma 1. *Using the notation introduced in [29],*

$$(45) \quad R1_i = R'1_i = h1_i = 1 - q_i, \quad r(P, P') = \sup_i (1 - q_i),$$

$$(46) \quad S1_{ik} = S'1_{ik} = T1_{ik} = 1 - g1_{ik} = 1 - q_{ik}, \quad \rho(P, P') = \sup_{i \neq k} (1 - q_{ik}).$$

In particular, if assumptions (1) and (2) hold, then

$$R1_i = R'1_i = h1_i \leq r(P, P') \leq \varepsilon,$$

$$S1_{ik} = S'1_{ik} = T1_{ik} = 1 - g1_{ik} \leq \rho(P, P') \leq \rho,$$

and

$$(47) \quad T^{(s)}1_{ik} \leq \rho^s \rightarrow 0, \quad s \rightarrow \infty.$$

Proof. We derive from (29) that

$$R1_i = (P - Q)1_i = 1 - \sum_j Q_{ij} = 1 - q_i, \quad R'1_i = 1 - q_i,$$

and

$$h1_i = \sum_{j,l} R_{ij} R'_{il} / (1 - q_i) = \sum_j R_{ij} \sum_l R'_{il} / (1 - q_i) = 1 - q_i.$$

Then

$$r(P, P') = \sup_i \sum_j |P_{ij} - P'_{ij}| / 2 = \sup_i \sum_j (R_{ij} + R'_{ij}) / 2 = \sup_i (1 - q_i),$$

which proves (45).

Similarly, we obtain from (34) and (35) that (46) holds and

$$S1_{ik} = (P - g)1_{ik} = 1 - g1_{ik} = 1 - q_{ik}, \quad S'1_{ik} = 1 - q_{ik},$$

$$T1_{ik} = \sum_{j,l} S_{ik,j} S'_{ik,l} / (1 - q_{ik}) = \sum_j S_{ik,j} \sum_l S'_{ik,l} / (1 - q_{ik}) = 1 - q_{ik},$$

$$\rho(P, P') = \sup_{i \neq k} \sum_j |P_{ij} - P'_{kj}| / 2 = \sup_{i \neq k} \sum_j (S_{ik,j} + S'_{ik,j}) / 2 = \sup_{i \neq k} (1 - q_{ik}).$$

Applying assumptions (1) and (2) to equalities (45) and (46) we prove the first two inequalities in (47).

Using the induction, the second inequality in (47) implies that

$$T^{(s)}1_{ik} = \left(T^{(s-1)}(T1) \right)_{ik} \leq \left(T^{(s-1)}(\rho 1) \right)_{ik} = \rho T^{(s-1)}1_{ik} \leq \dots \leq \rho^s 1,$$

since the matrix T is nonnegative. □

We agree that $\inf(\emptyset) = \infty$ in what follows.

Definition. *The switching moments for the coordinate d_n of the coupling process \overline{X} are defined by the following recurrence relation:*

$$(48) \quad \tau_0^{d_0} = 0, \quad \tau_m^d = \inf (n > \tau_{m-1}^{1-d} : d_n = d), \quad d = \text{mod}(m - d_0, 2), \quad m \geq 1.$$

If $d_0 = 1$, then we deal with a sequence of a Markov moment,

$$0 = \tau_0^1 < \tau_1^0 < \tau_2^1 < \tau_3^0 < \dots \leq \infty.$$

If $d_0 = 0$, then

$$0 = \tau_0^0 < \tau_1^1 < \tau_2^0 < \tau_3^1 < \dots \leq \infty.$$

Lemma 2. *Using the notation introduced in [29],*

$$(49) \quad \begin{aligned} \mathbb{P}_{ii1}(\tau_1^0 > t, \bar{X}_t = (k, k, 1)) &= Q_{ik}^{(t)}, & t \geq 1, \\ \mathbb{P}_{ii1}(\tau_1^0 = t, \bar{X}_t = (j, l, 0)) &= Q^{(t-1)} h_{i,jl}, & t \geq 1, \\ \mathbb{P}_{ik0}(\tau_1^1 > t, \bar{X}_t = (j, l, 0)) &= T_{ik,jl}^{(t)}, & t \geq 1, \end{aligned}$$

and

$$(50) \quad \mathbb{P}_{ik0}(\tau_1^1 = t, \bar{X}_t = (j, j, 1)) = T^{(t-1)} g_{ik,j}, \quad t \geq 1.$$

Proof. The proof follows from the Markov property of the chain \bar{X} , since the transition probability over several steps is equal to the product of one-step transition probabilities. According to (31), (32), (36), and (37) the latter probabilities for the coordinate d_n are given by the matrix Q for the transition from 1 to 1, by matrix h for the transition from 1 to 0, by matrix T for the transition from 0 to 0, and by matrix g for the transition from 0 to 1. \square

Lemma 3. *Assume that condition (7) holds. Then $\mathbb{P}_{ik0}(\tau_1^1 < \infty) = 1$ for all $i \neq k$.*

Proof. We deduce from Lemma 2 that the function

$$\varphi_{ik} = \mathbb{P}_{ik0}(\tau_1^1 < \infty) = \sum_{t \geq 1} \mathbb{P}_{ik0}(\tau_1^1 = t) = \sum_{t \geq 1} \sum_j T^{(t-1)} g_{ik,j} = \sum_{s \geq 0} T^{(s)} g_{1ik}$$

is the minimal solution of the equation $\varphi_{ik} = g_{1ik} + T\varphi_{ik}$. By Lemma 1, the function 1 satisfies this equation. Then (7) together with Lemma 1 of [27, 2009] implies that the above function is the minimal solution. \square

Definition. The number of complete coupling–decoupling cycles for the initial condition $d_0 = 1$ is defined by

$$(51) \quad \begin{aligned} \nu_n &= \sup\{m \geq 0: \tau_{2m}^1 \leq n\} < \infty, & n \geq 1, \\ \{\nu_n = m\} &= \{\tau_{2m}^1 \leq n < \tau_{2m+2}^1\}. \end{aligned}$$

This definition is correct, since $\tau_0^1 = 0$ and $\tau_m^d \geq m$.

Remark 8. In the general case, the random events $\{\tau_{2m}^1 = \infty\}$ can have positive probabilities. Note however that if $d_0 = 1$, then condition (7) implies that

$$\tau_{2m}^1 - \tau_{2m-1}^0 < \infty$$

almost surely. Hence if τ_{2m-1}^0 is finite almost surely, then τ_{2m}^1 is finite almost surely as well. Therefore only one sequence may have a positive probability, namely the sequence whose members d_n equal each other, that is, $d_n = 1$ for $n \geq \tau_{2\mu}^1$ on the event $\{\mu < \infty\}$, where $\mu = \sup_{n \geq 0} \nu_n$.

Lemma 4. *For all $B \subset E$ and $n \geq 1$, we have*

$$(52) \quad \mathbb{P}_{ii1}(X_n \in B, d_n = 1) = \mathbb{P}_{ii1}(X'_n \in B, d_n = 1).$$

Proof. Consider the random events

$$C_{sk}^{nm} = \{\nu_n = m, \tau_{2m}^1 = s, \bar{X}_s = (k, k, 1)\}$$

being disjoint for different $m \geq 0$, $s \geq 0$, and $k \in E$. Since

$$\{d_n = 1\} = \bigcup_{m \geq 0} \{\nu_n = m, \tau_{2m}^1 < \infty, d_n = 1\},$$

we get

$$\begin{aligned} \mathbb{P}_{ii1}(X_n \in B, d_n = 1) &= \sum_{m \geq 0} \sum_{s \leq n} \sum_k \mathbb{P}_{ii1}(C_{sk}^{nm}, X_n \in B, d_n = 1) \\ &= \sum_{m \geq 0} \sum_{s \leq n} \sum_k \mathbb{P}_{ii1}(C_{sk}^{sm}, X_n \in B, \tau_{2m}^1 \leq n < \tau_{2m+1}^0) \end{aligned}$$

because $\{\nu_n = m\} = \{\nu_s = m\}$ on the random event $\{d_n = 1, \tau_{2m}^1 = s\}$ for $s \leq n$ and because

$$C_{sk}^{nm} \cap \{d_n = 1\} = C_{sk}^{nm} \cap \{\tau_{2m}^1 \leq n < \tau_{2m+1}^0\}.$$

Taking into account the strong Markov property of the chain \bar{X} , we conclude that

$$\begin{aligned} \mathbb{P}_{ii1}(X_n \in B, d_n = 1) &= \sum_{m \geq 0} \sum_{s \leq n} \sum_k \mathbb{P}_{ii1}(C_{sk}^{sm}) \mathbb{P}(X_n \in B, s \leq n < \tau_{2m+1}^0 \mid C_{sk}^{sm}) \\ &= \sum_{m \geq 0} \sum_{s \leq n} \sum_k \mathbb{P}_{ii1}(C_{sk}^{sm}) \mathbb{P}_{kk1}(X_{n-s} \in B, n-s < \tau_1^0) \\ &= \sum_{m \geq 0} \sum_{s \leq n} \sum_k \mathbb{P}_{ii1}(C_{sk}^{sm}) \mathbb{P}_{kk1}(X'_{n-s} \in B, n-s < \tau_1^0), \end{aligned}$$

since $X_n = X'_n$ on the random event $\{n < \tau_1^0\}$ if $d_0 = 1$. Hence

$$\mathbb{P}_{ii1}(X_n \in B, d_n = 1) = \mathbb{P}_{ii1}(X'_n \in B, d_n = 1).$$

The latter equality is proved by a similar reasoning if X_n is changed by X'_n . \square

Lemma 5. *Let $A_n = \{X_n \in B\} \cap \{X'_n \in B'\}$, where $B, B' \subset E$. Then*

$$(53) \quad \mathbb{P}_{ii1}(A_n, d_n = 0) = \sum_{1 \leq t \leq n} \sum_k \mathbb{P}_{ii1}(\bar{X}_{t-1} = (k, k, 1)) r_k^{(n-t)},$$

where

$$(54) \quad \begin{aligned} r_k^{(s)} &\equiv \mathbb{P}_{kk1}(d_1 = 0, s+1 < \tau_2^1, A_{s+1}) = \left(hT^{(s)} 1_{B \times B'} \right)_k, \\ s &\geq 0, \quad k \in E, \quad (1_{B \times B'})_{jl} \equiv 1_{j \in B, l \in B'}. \end{aligned}$$

Proof. Since $\{\nu_n = m, d_n = 0\} \subset \{\tau_{2m+1}^0 \leq n\}$, the full probability formula implies that

$$\begin{aligned} \mathbb{P}_{ii1}(A_n, d_n = 0) &= \sum_{m \geq 0} \sum_{0 \leq s < t \leq n} \mathbb{P}_{ii1}(\nu_n = m, \tau_{2m}^1 = s, \tau_{2m+1}^0 = t, d_t = 0, d_n = 0, A_n) \\ &= \sum_{m \geq 0} \sum_{0 \leq s < t \leq n} \sum_k \mathbb{P}_{ii1}(\nu_{t-1} = m, \tau_{2m}^1 = s, \bar{X}_{t-1} = (k, k, 1), \\ &\quad d_t = 0, n < \tau_{2m+1}^1, A_n). \end{aligned}$$

The latter equality holds since $\{\nu_s = \nu_{t-1} = \nu_n = m\}$ on the random event

$$\{\nu_n = m, \tau_{2m}^0 = s < t \leq n, d_n = 0\}$$

and since

$$\{\tau_{2m+1}^0 = t\} = \bigcup_k \{d_t = 0, \bar{X}_{t-1} = (k, k, 1)\}.$$

The random events

$$D_{sk}^{tm} \equiv \{\nu_{t-1} = m, \tau_{2m}^1 = s, \bar{X}_{t-1} = (k, k, 1)\}$$

are disjoint. Hence the above equality yields

$$\begin{aligned}
 P_{ii1}(A_n, d_n = 0) &= \sum_{m \geq 0} \sum_{0 \leq s < t \leq n} \sum_k P_{ii1}(D_{sk}^{tm}, d_t = 0, n < \tau_{2m+1}^1, A_n) \\
 &= \sum_{m \geq 0} \sum_{0 \leq s < t \leq n} \sum_k P_{ii1}(D_{sk}^{tm}) P(\nu_{t-1} = m, d_t = 0, n < \tau_{2m+1}^1, A_n \mid D_{sk}^{tm}) \\
 &= \sum_{m \geq 0} \sum_{0 \leq s < t \leq n} \sum_k P_{ii1}(D_{sk}^{tm}) P_{kk1}(d_1 = 0, n - t + 1 < \tau_2^1, A_{n-t+1}) \\
 &= \sum_{t \leq n} \sum_k P_{ii1}(\bar{X}_{t-1} = (k, k, 1)) r_k^{(n-t)}
 \end{aligned}$$

in view of the Markov property of the chain \bar{X} at the moment $t - 1$. Here we used the definition of (54) and the property that the events $\{\nu_{t-1} = m, \tau_{2m}^1 = s\}$ are disjoint for different m and s .

To prove the equality in (54), we use Lemmas 1 and 2 together with the Markov property of the chain \bar{X} at the moment 1. Then we get

$$\begin{aligned}
 r_k^{(s)} &= \sum_{j \neq l} P_{kk1}(\bar{X}_1 = (j, l, 0)) P_{jl0}(s < \tau_1^1, (X_s, X'_s) \in B \times B', d_s = 0) \\
 &= \sum_{j \neq l} h_{k,jl} \left(T^{(s)} 1_{B \times B'} \right)_{jl} = \left(h T^{(s)} 1_{B \times B'} \right)_k. \quad \square
 \end{aligned}$$

Proof of Remark 1. The inequality for the increment of the distance ρ follows from the triangle inequality for the norm $\|\cdot\|$. The contraction inequality is equivalent to the same inequality for the difference of point measures $\mu_j = \delta_{ij} - \delta_{kj}$, since these measures generate the space of measures with $\mu(E) = 0$ under the closure of the linear span. \square

Proof of Theorem 1. Using the marginal distributions (43) of the chain \bar{X} together with Lemma 4 we obtain

$$\begin{aligned}
 (55) \quad |P_i(X_n \in B) - P'_i(X'_n \in B)| &= |P_{ii1}(X_n \in B) - P_{ii1}(X'_n \in B)| \\
 &= |P_{ii1}(X_n \in B, d_n = 0) - P_{ii1}(X'_n \in B, d_n = 0)| \\
 &\leq P_{ii1}(\{X_n \in B\} \Delta \{X'_n \in B\}, d_n = 0) \\
 &\leq P_{ii1}(d_n = 0).
 \end{aligned}$$

Then we choose $B = B' = E$ and $A_n = \Omega$ in Lemma 5 to prove bound (4):

$$\begin{aligned}
 (56) \quad P_{ii1}(d_n = 0) &= \sum_{1 \leq t \leq n} \sum_k P_{ii1}(\bar{X}_{t-1} = (k, k, 1)) r_k^{(n-t)} \\
 &\leq \sum_{1 \leq t \leq n} \sup_k r_k^{(n-t)} \sum_k P_{ii1}(\bar{X}_{t-1} = (k, k, 1)) \\
 &\leq \sum_{1 \leq t \leq n} \sup_k r_k^{(n-t)} \\
 &\leq \sum_{1 \leq t \leq n} \varepsilon \rho^{n-t} = \varepsilon(1 - \rho^n)/(1 - \rho).
 \end{aligned}$$

Here we used Lemmas 3 and 5 together with the relations

$$r_k^{(s)} = \sum_{j \neq l} h_{k,jl} (T^{(s)} 1)_{jl} \leq \left(\sum_{j \neq l} h_{k,jl} \right) \sup_{j \neq l} (T^{(s)} 1)_{jl} \leq (1 - q_k) \rho^s \leq \varepsilon \rho^s.$$

Substituting (56) to (55) we prove the first inequality in (4). The second inequality is obvious. \square

Proof of Corollary 1. Using the strong Markov property of the chain \overline{X} we see that

$$\begin{aligned} \mathbf{P}_{ik0}(X_n \in B) &= \mathbf{P}_{ik0}(\tau_1^1 > n, X_n \in B) \\ &\quad + \sum_{1 \leq t \leq n} \sum_j \mathbf{P}_{ik0}(\tau_1^1 = t, \overline{X}_t = (j, j, 1), X_n \in B) \\ &= \mathbf{P}_{ik0}(\tau_1^1 > n, X_n \in B) \\ &\quad + \sum_{1 \leq t \leq n} \sum_j \mathbf{P}_{ik0}(\tau_1^1 = t, \overline{X}_t = (j, j, 1)) \mathbf{P}_{jj1}(X_{n-t} \in B). \end{aligned}$$

Subtracting a similar equality for X'_n and using (43) and (4), we obtain for $i \neq k$ that

$$\begin{aligned} |\mathbf{P}_i(X_n \in B) - \mathbf{P}'_k(X'_n \in B)| &= |\mathbf{P}_{ik0}(X_n \in B) - \mathbf{P}_{ik0}(X'_n \in B)| \\ &\leq \mathbf{P}_{ik0}(\tau_1^1 > n, \{X_n \in B\} \Delta \{X'_n \in B\}) \\ &\quad + \sum_{t \leq n} \sum_j \mathbf{P}_{ik0}(\tau_1^1 = t, \overline{X}_t = (j, j, 1)) |\mathbf{P}_{jj1}(X_{n-t} \in B) - \mathbf{P}_{jj1}(X'_{n-t} \in B)| \\ &\leq \mathbf{P}_{ik0}(\tau_1^1 > n) + \sum_{t \leq n} \mathbf{P}_{ik0}(\tau_1^1 = t) \varepsilon / (1 - \rho) \\ &= \mathbf{P}_{ik0}(\tau_1^1 > n) + \mathbf{P}_{ik0}(\tau_1^1 \leq n) \varepsilon / (1 - \rho) \\ &= \varepsilon / (1 - \rho) + (1 - \varepsilon / (1 - \rho)) \mathbf{P}_{ik0}(\tau_1^1 > n) \\ &= \varepsilon / (1 - \rho) + (1 - \varepsilon / (1 - \rho)) T^{(n)} \mathbf{1}_{ik} \\ &\leq \varepsilon / (1 - \rho) + (1 - \varepsilon / (1 - \rho)) \rho^n \end{aligned}$$

by Lemmas 1 and 2. The bound (5) for $i = k$ follows from (4). \square

Proof of Corollary 2. Let μ be a probability measure on E . Consider the measure

$$(57) \quad \pi = \mu - \sum_{n \geq 0} \mu(I - P)P^{(n)}.$$

Since $\mu(I - P) \in l_1^0(E)$, relation (3) implies that the series in (57) converges in the full variation norm in view of

$$\left\| \mu(I - P)P^{(n)} \right\| \leq \rho^n \|\mu(I - P)\|.$$

In particular, the partial sums converge, namely

$$\begin{aligned} \mu - \sum_{t < n} \mu(I - P)P^{(t)} &= \mu P^{(n)} \rightarrow \pi, \\ n &\rightarrow \infty. \end{aligned}$$

This implies the invariance $\pi = \pi P$. The contracting mapping theorem together with (3) implies that the measure π is unique. Taking into account $\varepsilon + \rho < 1$ and Remark 1, we similarly deduce that the chain X' is ergodic.

Finally, inequality (6) is proved by passing to the limit as $n \rightarrow \infty$ in the bound (4) of Theorem 1. \square

Proof of Theorem 2. We use inequality (55) and equalities of Lemma 5 for $A_n = \Omega$ together with the inclusion $\{\bar{X}_s = (k, k, 0)\} \subset \{X_s = k\}$:

$$\begin{aligned}
 |P'_\pi(X'_n \in B) - \pi(B)| &= \left| \sum_i \pi_i (P'_i(X'_n \in B) - P_i(X_n \in B)) \right| \\
 &\leq \sum_i \pi_i |P'_i(X'_n \in B) - P_i(X_n \in B)| \leq \sum_i \pi_i P_{ii1}(d_n = 0) \\
 &= \sum_i \pi_i \sum_{1 \leq t \leq n} \sum_k P_{ii1}(\bar{X}_{t-1} = (k, k, 0)) r_k^{(n-t)} \\
 &\leq \sum_i \pi_i \sum_{1 \leq t \leq n} \sum_k P_i(X_{t-1} = k) r_k^{(n-t)} = \sum_{1 \leq t \leq n} \sum_k \pi_k r_k^{(n-t)} \\
 &= \sum_{0 \leq s < n} \sum_k \pi_k (hT^{(s)})_k \leq \sum_{0 \leq s} \pi hT^{(s)} 1 = \varepsilon_T.
 \end{aligned}$$

The first inequality in (10) follows in view of the ergodicity by passing to the limit in (8) as $n \rightarrow \infty$. The second inequality follows from (56). \square

Proof of Remark 3. Let n be fixed. Consider the functions

$$f_i = P_i(X_n \in B) = (P^{(n)} 1_B)_i, \quad f'_i = P'_i(X'_n \in B) = (P'^{(n)} 1_B)_i.$$

By definition,

$$P_i(X_{n+t} \in B) = P^{(t)} f_i \quad \text{and} \quad P'_i(X'_{n+t} \in B) = P'^{(t)} f'_i.$$

Note that

$$|P g_i - P' g_i| \leq \varepsilon \sup g$$

by condition (1) for nonnegative bounded functions g . Thus

$$P^{(t)} f - P'^{(t)} f' = P^{(t)}(f - f') + \sum_{0 \leq s < t} P^{(t-1-s)}(P - P')P'^{(s)} f'$$

implies that

$$\left| P^{(t)} f - P'^{(t)} f' \right| \leq \sup_i |f_i - f'_i| + \sum_{0 \leq s < t} \varepsilon \leq \sup_i |f_i - f'_i| + (m-1)\varepsilon$$

for $t < m$, $t+n = 0 \pmod m$, since the matrices P and P' are stochastic.

Inequalities (4) and (6) hold for $t+n = 0 \pmod m$, whence we derive the result.

The inequality $r(P^{(m)}, P'^{(m)}) \leq m r(P, P')$ follows from the above equality after the substitution $f = f' = 1_B$ and $t = m$. \square

Proof of Example 2. We evaluate the number defined by (2):

$$\begin{aligned}
 \rho(P, P') &= \sup_{i \neq k} \sum_j |P_{ij} - P'_{kj}| / 2 = \sup_{i \neq k} \left(1 - \sum_j \min(P_{ij}, P'_{kj}) \right) \\
 &\leq 1 - \inf_{i \neq k} \min(P_{io}, P'_{ko}) \leq 1 - d = \rho < 1.
 \end{aligned}$$

The rest of the proof follows from (4), (5), and Remark 3. \square

In what follows, the symbol $E(\xi, A)$ stands for $E(\xi 1_A)$.

Proof of Theorem 3. Put

$$W(j, l) \equiv V_j + V_l, \quad j, l \in E,$$

and let $|f_i| \leq V_i$, $i \in E$.

Consider the sum of the values of the function $f = (f_i)$ with respect to the measures of the sets $B \subset E$ in equality (52) of Lemma 4. We obtain

$$(58) \quad \mathbf{E}_{ii1}(f(X_n), d_n = 1) = \mathbf{E}_{ii1}(f(X'_n), d_n = 1),$$

whence

$$(59) \quad \begin{aligned} |\mathbf{E}_i f(X_n) - \mathbf{E}'_i f(X'_n)| &= |\mathbf{E}_{ii1} f(X_n) - \mathbf{E}_{ii1} f(X'_n)| \\ &= |\mathbf{E}_{ii1}(f(X_n), d_n = 0) - \mathbf{E}_{ii1}(f(X'_n), d_n = 0)| \\ &\leq \mathbf{E}_{ii1}(W(X_n, X'_n), d_n = 0). \end{aligned}$$

The finiteness of the right hand side proved below implies the same property for the left hand side.

Consider the sum of the values of the function $W(j, l)$ with respect to the measures of the sets $B \times B' \subset E \times E$ in equality (53) of Lemma 5. Taking into account the definition of (54) we obtain

$$(60) \quad \begin{aligned} \mathbf{E}_{ii1}(W(X_n, X'_n), d_n = 0) &= \sum_{1 \leq t \leq n} \sum_k \mathbf{P}_{ii1}(\bar{X}_{t-1} = (k, k, 1)) \left(hT^{(n-t)}W \right)_k \\ &\leq \sum_{1 \leq t \leq n} \sum_k \mathbf{P}_{ii1}(X_{t-1} = k) \left(hT^{(n-t)}W \right)_k \\ &\leq \sum_{1 \leq t \leq n} \sum_k \mathbf{P}_i(X_{t-1} = k) V_k C^{(n-t)} \\ &= \sum_{1 \leq t \leq n} \mathbf{E}_i V(X_{t-1}) C^{(n-t)} \leq K_i^{(n)} \sum_{s < n} C^{(s)}, \end{aligned}$$

where

$$(61) \quad \begin{aligned} C^{(s)} &\equiv \sup_k V_k^{-1} \left(hT^{(s)}W \right)_k \\ &= \sup_k V_k^{-1} \sum_{j \neq l} h_{k,jl} \left(T^{(s)}W \right)_{jl} = \sup_k V_k^{-1} \sum_{j \neq l} h_{k,jl} W_{jl} W_{jl}^{-1} \left(T^{(s)}W \right)_{jl} \\ &\leq \sup_k V_k^{-1} (hW)_k \sup_{j \neq l} W_{jl}^{-1} \left(T^{(s)}W \right)_{jl}. \end{aligned}$$

In view of the definition of (30) and Lemma 1,

$$(62) \quad \begin{aligned} V_k^{-1} (hW)_k &= V_k^{-1} \sum_{j \neq l} h_{k,jl} (V_j + V_l) = V_k^{-1} \sum_{j \neq l} R_{kj} R'_{kl} (V_j + V_l) / (1 - q_k) \\ &= V_k^{-1} \left(\sum_j R_{kj} V_j + \sum_l R'_{kl} V_l \right) = V_k^{-1} \sum_j |P_{kj} - P'_{kj}| V_j \leq \varepsilon_V \end{aligned}$$

by condition (12).

Further, Lemma 1 and the definitions of (34) and (35) imply that

$$(63) \quad \begin{aligned} TW_{ik} &= \sum_{j \neq l} T_{ik,jl} (V_j + V_l) = \sum_j \sum_l T_{ik,jl} V_j + \sum_l \sum_j T_{ik,jl} V_l \\ &= \sum_j (P_{ij} - g_{ik,j}) V_j + \sum_l (P'_{kl} - g_{ik,l}) V_l \\ &= \sum_j (P_{ij} - P'_{kj})^+ V_j + \sum_j (P'_{kj} - P_{ij})^+ V_j = \sum_j |P_{ij} - P'_{kj}| V_j \\ &\leq \rho_V (V_i + V_k) = \rho_V W_{ik} \end{aligned}$$

if condition (13) holds. Using the induction, we derive from here that

$$(64) \quad T^{(s)}W_{ik} \leq \rho_V^s W_{ik}, \quad s \geq 1, \quad i \neq k \in E.$$

Substituting this inequality into (60) and (61) and applying (62) we conclude that

$$\mathbf{E}_{i1}(W(X_n, X'_n), d_n = 0) \leq K_i^{(n)} \sum_{0 \leq s < n} \varepsilon_V \rho_V^s,$$

which proves (14). \square

Proof of Example 3. The uniform mixing condition fails in this case, since both the first and last rows of the matrix P are singular. Then system (13) can be rewritten in the following form:

$$\begin{aligned} \alpha + 1 - \beta + \gamma v &< 1 + 1, \\ 1 + 1 &< 1 + v, \\ 1 - \alpha + \beta + \gamma v &< 1 + v. \end{aligned}$$

This system has a solution $v \in (1, 1 + 2\beta/\gamma)$. \square

Proof of Corollary 3. We check the strong mixing condition (13) for the case of

$$\rho_V = \rho_O + \varepsilon_V.$$

Let $i \notin O$ and $k \notin O$, $i \neq k$. Then (16) implies that

$$(65) \quad \begin{aligned} \sum_j |P_{ij} - P_{kj}| V_j &\leq \sum_{j \in O} |P_{ij} - P_{kj}| V_j + \sum_{j \notin O} (P_{ij} + P_{kj}) V_j \\ &\leq \sum_{j \in O} |P_{ij} - P_{kj}| V_j + \rho_O V_i - \sum_{j \in O} P_{ij} V_j + \rho_O V_k - \sum_{j \in O} P_{kj} V_j \\ &\leq \rho_O (V_i + V_k). \end{aligned}$$

Assume that $i \neq k$ for $k \in O$. Then inequality (65) is equivalent to condition (17). The case $i \neq k$ for $i \in O$ is similar.

Thus inequalities of the form (65) and (12) proved above imply that, for all $i \neq k$,

$$\begin{aligned} \sum_j |P_{ij} - P'_{kj}| V_j &\leq \sum_j |P_{ij} - P_{kj}| V_j + \varepsilon_V V_k \\ &\leq \rho_O (V_i + V_k) + \varepsilon_V V_k < (\rho_O + \varepsilon_V)(V_i + V_k). \end{aligned}$$

Therefore (18) follows from (14).

To derive (19) from (15) put

$$k_i^{(n)} \equiv \mathbf{E}_i V(X_n), \quad K_O \equiv \sup_{i \in O} P V_i.$$

Note that $\mathbf{E}_i V(X_1) \leq \rho_O V_i$ for $i \notin O$ in view of (16).

Using the Markov property of the chain X ,

$$\begin{aligned} k_i^{(n)} &= \mathbf{E}_i (1_{\{X_{n-1} \in O\}} \mathbf{E}_{X_{n-1}} V(X_1) + 1_{\{X_{n-1} \notin O\}} \mathbf{E}_{X_{n-1}} V(X_1)) \\ &\leq \mathbf{E}_i (K_O + \rho_O V(X_{n-1})) = K_O + \rho_O k_i^{(n-1)}, \quad n \geq 1. \end{aligned}$$

By induction, this implies that

$$k_i^{(n)} \leq K_O (1 - \rho_O^n) / (1 - \rho_O) + \rho_O^n V_i, \quad n \geq 0.$$

Since the right hand side is monotone with respect to n ,

$$K_i^{(n)} \leq \sup_{m \geq 0} k_i^{(m)} \leq \max(k_i^{(0)}, k_i^{(\infty)}) = \max(V_i, K_O/(1 - \rho_O)),$$

and this completes the proof of (19). \square

Proof of Example 4. The equivalence of (21) and the condition imposed on the mean increment (22) is proved in [4, 1977] under the assumption that the exponential moment of a jump is bounded and by using the method based on finding power test functions. We apply this method below to check (16).

To prove condition (16) of Corollary 3 note that the function in (21) is convex in the interval $(0, \ln \beta_0)$ and equals 1 if $\beta_0 = 1$. Thus the inequality

$$\sum_j P_{ij} v^{j-i} \leq \rho < 1, \quad i \neq 0,$$

holds for all $v \in (1, \beta_0]$ and some $\rho < 1$. Thus

$$\rho v^i \geq \sum_j P_{ij} v^j = P_{i0} + \sum_{j \neq 0} P_{ij} v^j$$

and this proves (16). The value $V_0 = 1$ is chosen according to the definition.

Now we prove that bounds (24) hold for all sufficiently small numbers $v - 1 > 0$ and for some $\rho_V < 1$.

Define the functions

$$R_i(v) \equiv \left((1 + v^i)^{-1} \sum (P_{ij} + P_{0j}) v^j - 1 \right) / (v - 1)$$

and

$$(66) \quad S_i(v) \equiv \sum_j P_{ij} (v^{j-i} - 1) / (v - 1)$$

for $i \in E$ and $v \in (1, \beta_0]$.

Note that

$$(67) \quad S_i(v) - \Delta_i = o(1), \quad v \downarrow 1,$$

uniformly with respect to i under the majorization condition (20), where Δ_i is defined by equality (22).

Further, the definition of (66) implies that

$$\begin{aligned} (1 + v^i) R_i(v) - (v^i \Delta_i + \Delta_0) &= \sum_j (P_{ij} (v^j - v^i) + P_{0j} (v^j - 1)) - (v^i \Delta_i + \Delta_0) \\ &= v^i (S_i - \Delta_i) + S_0 - \Delta_0, \end{aligned}$$

whence

$$(68) \quad R_i(v) - (\Delta_i + \Delta_0)/2 = o(1), \quad v \downarrow 1,$$

uniformly with respect to $i \neq 0$ in view of (67).

Denote by $-2d < 0$ the left hand side of (23).

(a) Let $i \neq 0$ be a state such that the minimum in (23) is attained at the first index: $\Delta_i + \Delta_0 \leq -2d$. According to (66),

$$\begin{aligned} \rho_i(v) &\leq (1 + v^i)^{-1} \sum_j (P_{ij} + P_{0j}) v^j = 1 + (v - 1) R_i(v) \\ &= 1 + (v - 1) ((\Delta_i + \Delta_0)/2 + o(1)) \\ &\leq 1 - (v - 1)d + o(v - 1) \leq 1 - (v - 1)d/2 = \rho_v < 1, \end{aligned}$$

starting with a certain $v - 1 > 0$ and for all $i \neq 0$, since $o(1)$ in (68) is uniform with respect to i .

(b) Let $i \neq 0$ and let the inequality in (23) hold due to the second index under the sign of minimum, that is, $\sum_j |P_{ij} - P_{0j}| - 2 \leq -2d$. The latter condition is equivalent to the inequality

$$\sum_j \min(P_{ij}, P_{0j}) \geq d.$$

If the constant N is such that $\sum_{j>N} P_{0j} < d/2$, then

$$\sum_{j \leq N} P_{ij} \geq \sum_{j \leq N} \min(P_{ij}, P_{0j}) \geq d/2.$$

This inequality for a fixed N contradicts condition (20) that the sums

$$\sum_{j < i} (i - j) P_{ij} \geq (i - N)d/2 \rightarrow \infty$$

are bounded as $i \rightarrow \infty$ along a subsequence of states i . Hence $i \leq m$ for some fixed m if the second inequality in (23) holds.

Therefore

$$\begin{aligned} \rho_i(v) &= (1 + v^i)^{-1} \left(\sum_j |P_{ij} - P_{0j}| + \sum_j |P_{ij} - P_{0j}| (v^j - 1) \right) \\ &\leq (1 + v^i)^{-1} \left(2 - 2d + \sum_j (P_{ij} + P_{0j}) (v^j - 1) \right) \\ &= (1 + v^i)^{-1} (2 - 2d - 2 + (1 + v^i) (1 + (v - 1)R_i(v))) \\ &= 1 - 2d(1 + v^i)^{-1} + (v - 1)R_i(v) = 1 - d + o(1), \quad v \downarrow 1, \end{aligned}$$

uniformly with respect to $i \leq m$ by relations (66) and (68). This means that $\rho_i(v) \leq 1 - d/2 = \rho_v$, starting with some sufficiently small $v - 1 > 0$ and for all $i \leq m$. \square

Proof of Theorem 4. Consider the stopping time $\theta_1 \equiv \min(\theta, \tau_1^0)$ for the chain \overline{X} . Let

$$m \equiv \sup(n \geq 1: P_{ii1}(\theta_1 > n) > 0) + 1 \leq \infty.$$

Taking into account the extended Markov property of \overline{X} and the inequality $h1_i \leq \varepsilon 1_i$ of Lemma 1 we obtain

$$\begin{aligned} (69) \quad P_{ii1}(\theta \geq \tau_1^0 = n) &= P_{ii1}(\theta_1 > n - 1, \tau_1^0 = n) \\ &= \sum_j P_{ii1}(\theta_1 > n - 1, \tau_1^0 = n, \overline{X}_{n-1} = (j, j, 1)) \\ &= \sum_j P_{ii1}(\theta_1 > n - 1, \overline{X}_{n-1} = (j, j, 1)) \\ &\quad \times P(\tau_1^0 = n \mid \theta_1 > n - 1, \overline{X}_{n-1} = (j, j, 1)) \\ &= \sum_j P_{ii1}(\theta_1 > n - 1, \overline{X}_{n-1} = (j, j, 1)) P_{jj1}(\tau_1^0 = 1) \\ &\leq \sum_j P_{ii1}(\theta_1 > n - 1, \overline{X}_{n-1} = (j, j, 1)) \varepsilon = \varepsilon P_{ii1}(\theta_1 > n - 1) \end{aligned}$$

for $n \geq 1$, whence

$$(70) \quad P_{ii1}(\theta \geq \tau_1^0) = \sum_{1 \leq n < m} P_{ii1}(\theta \geq \tau_1^0 = n) \leq \varepsilon \sum_{1 \leq n < m} P_{ii1}(\theta_1 > n - 1) = \varepsilon E_{ii1} \theta_1.$$

Since $\{\theta' \geq \tau_1^0\} = \{\theta \geq \tau_1^0\}$ in view of the equality $\theta = \theta'$ being true in the set

$$\{\theta' < \tau_1^0\} \cap \{\theta < \tau_1^0\},$$

we get

$$(71) \quad \mathbf{P}_{ii1}(\theta' \geq \tau_1^0) = \mathbf{P}_{ii1}(\theta \geq \tau_1^0) \leq \varepsilon \mathbf{E}_{ii1} \theta_1.$$

If φ is a nonnegative and (\mathfrak{F}_t) -measurable random variable, then we use definition (25) and prove that the expectations are the same:

$$\begin{aligned} \mathbf{E}_{ii1}(\varphi 1_{\{\theta=n\}}, n < \tau_1^0) &= \mathbf{E}_{ii1}(\varphi_n(X_1, \dots, X_n), n < \tau_1^0) = \mathbf{E}_{ii1}(\varphi_n(X'_1, \dots, X'_n), n < \tau_1^0) \\ &= \mathbf{E}_{ii1}(\varphi' 1_{\{\theta'=n\}}, n < \tau_1^0), \end{aligned}$$

since $(X_1, \dots, X_n) = (X'_1, \dots, X'_n)$ for $n < \tau_1^0$.

Then

$$(72) \quad \begin{aligned} \mathbf{E}_{ii1}(\varphi, \theta < \tau_1^0) &= \sum_{n \geq 1} \mathbf{E}_{ii1}(\varphi 1_{\{\theta=n\}}, n < \tau_1^0) = \sum_{n \geq 1} \mathbf{E}_{ii1}(\varphi' 1_{\{\theta'=n\}}, n < \tau_1^0) \\ &= \mathbf{E}_{ii1}(\varphi', \theta' < \tau_1^0). \end{aligned}$$

By Hölder inequality,

$$(73) \quad \begin{aligned} \mathbf{E}_i \varphi - \mathbf{E}'_i \varphi' &= \mathbf{E}_{ii1} \varphi - \mathbf{E}_{ii1} \varphi' = \mathbf{E}_{ii1}(\varphi, \theta \geq \tau_1^0) - \mathbf{E}_{ii1}(\varphi', \theta' \geq \tau_1^0) \\ &\leq \mathbf{E}_{ii1}(\varphi 1_{\{\theta \geq \tau_1^0\}}) \leq (\mathbf{E}_{ii1} \varphi^\alpha)^{1/\alpha} \left(\mathbf{E}_{ii1} 1_{\{\theta \geq \tau_1^0\}}^\beta \right)^{1/\beta} \\ &\leq K_i^\alpha(\varphi) (\mathbf{P}_{ii1}(\theta \geq \tau_1^0))^{1/\beta}. \end{aligned}$$

A lower bound is derived similarly:

$$(74) \quad \mathbf{E}_i \varphi - \mathbf{E}'_i \varphi' \geq -K_i^\alpha(\varphi) (\mathbf{P}_{ii1}(\theta' \geq \tau_1^0))^{1/\beta}.$$

Finally, we apply bound (70) in (73) and bound (71) in (74) and derive the two sided inequalities (26) in view of the inequality $\mathbf{E}_{ii1} \theta_1 \leq \mathbf{E}_{ii1} \theta = \mathbf{E}_i \theta$. \square

Proof of Corollary 4. Substituting $\varphi = 1_A$ and $\varphi' = 1_{A'}$ to (26), we get $K_i^\alpha(\varphi) \leq 1$; thus (27) follows from (4) by passing to the limit as $\beta \downarrow 1$. \square

Proof of Example 5. Let $A \in \mathfrak{F}_\infty$, that is, $A = \{X_\infty \in B\}$, where

$$X_\infty = (X_n, n \geq 1) \in E^\infty$$

is a trajectory of the chain X and $B \subset E^\infty$.

Let $\pi_n(x) = (x_1, \dots, x_n)$ be the projection for a sequence $x = (x_n, n \geq 1)$.

Consider the sets $B_n = \bigcup_{x \in B} \{\pi_n(x)\}$ and

$$B^n = B_n \times \{(o, o, \dots)\} = \{(x_1, \dots, x_n, o, o, \dots), (x_1, \dots, x_n) \in B_n\}.$$

Put $C_n = \bigcap_{t > n} \{X_t = o\}$.

We also introduce the random event

$$\tilde{A} = \bigcup_{n \geq 1} \{\theta = n\} \cap \{(X_1, \dots, X_n) \in B_n\}.$$

By definition, $\tilde{A} \in \mathfrak{F}_\theta$. Since $\mathbf{P}_i(\theta = n, \overline{C}_n) = 0$ by the almost sure absorption property, we obtain

$$\begin{aligned} \mathbf{P}_i(\theta = n, A \Delta \tilde{A}) &= \mathbf{P}_i(\theta = n, C_n, A \Delta \tilde{A}) \\ &= \mathbf{P}_i(\theta = n, C_n, \{X_\infty \in B\} \Delta \{(X_1, \dots, X_n) \in B_n\}) \\ &= \mathbf{P}_i(\theta = n, C_n, X_\infty \in B \Delta B^n) = 0, \end{aligned}$$

since

$$P_i(\theta = n, C_n, X_\infty \in B) = P_i(\theta = n, C_n, X_\infty \in B^n)$$

by the definition of the sets B_n , B^n , and C_n .

Since $\theta < \infty$ almost surely, we conclude that $P_i(A\Delta\tilde{A}) = 0$.

Applying Theorem 4 to the random event $\tilde{A} \in \mathfrak{F}_\theta$, we complete the proof of the statement of Example 5. \square

Proof of Remark 7. The Markov property of \tilde{X} follows from (43), since

$$\bar{X}_n = (X_n, X'_n, d_n)$$

is a Markov chain and $d_n = 1_{\{X_n = X'_n\}}$ almost surely. Inequality (44) follows from the Monge theorem [12], [13] by using the definitions of (28) and (34) for transition probabilities of the chain \tilde{X} that include the minimum of two possible marginal distributions in each of the only two possible cases, namely $i = k$ and $i \neq k$. \square

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