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FUNCTIONAL LAW OF THE ITERATED LOGARITHM TYPE FOR A SKEW BROWNIAN MOTION

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Abstract. The functional law of the iterated logarithm is proved for a skew Brownian motion.

1. Introduction

The functional law of the iterated logarithm for the Wiener process was proved in a well-known paper by Strassen [13]. A modification of this result for more general normalizing functions was proposed by Bulinskiĭ [1]. A functional law of the iterated logarithm for solutions of Itô stochastic differential equations with a jump process was obtained by Makhno [11].

The skew Brownian process studied in this paper was introduced by Itô and McKean [9] in terms of elliptic differential operators of the first order according to the Feller classification of one-dimensional diffusion processes. The skew Brownian motion has been studied by many authors since then. Among those authors are, to mention a few, Harrison and Shepp [8] and Le Gall [10], who considered this process as a solution of a stochastic equation with local time. In [10], as well as in [4] and [7], some interrelations were proposed between the solutions of stochastic equations with local time and solutions of Itô's equations.

The functional law of the iterated logarithm for a skew Brownian motion is studied in this paper. In doing so, we follow the approach of the paper [4].

The paper is organized as follows. Notation and the main results are given in Section 2. An auxiliary Theorem 2 is proved in Section 3. Section 4 is devoted to the proof of some lemmas and Theorem 1.

2. Main results

Consider a skew Brownian motion as a solution of the following stochastic differential equation with local time:

$$\xi(t) = x + \beta L^{\xi}(t, 0) + w(t), \qquad t \in [0, 1]. \tag{1}$$

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If $|\beta| \le 1$, then equation (1) has a strong solution [8]. This means that there exists a continuous semimartingale $(\xi(t), \Im_t)$ on the probability space $(\Omega, \Im, \Im_t, \mathsf{P})$ equipped with a flow of σ-algebras $\Im_t$, $t \in [0,1]$, where a standard one-dimensional Wiener process $(w(t), \Im_t)$ leaves, such that the symmetric local time

$$L^{\xi}(t,0) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t I_{(-\delta,\delta)}(\xi(s))\, ds \tag{2}$$

exists almost surely and equation (1) is satisfied almost surely.

In relation (2) and throughout this paper, $I_A(x)$ denotes the indicator of a set A. Let $\mathbf{R}$ be the real line and let $\mathcal{B}(\mathbf{R})$ be the Borel σ-algebra in $\mathbf{R}$. The space of continuous functions f on $[0,1]$ assuming values in $\mathbf{R}$ is denoted by $C[0,1]$. Let $\mathcal{B}(C[0,1])$ be the Borel σ-algebra of $C[0,1]$ and let the norm in $C[0,1]$ be given by $\|x\| = \sup_{t\in[0,1]} |x(t)|$. In what follows we use the standard notation $\dot f$ for the density of an absolutely continuous function f, namely

$$f(t) = f(a) + \int_a^t \dot f(s)\, ds.$$

Further, let

$$H^2[0,1] = \left\{ f\colon f(t) \text{ is absolutely continuous and such that } \int_0^1 |\dot f(t)|^2\, dt < \infty \right\}.$$

Recall the following property of absolutely continuous functions (throughout this paper, the symbol $\operatorname{Leb}(A)$ denotes the Lebesgue measure of a set A):

$$\operatorname{Leb}\Big(t \in [0,1]\colon f(t) = 0, \dot f(t) \ne 0\Big) = 0. \tag{3}$$

We put

$$\operatorname{sgn} x = \begin{cases} -1, & \text{for } x < 0,\\ 0, & \text{for } x = 0,\\ 1, & \text{for } x > 0. \end{cases}$$

Let $(X, \mathcal{B}(X))$ be a metric space equipped with a metric ρ, where $\mathcal{B}(X)$ is the Borel σ-algebra in the space X. Let $I(x)\colon X \to [0,\infty]$ be a lower semicontinuous functional such that $\{x\colon I(x) \le a\}$ is a compact set for all $a > 0$.

We say that a family of probability measures $\{\mu_\varepsilon\}$, $\varepsilon > 0$, defined on X satisfies the large deviation principle with a normalizing coefficient $k(\varepsilon)$ such that $\lim_{\varepsilon\to 0} k(\varepsilon) = +\infty$ and with an action functional $I(x)$ if

a) for every open set $G \in \mathcal{B}(X)$,

$$\liminf_{\varepsilon \to 0} \frac{1}{k(\varepsilon)} \ln \mu_\varepsilon(G) \ge -\inf\{I(x), x \in G\};$$

b) for every closed set $F \in \mathcal{B}(X)$,

$$\limsup_{\varepsilon \to 0} \frac{1}{k(\varepsilon)} \ln \mu_\varepsilon(F) \le -\inf\{I(x), x \in F\}.$$

Next we formulate the contraction principle (see [2, Theorem 5.3.1]). Let measures $\{\mu_\varepsilon\}$ on X be generated by some random elements $\{X_\varepsilon\}$. Assume that the family $\{\mu_\varepsilon\}$ satisfies the large deviation principle with an action functional $I(x)$. Further, let $F(x)$ be a continuous mapping acting from X to X'. Then the family of measures $\{\mu'_\varepsilon\}$ on X' generated by the random elements $\{F(X_\varepsilon)\}$ satisfy the large deviation principle with the action functional

$$I'(x) = \inf_{y\colon F(y)=x} \{I(y)\}.$$

Now we introduce the class Φ of increasing functions $\phi(T)$ such that

$$\lim_{T\to\infty}\phi(T)=\infty, \qquad \lim_{T\to\infty}\frac{\phi(T)}{\sqrt{T}}=0.$$

Throughout the paper we use the notation $\psi(T)=\phi(T)\sqrt{T}$.

Consider the functional

$$J^*(\phi,h,c)=\sum_{k=1}^{\infty}\exp\left\{\frac{-h\phi^2\left(c^k\right)}{2}\right\}, \qquad c>1.$$

Note that if $J^*(\phi,h,c_0)<\infty$ for some number $c_0>1$, then $J^*(\phi,h,c)<\infty$ for all $c>1$.

Given $\phi\in\Phi$, let

$$G^2(\phi)=\inf\{h>0\colon J^*(\phi,h,c)<\infty\}. \tag{4}$$

We agree that $G^2(\phi)=\infty$ if there is no $h<\infty$ such that $J^*(\phi,h,c)<\infty$. In what follows, the numbers G, G^2, or $G^2(\cdot)$ are always defined according to relation (4).

Put

$$Y(f)=\begin{cases}\frac12\int_0^1|\dot f(t)|^2\,dt, & \text{if } f\in H^2[0,1],\ f(0)=0,\\ +\infty, & \text{otherwise}\end{cases} \tag{5}$$

and

$$\mathcal{F}_D=\left\{h\in C[0,1]\colon h(0)=0; Y(h)\le\frac{D^2}{2}\right\}.$$

If $D^2=\infty$, then $\mathcal{F}_\infty=\big\{h\in C[0,1]\colon h(0)=0\big\}$.

For an arbitrary $T>0$, consider the following stochastic process:

$$\xi_T(t)=\frac{\xi(Tt)-x}{\sqrt{T}\phi(T)}=\frac{\beta L^\xi(tT,0)+w(tT)}{\sqrt{T}\phi(T)}. \tag{6}$$

The following is the main result of the paper.

Theorem 1. *Let $|\beta|<1$, $\phi\in\Phi$, and let G be defined by (4). Then the set of cluster points of the family $\{\xi_T(t)\}$ for the almost sure convergence as $T\to\infty$ coincides in $C[0,1]$ with $\mathcal{F}_G$.*

3. Auxiliary results

The solution of equation (1) is closely related to the solution of the Itô stochastic differential equation. Put

$$\kappa(x)=\begin{cases}(1-\beta)x, & x\le 0,\\ (1+\beta)x, & x\ge 0\end{cases} \tag{7}$$

and let

$$\varphi(x)=\begin{cases}\frac{x}{1-\beta}, & x\le 0,\\ \frac{x}{1+\beta}, & x\ge 0\end{cases}$$

be the inverse function to $\kappa(x)$.

Consider the following Itô stochastic differential equation:

$$\eta(t)=\varphi(x)+\int_0^t\frac{dw(s)}{1+\beta\operatorname{sgn}\eta(s)}, \qquad t\in[0,1]. \tag{8}$$

Note that the diffusion coefficient of this equation is a discontinuous function of bounded variation for which a unique strong solution of equation (8) exists according to a result from [12].

It is known that

$$\eta(t) = \varphi(\xi(t)) \quad \text{or} \quad \xi(t) = \kappa(\eta(t)) \tag{9}$$

(see [4]).

Now we consider the processes

$$\eta_T(t) = \frac{\eta(Tt) - \varphi(x)}{\sqrt{T}\phi(T)} = \frac{1}{\sqrt{T}\phi(T)} \int_0^{Tt} \frac{dw(s)}{1 + \beta \operatorname{sgn} \eta(s)}, \qquad t \in [0,1].$$

Let

$$L(f(s), \dot{f}(s)) = (1 + \beta \operatorname{sgn} f(s))^2 \dot{f}^2(s) \tag{10}$$

and introduce the functional $J(f)$ as follows:

$$J(f) = \begin{cases} \frac{1}{2}\int_0^1 L(f(t), \dot{f}(t))\, dt, & \text{if } f \in H^2[0,1],\ f(0) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Further, let

$$\mathcal{K}_D = \left\{ f \in C[0,1] \colon f(0) = 0; J(f) \le \frac{D^2}{2} \right\}.$$

If $D^2 = \infty$, then $\mathcal{K}_\infty = \{f \in C[0,1] \colon f(0) = 0\}$ and

$$L(f(s), \dot{f}(s)) = \left(\frac{d\kappa(f(s))}{ds} \right)^2. \tag{11}$$

Remark 1. It follows from relation (11) that $Y(\kappa(f)) = J(f)$, whence $\kappa(f) \in \mathcal{F}_D$ in view of $f \in \mathcal{K}_D$.

Lemma 1. *Let $|\beta| < 1$ and let the measures $\{\nu^T\}$ be generated by the processes $\{\eta_T(t)\}$. Then the family of measures $\{\nu^T\}$ satisfies the large deviation principle in the space $(C[0,1], \mathcal{B}(C[0,1]))$ with the normalizing coefficient $\phi^2(T)$ and action functional $J(\phi)$.*

Proof. Using relation (3), the proof follows from [6, Theorem B] with $\varepsilon = 1/\phi(T)$, since the infimum in Theorem B is attained at either $\rho = 0$ or $\rho = 1$ (note that the infimum itself equals 0).

Lemma 1 is proved. □

Consider the sequence of functions $z_k(t) = \eta_{c^k}(t)$, that is,

$$z_k(t) = \frac{1}{\psi\left(c^k\right)} \int_0^{c^k t} \frac{dw(s)}{1 + \beta \operatorname{sgn} \eta(s)}.$$

Put

$$u(t) = \int_0^t \frac{dw(s)}{1 + \beta \operatorname{sgn} \eta(s)}.$$

Then

$$z_k(t) = \frac{u(c^k t)}{\psi\left(c^k\right)}. \tag{12}$$

Theorem 2. *Let $|\beta| < 1$, $\phi \in \Phi$, and let G be defined by equality (4). Then the set of cluster points of the family $\{\eta_T(t)\}$ with respect to the almost sure convergence as $T \to \infty$ coincides with $\mathcal{K}_G$ in $C[0,1]$.*

Proof. The proof consists of the following three standard steps.

Step 1. First we prove that, for $G^2(\phi) < \infty$, for all $c > 1$, and for an arbitrary $\varepsilon > 0$, there exists a number k_0 such that

$$\rho(z_k, \mathcal{K}_G) < \varepsilon$$

almost surely for all $k > k_0$. Note that $\{f\colon J(f) \le a\}$ is a compact set in $C[0,1]$ whatever a number $a < \infty$.

Put $N_\varepsilon = \{f\colon \rho(f, \mathcal{K}_G) \ge \varepsilon\}$. Then there exists $\delta > 0$ such that

$$\inf_{f \in N_\varepsilon} J(f) \ge \frac{G^2(\phi)}{2} + \delta.$$

By Lemma 1, the family $\{\eta_T(t)\}$ satisfies the large deviation principle. Using property b) of the large deviation principle we get

$$\mathsf{P}\{z_k \in N_\varepsilon\} \le \exp\left\{-\phi^2\left(c^k\right)\left(\frac{G^2(\phi)}{2} + \delta\right)\right\}$$

for sufficiently large k. Then the definition of $G^2(\phi)$ and Borel–Cantelli lemma complete the proof of Step 1.

Step 2. We prove that every limit point of the family $\{\eta_T(t)\}$ almost surely belongs to $\mathcal{K}_G$ if $G^2(\phi) < \infty$. This result is proved in Step 1 for $\{T\} = \{c^k\}$. Now let $T \in [c^k, c^{k+1}]$. Since the function $\psi(T)$ is non-decreasing with respect to T, we write

$$\frac{1}{\psi(T)} = \frac{\alpha(T,k)}{\psi\left(c^k\right)} + \frac{\beta(T,k)}{\psi(c^{k+1})}, \tag{13}$$

where $\alpha(T,k) \ge 0$, $\beta(T,k) \ge 0$, and $\alpha(T,k) + \beta(T,k) = 1$. Put

$$\widehat{\eta}_{T,k}(t) = \alpha(T,k) z_k(t) + \beta(T,k) z_{k+1}(t).$$

The desired result follows from the following bound: for every $\varepsilon > 0$, there exist two numbers $c_\varepsilon > 1$ and k_0 such that

$$\sup_{t \in [0,1], T \in [c^k, c^{k+1}]} \left|\eta_T(t) - \widehat{\eta}_{T,k}(t)\right| < \varepsilon \tag{14}$$

almost surely for all $k > k_0$ and $c \in (1, c_\varepsilon)$.

It follows from the definition of the family $\{\eta_T(t)\}$ and equality (13) that

$$\eta_T(t) = z_k\left(t\frac{T}{c^k}\right)\frac{\psi\left(c^k\right)}{\psi(T)} = \alpha(T,k) z_k\left(t\frac{T}{c^k}\right) + \beta(T,k) z_{k+1}\left(t\frac{T}{c^{k+1}}\right).$$

Note that $z_k, z_{k+1} \in \{f\colon \rho(f, \mathcal{K}_G) < \delta\}$ for sufficiently large k and for all δ.

Then

$$\left|\eta_T(t) - \widehat{\eta}_{T,k}(t)\right| \le \alpha(T,k)\left|z_k(t) - z_k\left(t\frac{T}{c^k}\right)\right| + \beta(T,k)\left|z_{k+1}(t) - z_{k+1}\left(t\frac{T}{c^{k+1}}\right)\right|$$

and

$$\begin{aligned}\sup_{t \in [0,1], T \in [c^k, c^{k+1}]} &\left|\eta_T(t) - \widehat{\eta}_{T,k}(t)\right| \\ &\le \sup_{t \in [0,1], s \in [t, ct \wedge 1]} |z_k(t) - z_k(s)| + \sup_{t \in [0,1], s \in [t/c, t]} |z_{k+1}(t) - z_{k+1}(s)|.\end{aligned}$$

This implies that

$$
\begin{aligned}
&\mathsf{P}\left\{\sup_{t\in[0,1],T\in[c^k,c^{k+1}]}|\eta_T(t)-\widehat{\eta}_{T,k}(t)|\ge\varepsilon\right\}\\
&\le\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t,ct\wedge1]}|z_k(t)-z_k(s)|\ge\frac{\varepsilon}{2}\right\}\\
&\quad+\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t/c,t]}|z_{k+1}(t)-z_{k+1}(s)|\ge\frac{\varepsilon}{2}\right\}.
\end{aligned}\tag{15}
$$

To estimate the probabilities on the right hand side of (15) we apply Lemma 2 of [1]: there exists a constant C such that

$$
\mathsf{P}\left(\sup_{a\le t,s\le b;|t-s|\le h}|w(s)-w(t)|>x\sqrt{h}\right)\le\frac{C(b-a)}{hx}\exp\left\{-\frac{x^2}{4}\right\}
$$

for all $0\le a<b<\infty$, $h\le b-a$, and for an arbitrary $x>0$.

Thus we get, for another Wiener process $\widetilde{w}(t)=w(c^kt)/\sqrt{c^k}$, that

$$
\begin{aligned}
&\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t,ct\wedge1]}|z_k(t)-z_k(s)|\ge\frac{\varepsilon}{2}\right\}\\
&\quad=\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t,ct\wedge1]}\left|\int_t^s\frac{d\widetilde{w}(u)}{1+\beta\operatorname{sgn}\eta(c^ku)}\right|\ge\frac{\phi\left(c^k\right)\varepsilon}{2}\right\}.
\end{aligned}
$$

Next we make a random change of time. Consider the function

$$
\tau(u)=\int_0^u\frac{ds}{(1+\beta\operatorname{sgn}\eta(c^ks))^2}.
$$

Let $\gamma(u)$ be the inverse function to $\tau(u)$. It is clear that $\gamma(u)$ and $\tau(u)$ are increasing functions and that $\gamma(0)=\tau(0)=0$. Moreover, the derivatives

$$
\tau'(u)=\frac{1}{(1+\beta\operatorname{sgn}\eta(c^ku))^2},\qquad\gamma'(u)=\frac{1}{\tau'(\gamma(u))}=\left(1+\beta\operatorname{sgn}\eta(c^k\gamma(u))\right)^2
$$

exist almost surely. Letting $P_1=(1-|\beta|)^2$ and $P_2=(1+|\beta|)^2$, we prove that

$$
P_1u\le\gamma(u)\le P_2u,\qquad\frac{u}{P_2}\le\tau(u)\le\frac{u}{P_1}.
$$

According to the change of time made above, we get, for yet another Wiener process $\hat{w}(t)$, that

$$
\int_t^s\frac{d\widetilde{w}(u)}{1+\beta\operatorname{sgn}\eta(c^ku)}=\hat{w}(\gamma(s))-\hat{w}(\gamma(t)).
$$

Further,

$$
\begin{aligned}
&\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t,ct\wedge1]}\left|\int_t^s\frac{d\widetilde{w}(u)}{1+\beta\operatorname{sgn}\eta(c^ku)}\right|\ge\frac{\phi\left(c^k\right)\varepsilon}{2}\right\}\\
&\quad=\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t,ct\wedge1]}|\hat{w}(\gamma(s))-\hat{w}(\gamma(t))|\ge\frac{\phi\left(c^k\right)\varepsilon}{2}\right\}\\
&\quad=\mathsf{P}\left\{\sup_{\gamma(t)\in[\gamma(0),\gamma(1)],\gamma(s)\in[\gamma(t),\gamma(ct\wedge1)]}|\hat{w}(\gamma(s))-\hat{w}(\gamma(t))|\ge\frac{\phi\left(c^k\right)\varepsilon}{2}\right\}\\
&\quad\le\mathsf{P}\left\{\sup_{u,v\in[0,P_2],|v-u|\le P_2\frac{c-1}{c}}|\hat{w}(v)-\hat{w}(u)|\ge\frac{\phi\left(c^k\right)\varepsilon}{2}\right\}.
\end{aligned}
$$

In view of the result of [1] mentioned above,

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t,ct\wedge 1]}|z_k(t)-z_k(s)|\geq\frac{\varepsilon}{2}\right\}\\
&\quad\leq\mathsf{P}\left\{\sup_{u,v\in[0,P_2],|v-u|\leq P_2\frac{c-1}{c}}|\hat{w}(v)-\hat{w}(u)|\geq\frac{\phi\left(c^k\right)\varepsilon\sqrt{c}\sqrt{P_2(c-1)}}{2\sqrt{P_2(c-1)}\sqrt{c}}\right\}\\
&\quad\leq\frac{2CP_2\sqrt{c}}{\phi\left(c^k\right)\varepsilon\sqrt{P_2(c-1)}}\exp\left\{-\frac{\phi^2\left(c^k\right)\varepsilon^2 c}{16P_2(c-1)}\right\}.
\end{aligned}$$

Here we used the property

$$|v-u|\leq\gamma(ct\wedge 1)-\gamma(t)=\int_t^{ct\wedge 1}\gamma'(x)\,dx\leq(1+|\beta|)^2\int_t^{ct\wedge 1}dx\leq P_2\frac{c-1}{c},$$

since $ct\wedge 1-t\leq(c-1)/c$ under the assumptions of the theorem.

Choosing

$$c_\varepsilon=1+\frac{\varepsilon^2}{8(1+|\beta|)^2G^2(\phi)}$$

we get

$$\exp\left\{-\frac{\phi^2\left(c^k\right)\varepsilon^2 c}{16P_2(c-1)}\right\}\leq\exp\left\{-\frac{\phi^2\left(c^k\right)}{2}G^2(\phi)\right\}$$

for $c\in(1,c_\varepsilon)$. Next, for every positive constant C_1, there exists a positive integer k_0 such that

$$\frac{2CP_2\sqrt{c}}{\phi\left(c^k\right)\varepsilon\sqrt{P_2(c-1)}}\leq C_1$$

for all $k\geq k_0$. Thus

$$\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t,ct\wedge 1]}|z_k(t)-z_k(s)|\geq\frac{\varepsilon}{2}\right\}\leq C_1\exp\left\{-\frac{\phi^2\left(c^k\right)}{2}G^2(\phi)\right\}. \tag{16}$$

In a similar way we prove that, for any positive constant C_2, there exists a positive integer k_0 such that

$$\mathsf{P}\left\{\sup_{t\in[0,1],s\in[t/c,t]}|z_{k+1}(t)-z_{k+1}(s)|\geq\frac{\varepsilon}{2}\right\}\leq C_2\exp\left\{-\frac{\phi^2\left(c^k\right)}{2}G^2(\phi)\right\} \tag{17}$$

for all $k\geq k_0$. Hence (4), (15)–(17), and Borel–Cantelli imply (14).

Step 3. To complete the proof of Theorem 2 it is sufficient to prove that if $G^2(\phi)\leq\infty$, then every function $f\in\mathcal{K}_G$ such that $2J(f)=h^2<G^2(\phi)$ is a limit point of the sequence $\{z_k(t)\}$. Therefore it is sufficient to prove that, for every function $f\colon 2J(f)=h^2$, there exists a number $c>1$ such that the random events

$$B_k=\left\{\omega\colon\sup_{t\in[0,1]}|z_k(t)-f(t)|<\delta\right\}$$

occur infinitely often for every $\delta>0$. This means that

$$\mathsf{P}\left\{\limsup_{k\to\infty}B_k\right\}=1. \tag{18}$$

We use the Borel–Cantelli–Lévy lemma [5] to prove relation (18). Introduce the family of σ-algebras $\Im_j = \sigma\{\eta(s), s \le c^j\}$. Put

$$A_k = \left\{\omega\colon \sup_{t\in[0,1/c]} |z_k(t) - f(t)| < \delta\right\}$$

and

$$D_k = \left\{\omega\colon \sup_{t\in[1/c,1]} |z_k(t) - f(t)| < \delta\right\}.$$

Note that $B_k = A_k \cap D_k$ and $D_k \prec \Im_k$. Since

$$z_k(t) = z_{k-1}(tc)\frac{\psi\left(c^{k-1}\right)}{\psi\left(c^k\right)},$$

$z_k(t) \prec \Im_{k-1}$ for $t \in [0, 1/c]$. Then $A_k \prec \Im_{k-1}$ and

$$\mathsf{P}(B_k|\Im_{k-1}) = I(A_k)\,\mathsf{P}(D_k|\Im_{k-1}). \tag{19}$$

It follows from the Borel–Cantelli–Lévy lemma that relation (18) holds if

$$\sum_k I(A_k)\,\mathsf{P}(D_k|\Im_{k-1}) = \infty. \tag{20}$$

We construct a partition of the interval $[1/c, 1]$ consisting of smaller intervals of length Δ as follows: let Δ be a sufficiently small positive number such that $n(\Delta) = \frac{c-1}{c\Delta}$ is a positive integer number. Then the members of the partition of the interval $[1/c, 1]$ are

$$\Delta_i = [d_i, d_{i+1}], \qquad d_i = \frac{1}{c} + i\Delta, \qquad i = 0, \dots, n(\Delta) - 1.$$

In what follows we construct all the partitions of the interval $[1/c, 1]$ in the way described above.

Now we consider the set

$$\begin{aligned}\overline{D}_k &= \left\{\sup_{t\in[1/c,1]} |z_k(t) - f(t)| \ge \delta\right\}\\ &\subset \left\{\sup_i \sup_{t\in\Delta_i} |z_k(t) - z_k(d_i)| \ge \frac{\delta}{3}\right\} \cup \left\{\sup_i |z_k(d_i) - f(d_i)| \ge \frac{\delta}{3}\right\}\\ &\quad\cup \left\{\sup_i \sup_{t\in\Delta_i} |f(d_i) - f(t)| \ge \frac{\delta}{3}\right\}.\end{aligned}$$

By the Cauchy–Bunyakovskiĭ inequality,

$$|f(t) - f(d_i)|^2 = \left|\int_{d_i}^t \dot f(s)\,ds\right|^2 \le (t - d_i)\int_s^t \left|\dot f(s)\right|^2 ds \le \Delta h^2.$$

If $\Delta < \Delta_* = \delta^2/(9h^2)$, then

$$\left\{\sup_i \sup_{t\in\Delta_i} |f(d_i) - f(t)| \ge \frac{\delta}{3}\right\}$$

is an empty set. For such a number Δ,

$$\begin{aligned}\mathsf{P}(D_k|\Im_{k-1}) &\ge \mathsf{P}\left\{\sup_i \left|u\left(c^k d_i\right) - f(d_i)\psi\left(c^k\right)\right| < \frac{\delta}{3}\psi\left(c^k\right) \Big| \Im_{k-1}\right\}\\ &\quad - \mathsf{P}\left\{\sup_i \sup_{t\in\Delta_i} |z_k(t) - z_k(d_i)| \ge \frac{\delta}{3}\Big|\Im_{k-1}\right\}.\end{aligned} \tag{21}$$

Now we make use of several auxiliary results stated below. Lemma 2 (see Section 4) implies that there exists a constant $c > 1$ such that

$$I_{A_k}(\omega) = 1 \tag{22}$$

almost surely for all sufficiently large k.

Now Lemma 3 (see Section 4) implies that, for a fixed $c > 1$ and all $\delta > 0$ and $Q > 0$, there exists a partition of the interval $[1/c, 1]$ consisting of smaller intervals of length Δ_{**} such that

$$\mathsf{P}\left\{\sup_i \sup_{t\in\Delta_i} |z_k(t) - z_k(d_i)| \geq \frac{\delta}{3} \Big| \Im_{k-1}\right\} \leq 2n(\Delta_{**}) \exp\left\{-\phi^2\left(c^k\right) Q\right\} \tag{23}$$

almost surely. Lemma 8 (see Section 4) implies that

$$\begin{aligned} &\mathsf{P}\left\{\sup_i \left|u(c^k d_i) - f(d_i)\psi\left(c^k\right)\right| < \frac{\delta}{3}\psi\left(c^k\right) \Big| \Im_{k-1}\right\} \\ &\qquad \geq \frac{1}{2}\exp\left\{-\phi^2\left(c^k\right)\left(\frac{G^2(\phi)}{2} - q\right)\right\} \end{aligned} \tag{24}$$

almost surely for the constant c defined in Lemma 2 and for an arbitrary $q > 0$ if k is sufficiently large.

Now we turn back to the proof of the theorem. We pick up a number $c > 1$ such that equality (22) holds. Then we choose

$$Q = \frac{G^2(\phi)}{2} - q + 1$$

in inequality (23), where the constant q is the same as in (24), and a partition of the interval $[1/c, 1]$ with $\Delta < \min(\Delta_*, \Delta_{**})$. If the number k is sufficiently large, namely, if

$$8n(\Delta) \leq \exp\left\{\phi^2\left(c^k\right)\right\},$$

then

$$\begin{aligned} \mathsf{P}\left\{\sup_i \sup_{t\in\Delta_i} |z_k(t) - z_k(d_i)| \geq \frac{\delta}{3}\Big|\Im_{k-1}\right\} &\leq 2n(\Delta)\exp\left\{-\phi^2\left(c^k\right)\left(\frac{G^2(\phi)}{2} - q + 1\right)\right\} \\ &\leq \frac{2n(\Delta)}{\exp\{\phi^2\left(c^k\right)\}}\exp\left\{-\phi^2\left(c^k\right)\left(\frac{G^2(\phi)}{2} - q\right)\right\} \\ &\leq \frac{1}{4}\exp\left\{-\phi^2\left(c^k\right)\left(\frac{G^2(\phi)}{2} - q\right)\right\} \end{aligned}$$

almost surely, whence

$$\mathsf{P}(D_k|\Im_{k-1}) \geq \frac{1}{4}\exp\left\{-\phi^2\left(c^k\right)\left(\frac{G^2(\phi)}{2} - q\right)\right\}$$

almost surely by (24) and (21) for sufficiently large k and some $q > 0$.

Taking into account equalities (22) and (19) together with the definition of $G^2(\phi)$ we obtain (20). The proof of Step 3 is complete and thus Theorem 2 is proved. □

4. Proof of Theorem 1 and further auxiliary results

We start with the auxiliary results.

Lemma 2. *For all $\delta > 0$ and all $h < \infty$, there exist a constant $c > 1$ and a positive integer number k_0 such that*

$$\sup_{t\in[0,1/c]} |z_k(t) - g(t)| < \delta \tag{25}$$

almost surely for all $k > k_0$ and $g \in K_h$.

Proof. According to Step 1 in the proof of Theorem 2, for all $c > 1$ and an arbitrary $\delta > 0$ there exists a number k_0 such that

$$\inf_{g\in K_h} \sup_{t\in[0,1/c]} |z_k(t) - g(t)| < \frac{\delta}{3} \tag{26}$$

almost surely for all $k > k_0$.

On the other hand, for every $g \in K_h$,

$$|g(t)|^2 = \left| \int_0^t \dot g(s)\, ds \right|^2 \le 2th^2.$$

Let $c > \max(1, 18h^2/\delta^2)$; then

$$\sup_{t\in[0,1/c]} |g(t)| < \frac{\delta}{3}. \tag{27}$$

We deduce from (26) and (27) that

$$\sup_{t\in[0,1/c]} |z_k(t)| < \frac{2\delta}{3}$$

almost surely. The latter inequality together with (27) proves (25), which completes the proof of Lemma 2. □

The following result uses the partition of the interval $[1/c, 1]$ consisting of smaller intervals of length Δ described above.

Lemma 3. *Let $c > 1$ be fixed. Then, for all $\delta > 0$ and an arbitrary $Q > 0$, there exists a partition of the interval $[1/c, 1]$ consisting of smaller intervals of length Δ such that*

$$\mathsf{P}\left\{ \sup_i \sup_{t\in\Delta_i} |z_k(t) - z_k(d_i)| \ge \delta \middle| \Im_{k-1} \right\} \le 2n(\Delta) \exp\left\{ -\phi^2\left(c^k\right) Q \right\} \tag{28}$$

almost surely.

Proof. Consider the σ-algebras $\mathcal{G}_{c^k d_i} = \sigma\{\eta(s), s \le c^k d_i\}$. Then

$$\mathsf{P}\left\{ \sup_{t\in\Delta_i} |z_k(t) - z_k(d_i)| \ge \delta \middle| \mathcal{G}_{c^k d_i} \right\} \le 2 \exp\left\{ -\phi^2\left(c^k\right) Q \right\}$$

almost surely. The latter bound is proved similarly to the proof of Theorem 5 in [3, p. 172].

Since $\Im_{k-1} \subseteq \mathcal{G}_{c^k d_i}$ for $i = 0, 1, \ldots, n(\Delta) - 1$,

$$\mathsf{P}\left\{ \sup_{t\in\Delta_i} |z_k(t) - z_k(d_i)| \ge \delta \middle| \Im_{k-1} \right\} \le 2 \exp\left\{ -\phi^2\left(c^k\right) Q \right\}$$

almost surely.

This implies inequality (28) and completes the proof of Lemma 3. □

Lemma 4. *Let $h(x)$ be a positive increasing function for $x \ge 0$. Then*

$$\mathsf{E}\, \zeta I_{(|\xi|>a)} \le \frac{1}{h(a)}\, \mathsf{E}\, \zeta h(|\xi|)$$

for $\zeta \ge 0$ and $a > 0$.

Proof. We have

$$\mathsf{E}\,\zeta I_{(|\xi|>a)} = \int_{(\omega:\,|\xi|>a)} \zeta\, \mathsf{P}(d\omega) = \int_{(\omega:\,h(|\xi|)>h(a))} \zeta\, \mathsf{P}(d\omega)$$
$$\le \int_{(\omega:\,h(|\xi|)>h(a))} \zeta \frac{h(|\xi|)}{h(a)}\, \mathsf{P}(d\omega) \le \frac{1}{h(a)} \int_\Omega \zeta h(|\xi|)\, \mathsf{P}(d\omega) \le \frac{1}{h(a)}\, \mathsf{E}\,\zeta h(|\xi|).$$

Lemma 4 is proved. □

Put

$$M_k(f;x) = \frac{1}{\phi^2\left(c^k\right)} \ln \left\{ \mathsf{E} \left\{ \exp\left[\frac{\phi\left(c^k\right)}{\sqrt{c^k}} \int_{c^{k-1}}^{c^k} \frac{f(\frac{s}{c^k})}{1+\beta\,\mathrm{sgn}\,\eta(s)}\, dw(s) \right] \Big| \eta\left(c^{k-1}\right) = x \right\} \right\}.$$

Lemma 5. *Let $|\beta|<1$ and let $c>1$ be fixed. Then*

$$M_k(f;x) \le \frac{1}{2(1-|\beta|)^2} \int_{1/c}^1 f^2(s)\, ds$$

almost surely for $f \in C[0,1]$.

Proof. Since

$$\frac{\phi\left(c^k\right)}{\sqrt{c^k}} \int_{c^{k-1}}^{c^k} \frac{f(\frac{s}{c^k})}{1+\beta\,\mathrm{sgn}\,\eta(s)}\, dw(s)$$
$$= \frac{\phi\left(c^k\right)}{\sqrt{c^k}} \int_{c^{k-1}}^{c^k} \frac{f(\frac{s}{c^k})}{1+\beta\,\mathrm{sgn}\,\eta(s)}\, dw(s) \mp \frac{\phi^2\left(c^k\right)}{2c^k} \int_{c^{k-1}}^{c^k} \frac{f^2(\frac{s}{c^k})}{(1+\beta\,\mathrm{sgn}\,\eta(s))^2}\, ds$$

and

$$\frac{1}{\left(1+\beta\,\mathrm{sgn}\,\eta(s)\right)^2} \le \frac{1}{(1-|\beta|)^2},$$

the Girsanov theorem implies

$$M_k(f;x) \le \frac{1}{\phi^2\left(c^k\right)} \ln \left\{ \exp\left\{ \frac{\phi^2\left(c^k\right)}{2(1-|\beta|)^2 c^k} \int_{c^{k-1}}^{c^k} f^2\left(\frac{s}{c^k}\right)\, ds \right\} \right\}$$
$$= \frac{1}{2(1-|\beta|)^2 c^k} \int_{c^{k-1}}^{c^k} f^2\left(\frac{s}{c^k}\right)\, ds = \frac{1}{2(1-|\beta|)^2} \int_{1/c}^1 f^2(s)\, ds$$

almost surely. Lemma 5 is proved. □

Put

$$C_k = \left\{ \sup_i |u(c^k d_i) - f(d_i)\psi\left(c^k\right)| < \frac{\delta}{3}\psi\left(c^k\right) \right\};$$
$$C_k(i) = \left\{ \left|u\left(c^k d_i\right) - f(d_i)\psi\left(c^k\right)\right| < \frac{\delta}{3}\psi\left(c^k\right) \right\}, \qquad i = 0,1,\dots,n(\Delta)-1;$$
$$J_c(f) = \frac{1}{2} \int_{1/c}^1 \left(1+\beta\,\mathrm{sgn}\, f(s)\right)^2 \dot f^2(s)\, ds.$$

For $|\beta|<1$, we choose the constants l, m, and p such that

A_1. $0 < m < \frac{(1-|\beta|)^2}{(1+|\beta|)^2}$.

A_2. If $\beta \ne 0$, then $\sqrt{m}\frac{1-|\beta|}{1+|\beta|} < l < \sqrt{\frac{m-m^2}{4|\beta|}}(1-|\beta|)$; otherwise $l=m$.

A_3. $p = \frac{(1-|\beta|)^2}{l}\left(\frac{l^2-m^2+m}{(1+|\beta|)^2}+1\right)$.

Put

$$K_1 = \frac{l^2 - m^2 + m}{(1+|\beta|)^2} - \frac{l^2}{(1-|\beta|)^2}. \tag{29}$$

Remark 2. It is not hard to check that the following properties hold:

1. Condition A_1 implies that $0 < m < 1$, that is, the expression under the square root in condition A_2 is positive.
2. Condition A_2 implies that $K_1 > 0$.
3. The set of numbers l satisfying the inequality in condition A_2 is nonempty, since this inequality is equivalent to

$$\frac{1}{1+|\beta|} < \sqrt{\frac{1-m}{4|\beta|}};$$

the latter inequality holds by condition A_1.

For the constants l, m, and p put

$$\begin{aligned}\rho_k(l,m) = \exp\Bigg\{ & l\frac{\phi\left(c^k\right)}{\sqrt{c^k}} \int_{c^{k-1}}^{c^k} \frac{1+\beta \operatorname{sgn} f(\frac{s}{c^k})}{1+\beta \operatorname{sgn} \eta(s)} \dot{f}\left(\frac{s}{c^k}\right)\, dw(s) \\ & - m\frac{\phi^2\left(c^k\right)}{2c^k} \int_{c^{k-1}}^{c^k} \frac{(1+\beta \operatorname{sgn} f(\frac{s}{c^k}))^2}{(1+\beta \operatorname{sgn} \eta(s))^2} \dot{f}^2\left(\frac{s}{c^k}\right)\, ds \Bigg\}\end{aligned} \tag{30}$$

and

$$\begin{aligned}L_{k,p}(\delta) = \Bigg\{ \Bigg| & \int_{c^{k-1}}^{c^k} \frac{1+\beta \operatorname{sgn} f\left(\frac{s}{c^k}\right)}{1+\beta \operatorname{sgn} \eta(s)} \dot{f}\left(\frac{s}{c^k}\right)\, dw(s) \\ & - p\frac{\phi\left(c^k\right)}{2\sqrt{c^k}(1-|\beta|)^2} \int_{c^{k-1}}^{c^k} \left(1+\beta \operatorname{sgn} f\left(\frac{s}{c^k}\right)\right)^2 \dot{f}^2\left(\frac{s}{c^k}\right)\, ds \Bigg| \\ & \le \frac{\delta\psi\left(c^k\right)}{(1-|\beta|)^2} J_c(f) \Bigg\}.\end{aligned}$$

Lemma 6. *Let $|\beta| < 1$. Then, for the constants l and m chosen above, there exists a constant $c > 1$ such that*

$$\mathsf{P}\left\{\rho_k(l,m) I_{\Omega\setminus C_k(i)}(\omega) | \Im_{k-1}\right\} \le \exp\left\{\phi^2\left(c^k\right) \frac{l^2-m^2}{(1+|\beta|)^2} J_c(f)\right\} a_k(i)$$

almost surely, where the numbers $a_k(i)$ do not depend on θ and Δ and are such that

$$\lim_{k\to\infty} a_k(i) = 0, \qquad i = 0, 1, \ldots, n(\Delta) - 1.$$

Proof. Let $\theta \prec \Im_{k-1}$ be an arbitrary positive bounded random variable. We apply Lemma 4 to the function

$$h(x) = \exp\left\{\frac{\phi\left(c^k\right) N}{\sqrt{c^k}} x\right\}$$

with some constant N to be specified later. Then

$$
\begin{aligned}
&\mathsf{E}\,\theta\rho_k(l,m)I_{\Omega\setminus C_k(i)}(\omega)\\
&\quad\le \mathsf{E}\,\theta\rho_k(l,m)\exp\left\{\frac{N\phi\left(c^k\right)}{\sqrt{c^k}}\left|u\left(c^k d_i\right)-f(d_i)\psi\left(c^k\right)\right|-\frac{\delta}{3}N\phi^2\left(c^k\right)\right\}\\
&\quad\le \mathsf{E}\,\theta\rho_k(l,m)\exp\left\{\frac{N\phi\left(c^k\right)}{\sqrt{c^k}}\left(u\left(c^k d_i\right)-f(d_i)\psi\left(c^k\right)\right)-\frac{\delta}{3}N\phi^2\left(c^k\right)\right\}\\
&\qquad+\mathsf{E}\,\theta\rho_k(l,m)\exp\left\{-\frac{N\phi\left(c^k\right)}{\sqrt{c^k}}\left(u\left(c^k d_i\right)-f(d_i)\psi\left(c^k\right)\right)-\frac{\delta}{3}N\phi^2\left(c^k\right)\right\}\\
&\quad= J_k^1(i)+J_k^2(i).
\end{aligned}
$$

First we consider the term $J_k^1(i)$. Using equality (30) together with

$$
\frac{1}{(1+\beta\operatorname{sgn}\eta(sc^k))^2}\ge\frac{1}{(1+|\beta|)^2},
$$

we get

$$
\begin{aligned}
J_k^1(i)&=\exp\left\{-\phi^2\left(c^k\right)\left[Nf(d_i)+N\frac{\delta}{3}+\frac{m}{2}\int_{1/c}^1\frac{(1+\beta\operatorname{sgn}f(s))^2}{(1+\beta\operatorname{sgn}\eta(sc^k))^2}\dot f^2(s)\,ds\right]\right\}\\
&\quad\times\mathsf{E}\,\theta\exp\left\{\frac{\phi\left(c^k\right)}{\sqrt{c^k}}\left(Nu\left(c^k d_i\right)+\int_{c^{k-1}}^{c^k} l\frac{\left(1+\beta\operatorname{sgn}f\left(\frac{s}{c^k}\right)\right)\dot f\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}dw(s)\right)\right\}\\
&\le\exp\left\{-\phi^2\left(c^k\right)\left[Nf(d_i)+N\frac{\delta}{3}+\frac{m}{(1+|\beta|)^2}J_c(f)\right]\right\}\\
&\quad\times\mathsf{E}\Bigg\{\theta\,\mathsf{E}\Bigg\{\exp\Bigg[\frac{\phi\left(c^k\right)}{\sqrt{c^k}}\Bigg(Nu\left(c^{k-1}\right)\\
&\qquad\qquad+\int_{c^{k-1}}^{c^k}\Bigg(\frac{l\left(1+\beta\operatorname{sgn}f\left(\frac{s}{c^k}\right)\right)\dot f\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\\
&\qquad\qquad\qquad+\frac{NI_{[c^{k-1},c^k d_i]}\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\Bigg)\,dw(s)\Bigg)\Bigg]\,\Bigg|\,\Im_{k-1}\Bigg\}\Bigg\}.
\end{aligned}
$$

The Markov property of the process $\eta(t)$ implies that

$$
\begin{aligned}
&\mathsf{E}\left\{\exp\left[\frac{\phi\left(c^k\right)}{\sqrt{c^k}}\int_{c^{k-1}}^{c^k}\frac{l\left(1+\beta\operatorname{sgn}f\left(\frac{s}{c^k}\right)\right)\dot f\left(\frac{s}{c^k}\right)+NI_{[c^{k-1},c^k d_i]}\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\,dw(s)\right]\,\middle|\,\Im_{k-1}\right\}\\
&\quad=\exp\left\{\phi^2\left(c^k\right)M_k\Big(l\big(1+\beta\operatorname{sgn}f\big)\dot f+NI_{[c^{k-1},c^k d_i]}(\cdot);\eta\left(c^{k-1}\right)\Big)\right\}.
\end{aligned}
$$

Applying Lemma 5 we obtain

$$
\begin{aligned}
J_k^1(i) \le \exp&\left\{-\phi^2\left(c^k\right)\left[N\frac{\delta}{3}+\frac{m}{(1+|\beta|)^2}J_c(f)\right]\right\}\\
&\times \mathsf{E}\,\theta\exp\left\{\frac{\phi\left(c^k\right)}{\sqrt{c^k}}N\left(u\left(c^{k-1}\right)-f(d_i)\psi\left(c^k\right)\right)\right\}\\
&\times\exp\left\{\phi^2\left(c^k\right)M_k\left(l\big(1+\beta\operatorname{sgn} f\big)\dot f+NI_{[c^{k-1},c^kd_i]}(\cdot);\eta\left(c^{k-1}\right)\right)\right\}\\
\le \exp&\left\{-\phi^2\left(c^k\right)\left[N\frac{\delta}{3}+\frac{m}{(1+|\beta|)^2}J_c(f)\right]\right\}\\
&\times \mathsf{E}\,\theta\exp\left\{\frac{\phi\left(c^k\right)}{\sqrt{c^k}}N\left(u\left(c^{k-1}\right)-f(1/c)\psi\left(c^k\right)\right)\right\}\\
&\times\exp\left\{\phi^2\left(c^k\right)N\left(f(1/c)-f(d_i)\right)\right\}\\
&\times\exp\left\{\frac{\phi^2\left(c^k\right)}{2(1-|\beta|)^2}\int_{1/c}^1\left(l\big(1+\beta\operatorname{sgn} f(s)\big)\dot f(s)+NI_{[1/c,d_i]}(s)\right)^2ds\right\}.
\end{aligned}
$$

By Lemma 2, there exists a constant $c>1$ such that

$$
\exp\left\{\frac{\phi\left(c^k\right)}{\sqrt{c^k}}N\left(u\left(c^{k-1}\right)-f(1/c)\psi\left(c^k\right)\right)\right\}\le\exp\left\{\frac{\delta}{6}N\phi^2\left(c^k\right)\right\} \tag{31}
$$

almost surely for sufficiently large k. Then we estimate

$$
\begin{aligned}
&\int_{1/c}^1\left(l\big(1+\beta\operatorname{sgn} f(s)\big)\dot f(s)+NI_{[1/c,d_i]}(s)\right)^2ds\\
&\qquad\le 2l^2J_c(f)+2Nl\int_{1/c}^{d_i}\big(1+\beta\operatorname{sgn} f(s)\big)\dot f(s)\,ds+N^2(1-1/c).
\end{aligned} \tag{32}
$$

Denote the right hand side of inequality (32) by $A_c(J_c,N,l)$. Then (31) and (32) imply

$$
\begin{aligned}
J_k^1(i) &\le \exp\left\{-\phi^2\left(c^k\right)\left[N\frac{\delta}{6}+\frac{m}{(1+|\beta|)^2}J_c(f)\right]\right\}\exp\left\{\frac{\phi^2\left(c^k\right)}{2(1-|\beta|)^2}A_c(J_c,N,l)\right\}\\
&\quad\times\exp\left\{\phi^2\left(c^k\right)N\left(f(1/c)-f(d_i)\right)\right\}\mathsf{E}\,\theta\\
&=\exp\Bigg\{-\phi^2\left(c^k\right)\Bigg[N\frac{\delta}{6}+\frac{m}{(1+|\beta|)^2}J_c(f)-\frac{l^2}{(1-|\beta|)^2}J_c(f)\\
&\qquad\qquad\qquad-\frac{N^2}{2(1-|\beta|)^2}(1-1/c)\Bigg]\Bigg\}\\
&\quad\times\exp\left\{\phi^2\left(c^k\right)N\int_{1/c}^{d_i}\left(\frac{l(1+\beta\operatorname{sgn} f(s))}{(1-|\beta|)^2}-1\right)\dot f(s)\,ds\right\}\mathsf{E}\,\theta.
\end{aligned} \tag{33}
$$

The expression written in the parentheses in the integral in (33) does not exceed

$$
\frac{l(1+|\beta|)}{(1-|\beta|)^2},
$$

while

$$
\int_{1/c}^{d_i}\dot f(s)\,ds\le\left|\int_{1/c}^{d_i}\frac{1+\beta\operatorname{sgn} f}{1+\beta\operatorname{sgn} f}\dot f(s)\,ds\right|\le\frac{\sqrt{2J_c(f)}\sqrt{1-1/c}}{1-|\beta|}.
$$

Thus we deduce from inequality (33) that

$$\begin{aligned} J_k^1(i) \le \exp\Bigg\{ -\phi^2\left(c^k\right) \Bigg[\frac{N\delta}{6} + J_c(f) \left(\frac{m}{(1+|\beta|)^2} - \frac{l^2}{(1-|\beta|)^2} \right) - \frac{N^2(1-1/c)}{2(1-|\beta|)^2} \\ - \frac{lN(1+|\beta|)}{(1-|\beta|)^3} \sqrt{2J_c(f)}\sqrt{1-1/c} \Bigg] \Bigg\} \mathsf{E}\,\theta \\ = \exp\left\{ \phi^2\left(c^k\right) \frac{l^2-m^2}{(1+|\beta|)^2} J_c(f) \right\} \\ \times \exp\Bigg\{ -\phi^2\left(c^k\right) \Bigg[\frac{N\delta}{6} - \frac{N^2(1-1/c)}{2(1-|\beta|)^2} \\ + J_c(f) \left(\frac{l^2-m^2}{(1+|\beta|)^2} + \frac{m}{(1+|\beta|)^2} - \frac{l^2}{(1-|\beta|)^2} \right) \\ - \frac{lN(1+|\beta|)}{(1-|\beta|)^3} \sqrt{2J_c(f)}\sqrt{1-1/c} \Bigg] \Bigg\} \mathsf{E}\,\theta. \end{aligned}$$

Taking into account equality (29) we put

$$\begin{aligned} \hat{a}_k(i) = \exp\Bigg\{ -\phi^2\left(c^k\right) \Bigg[\frac{N\delta}{6} \\ - \frac{N^2(1-1/c)}{2(1-|\beta|)^2} + K_1 J_c(f) - \frac{lN(1+|\beta|)}{(1-|\beta|)^3} \sqrt{2J_c(f)}\sqrt{1-1/c} \Bigg] \Bigg\}. \end{aligned}$$

Since $K_1 > 0$, the expression in the square brackets is positive for some $N > 0$. For such a number N,

$$\lim_{k\to\infty} \hat{a}_k(i) = 0$$

and

$$J_k^1(i) \le \exp\left\{ \phi^2\left(c^k\right) \frac{l^2-m^2}{(1+|\beta|)^2} J_c(f) \right\} \hat{a}_k(i)\, \mathsf{E}\,\theta. \tag{34}$$

Similarly

$$J_k^2(i) \le \exp\left\{ \phi^2\left(c^k\right) \frac{l^2-m^2}{(1+|\beta|)^2} J_c(f) \right\} \check{a}_k(i)\, \mathsf{E}\,\theta, \tag{35}$$

where

$$\lim_{k\to\infty} \check{a}_k(i) = 0.$$

Now Lemma 6 follows from bounds (34) and (35) with $a_k(i) = \hat{a}_k(i) + \check{a}_k(i)$. □

Lemma 7. *Let $|\beta| < 1$. Then*

$$\mathsf{P}\left\{ \rho_k(l,m) I_{\Omega\setminus L_{k,p}(\delta)}(\omega) | \Im_{k-1} \right\} \le \exp\left\{ \phi^2\left(c^k\right) \frac{l^2-m^2}{(1+|\beta|)^2} J_c(f) \right\} b_k(\delta)$$

almost surely for the constants l, m, and p chosen above and for all $\delta > 0$, where $b_k(\delta)$ does not depend on θ and is such that $\lim_{k\to\infty} b_k(\delta) = 0$.

Proof. Let $\theta \prec \Im_{k-1}$ be an arbitrary positive bounded random variable. We use Lemma 4 with the function

$$h(x) = \exp\left\{ \frac{\phi\left(c^k\right) N}{\sqrt{c^k}} x \right\}$$

and with some constant $0 < N < 1$ to be specified later. Then

$$\begin{aligned}
&\mathsf{E}\,\theta\rho_k(l,m)I_{\Omega\setminus L_{k,p}(\delta)}(\omega)\\
&\quad\le \mathsf{E}\,\theta\rho_k(l,m)\\
&\qquad\times\exp\Bigg\{\Bigg|\frac{\phi\left(c^k\right)}{\sqrt{c^k}}N\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn} f\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\dot f\left(\frac{s}{c^k}\right)\,dw(s)\\
&\qquad\qquad-\frac{p\phi^2\left(c^k\right)}{2(1-|\beta|)^2c^k}N\int_{c^{k-1}}^{c^k}\left(1+\beta\operatorname{sgn} f\left(\frac{s}{c^k}\right)\right)^2\dot f^2\left(\frac{s}{c^k}\right)\,ds\Bigg|\Bigg\}\\
&\qquad\times\exp\left\{-\phi^2\left(c^k\right)N\frac{\delta}{(1-|\beta|)^2}J_c(f)\right\}\\
&\quad\le \mathsf{E}\,\theta\rho_k(l,m)\exp\Bigg\{\frac{\phi\left(c^k\right)}{\sqrt{c^k}}N\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn} f\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\dot f\left(\frac{s}{c^k}\right)\,dw(s)\\
&\qquad\qquad-\frac{(\delta+p)\phi^2\left(c^k\right)N}{(1-|\beta|)^2}J_c(f)\Bigg\}\\
&\qquad+\mathsf{E}\,\theta\rho_k(l,m)\exp\Bigg\{-\frac{\phi\left(c^k\right)}{\sqrt{c^k}}N\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn} f\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\dot f\left(\frac{s}{c^k}\right)\,dw(s)\\
&\qquad\qquad-\frac{(\delta-p)\phi^2\left(c^k\right)N}{(1-|\beta|)^2}J_c(f)\Bigg\}\\
&\quad= J_k^1(\delta)+J_k^2(\delta).
\end{aligned}$$

Substituting $\rho_k(l,m)$, we consider the term $J_k^1(\delta)$. We see from the Markov property of the process $\eta(t)$ that

$$\begin{aligned}
J_k^1(\delta)&=\mathsf{E}\,\theta\exp\Bigg\{\frac{\phi\left(c^k\right)(N+l)}{\sqrt{c^k}}\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn} f\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\dot f\left(\frac{s}{c^k}\right)\,dw(s)\\
&\qquad-\phi^2\left(c^k\right)\left(\frac{(\delta+p)N}{(1-|\beta|)^2}J_c(f)+\frac{m}{2}\int_{1/c}^1\frac{(1+\beta\operatorname{sgn} f(s))^2}{(1+\beta\operatorname{sgn}\eta(sc^k))^2}\dot f^2(s)\,ds\right)\Bigg\}\\
&\le\exp\left\{-\phi^2\left(c^k\right)J_c(f)\left(\frac{(\delta+p)N}{(1-|\beta|)^2}+\frac{m}{(1+|\beta|)^2}\right)\right\}\\
&\quad\times\mathsf{E}\left\{\theta\,\mathsf{E}\left\{\exp\left[\frac{\phi\left(c^k\right)(N+l)}{\sqrt{c^k}}\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn} f\left(\frac{s}{c^k}\right)}{1+\beta\operatorname{sgn}\eta(s)}\dot f\left(\frac{s}{c^k}\right)\,dw(s)\right]\,\Bigg|\,\Im_{k-1}\right\}\right\}\\
&=\exp\left\{-\phi^2\left(c^k\right)J_c(f)\left(\frac{(\delta+p)N}{(1-|\beta|)^2}+\frac{m}{(1+|\beta|)^2}\right)\right\}\mathsf{E}\,\theta\\
&\quad\times\exp\left\{\phi^2\left(c^k\right)M_k\left((l+N)(1+\beta\operatorname{sgn} f)\dot f;\eta\left(c^{k-1}\right)\right)\right\}.
\end{aligned}$$

By Lemma 5, we get

$$\begin{aligned}
M_k\left((l+N)(1+\beta\operatorname{sgn} f)\dot f;\eta\left(c^{k-1}\right)\right)&\le\frac{(l+N)^2}{2(1-|\beta|)^2}\int_{1/c}^1(1+\beta\operatorname{sgn} f)^2\dot f^2\,ds\\
&=\frac{(l+N)^2}{(1-|\beta|)^2}J_c(f)
\end{aligned}$$

almost surely. Hence

$$
\begin{aligned}
J_k^1(\delta) &\le \exp\left\{-\phi^2\left(c^k\right)J_c(f)\left[\frac{(\delta+p)N}{(1-|\beta|)^2}+\frac{m}{(1+|\beta|)^2}-\frac{(l+N)^2}{(1-|\beta|)^2}\right]\right\}\mathsf{E}\,\theta \\
&= \exp\Biggl\{\phi^2\left(c^k\right)J_c(f) \\
&\qquad\times\left(\frac{l^2-m^2}{(1+|\beta|)^2}-\left[\frac{(\delta+p)N-(l+N)^2}{(1-|\beta|)^2}+\frac{m+l^2-m^2}{(1+|\beta|)^2}\right]\right)\Biggr\}\mathsf{E}\,\theta \\
&= \exp\left\{\phi^2\left(c^k\right)\frac{l^2-m^2}{(1+|\beta|)^2}J_c(f)\right\} \\
&\qquad\times\exp\left\{-\phi^2\left(c^k\right)J_c(f)\left[\frac{(\delta+p-2l)N-N^2}{(1-|\beta|)^2}+K_1\right]\right\}\mathsf{E}\,\theta \\
&= \exp\left\{\phi^2\left(c^k\right)\frac{l^2-m^2}{(1+|\beta|)^2}J_c(f)\right\}\check{b}_k(\delta)\,\mathsf{E}\,\theta
\end{aligned}
\tag{36}
$$

for

$$\check{b}_k(\delta)=\exp\left\{-\phi^2\left(c^k\right)J_c(f)\left[\frac{(\delta+p-2l)N-N^2}{(1-|\beta|)^2}+K_1\right]\right\}.$$

Since $K_1>0$, the expression in the square brackets on the right hand side of the definition of $\check{b}_k(\delta)$ is positive for some $N>0$. For such a number N,

$$\lim_{k\to\infty}\check{b}_k(\delta)=0.$$

Similarly,

$$J_k^2(\delta)\le\exp\left\{\phi^2\left(c^k\right)\frac{l^2-m^2}{(1+|\beta|)^2}J_c(f)\right\}\hat{b}_k(\delta)\,\mathsf{E}\,\theta,$$

where

$$\lim_{k\to\infty}\hat{b}_k(\delta)=0.$$

Now Lemma 7 holds with $b_k(\delta)=\check{b}_k(\delta)+\hat{b}_k(\delta)$ for some N. □

Lemma 8. *Let $f\in\mathcal{K}_G$ be an arbitrary function such that $2J(f)=h^2<G^2$. Then there are numbers $c>1$ and $v>0$ such that*

$$\mathsf{P}\left(C_k|\Im_{k-1}\right)\ge\frac{1}{2}\exp\left\{-\phi^2\left(c^k\right)\left(\frac{G^2}{2}-v\right)\right\}$$

almost surely for sufficiently large k.

Proof. Let $\theta\prec\Im_{k-1}$ be an arbitrary positive bounded random variable. Then

$$
\begin{aligned}
\mathsf{E}\,\theta I_{C_k}(\omega) &= \mathsf{E}\,\theta\rho_k(l,m)I_{C_k}(\omega) \\
&\quad\times\exp\Biggl\{-l\frac{\phi\left(c^k\right)}{\sqrt{c^k}}\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn}f(\frac{s}{c^k})}{1+\beta\operatorname{sgn}\eta(s)}\dot{f}\left(\frac{s}{c^k}\right)dw(s) \\
&\qquad\qquad+\frac{m\phi^2\left(c^k\right)}{2c^k}\int_{c^{k-1}}^{c^k}\frac{(1+\beta\operatorname{sgn}f(\frac{s}{c^k}))^2}{(1+\beta\operatorname{sgn}\eta(s))^2}\dot{f}^2\left(\frac{s}{c^k}\right)ds\Biggr\}
\end{aligned}
$$

$$\geq \exp\left\{\phi^2\left(c^k\right)\left[\frac{mJ_c(f)}{(1+|\beta|)^2}-\frac{plJ_c(f)}{(1-|\beta|)^2}\right]\right\}$$
$$\times \mathsf{E}\,\theta\rho_k(l,m)I_{C_k}(\omega)$$
$$\times \exp\Bigg\{-l\frac{\phi\left(c^k\right)}{\sqrt{c^k}}\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn} f(\frac{s}{c^k})}{1+\beta\operatorname{sgn}\eta(s)}\dot f\left(\frac{s}{c^k}\right)\,dw(s)$$
$$+pl\frac{\phi^2\left(c^k\right)}{2(1-|\beta|)^2c^k}\int_{c^{k-1}}^{c^k}\left(1+\beta\operatorname{sgn} f\left(\frac{s}{c^k}\right)\right)^2\dot f^2\left(\frac{s}{c^k}\right)\,ds\Bigg\}$$
$$\geq \exp\left\{\phi^2\left(c^k\right)J_c(f)\left[\frac{m}{(1+|\beta|)^2}-\frac{pl}{(1-|\beta|)^2}\right]\right\}$$
$$\times \mathsf{E}\,\theta\rho_k(l,m)I_{C_k}(\omega)I_{L_{k,p}}(\omega)$$
$$\times \exp\Bigg\{-l\Bigg(\frac{\phi\left(c^k\right)}{\sqrt{c^k}}\int_{c^{k-1}}^{c^k}\frac{1+\beta\operatorname{sgn} f(\frac{s}{c^k})}{1+\beta\operatorname{sgn}\eta(s)}\dot f\left(\frac{s}{c^k}\right)\,dw(s)$$
$$-p\frac{\phi^2\left(c^k\right)}{2(1-|\beta|)^2c^k}\int_{c^{k-1}}^{c^k}\left(1+\beta\operatorname{sgn} f(\frac{s}{c^k})\right)^2\dot f^2\left(\frac{s}{c^k}\right)\,ds\Bigg)\Bigg\}$$
$$\geq \exp\left\{\phi^2\left(c^k\right)J_c(f)\left[\frac{m}{(1+|\beta|)^2}-\frac{pl}{(1-|\beta|)^2}-\frac{\delta l}{(1-|\beta|)^2}\right]\right\}$$
$$\times \mathsf{E}\,\theta\rho_k(l,m)I_{C_k}(\omega)I_{L_{k,p}}(\omega).$$

In the above reasoning we used the inequalities $1\geq I_{L_{k,p}}(\omega)$ and

$$\exp\{-a\}I_{(|a|<b)}\geq \exp\{-b\}I_{(|a|<b)}.$$

Since $I_{C_k}(\omega)I_{L_{k,p(\delta)}}(\omega)\geq 1-I_{\Omega\setminus C_k}(\omega)-I_{\Omega\setminus L_{k,p(\delta)}}(\omega)$, we obtain

$$\mathsf{E}\,\theta I_{C_k}(\omega)\geq \exp\left\{\phi^2\left(c^k\right)J_c(f)\left(\frac{m}{(1+|\beta|)^2}-\frac{pl}{(1-|\beta|)^2}-\frac{\delta l}{(1-|\beta|)^2}\right)\right\}$$
$$\times \mathsf{E}\,\theta\rho_k(l,m)\left(1-I_{\Omega\setminus C_k}(\omega)-I_{\Omega\setminus L_{k,p(\delta)}}(\omega)\right).$$

Then equality (30) implies that

$$\mathsf{E}\,\theta\rho_k(l,m)=\mathsf{E}\left\{\theta\,\mathsf{E}\left\{\rho_k(l,m)\mid\Im_{k-1}\right\}\right\}$$
$$=\mathsf{E}\left\{\theta\widetilde{\mathsf{E}}\left\{\exp\left[\frac{\phi^2\left(c^k\right)}{2c^k}\int_{c^{k-1}}^{c^k}\left(l^2-m^2\right)\frac{(1+\beta\operatorname{sgn} f)^2}{(1+\beta\operatorname{sgn}\eta)^2}\dot f^2\,ds\;\middle|\;\Im_{k-1}\right]\right\}\right\}.$$

It is clear that

$$l^2-m^2>l^2\frac{(1+|\beta|)^2}{(1-|\beta|)^2}-m.$$

Considering the left hand side of property A_2, we conclude that

$$l^2\frac{(1+|\beta|)^2}{(1-|\beta|)^2}-m>0,$$

whence

$$l^2-m^2>0.$$

Hence

$$\mathsf{E}\,\theta\rho_k(l,m)\geq \exp\left\{\phi^2\left(c^k\right)\frac{l^2-m^2}{(1+|\beta|)^2}J_c(f)\right\}\mathsf{E}\,\theta.$$

We continue the proof by using the latter bound and applying Lemmas 6 and 7:

$$\begin{aligned}\mathsf{E}\,\theta I_{C_k}(\omega) &\geq \exp\left\{\phi^2\left(c^k\right)J_c(f)\left(\frac{m}{(1+|\beta|)^2}-\frac{pl+\delta l}{(1-|\beta|)^2}\right)\right\}\\ &\quad\times\exp\left\{\phi^2\left(c^k\right)\frac{l^2-m^2}{(1+|\beta|)^2}J_c(f)\right\}\left(1-\sum_{i=1}^{n(\Delta)}a_k(i)-b_k(\delta)\right)\mathsf{E}\,\theta\\ &\geq \exp\left\{-\phi^2\left(c^k\right)J_c(f)\left(\frac{m^2-l^2-m}{(1+|\beta|)^2}+\frac{pl+\delta l}{(1-|\beta|)^2}\right)\right\}\mathsf{E}\,\theta.\end{aligned}$$

Then we use property A_3:

$$\mathsf{E}\,\theta I_{C_k}(\omega) \geq \exp\left\{-\phi^2\left(c^k\right)J_c(f)\left(1+\frac{\delta l}{(1-|\beta|)^2}\right)\right\}\mathsf{E}\,\theta. \tag{37}$$

It is clear that

$$J_c(f)\left(1+\frac{\delta l}{(1-|\beta|)^2}\right)\leq\left(1+\frac{\delta l}{(1-|\beta|)^2}\right)\frac{h^2}{2}.$$

Choose

$$\delta<\frac{G^2-h^2}{3h^2}\frac{(1-|\beta|)^2}{l}.$$

The latter inequality implies that

$$J_c(f)\left(1+\frac{\delta l}{(1-|\beta|)^2}\right)\leq\frac{G^2}{2}-v, \tag{38}$$

where $v=\frac{1}{3}(G^2-h^2)$. Now Lemma 8 follows from inequalities (37) and (38). □

The Lipschitz property of the function κ (see definition (7)) yields the following result.

Lemma 9. *Assume that*

$$\mathsf{P}\left\{\lim_{n\to\infty}\sup_{t\in[0,1]}|f_n(t)-g(t)|=0\right\}=1$$

for all one-dimensional functions $\{f_n\}$ and g. Then

$$\mathsf{P}\left\{\lim_{n\to\infty}\sup_{t\in[0,1]}|\kappa(f_n(t))-\kappa(g(t))|=0\right\}=1,$$

where the function κ is defined by (7).

Proof of Theorem 1. Using Theorem 2 we prove that, for an arbitrary function $f\in\mathcal{K}_G$, there exists a subsequence $\{T_m\}$ such that

$$\mathsf{P}\left\{\lim_{T_m\to\infty}\sup_{t\in[0,1]}|\eta_{T_m}(t)-f(t)|=0\right\}=1.$$

Then Lemma 9 and relations (7)–(9) complete the proof of Theorem 1. □

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