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INTERPOLATION OF FUNCTIONALS OF STOCHASTIC SEQUENCES WITH STATIONARY INCREMENTS

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Abstract. The problem of optimal estimation of a linear functional

$$A_N\xi = \sum_{k=0}^{N} a(k)\xi(k)$$

that depends on unknown values of a stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ with stationary increments of order n by observations of the sequence at points

$$m \in \mathbb{Z} \setminus \{0, 1, \dots, N\}$$

is considered. Formulas for calculating the mean square error and spectral characteristic of the optimal linear estimator of the above functional are derived in the case where the spectral density is known. In the case where the spectral density is not known, but a set of admissible spectral densities is given, the minimax-robust approach is applied to the problem of optimal estimation of a linear functional. Formulas that determine the least favorable spectral densities and the minimax spectral characteristics are proposed for a given set of admissible spectral densities.

1. Introduction

Stochastic processes with stationary increments of order n have been studied by Yaglom in [18]. A spectral representation of stochastic increments is obtained and the problem of prediction of a stochastic increment by known observations is solved in the paper [18]. The method in [18] is to reduce the problem of prediction for processes with stationary increments of order n to the problem of prediction in the case of ordinary stationary increments. Stochastic processes with stationary increments of order n were also studied by Pinsker [12] and Yaglom and Pinsker [11].

The classical methods for solving the problems of extrapolation, interpolation, and filtration for stationary processes were proposed by Kolmogorov [5], Wiener [15], and Yaglom [16, 17] for the case where the spectral densities of the processes are known. If the spectral density is unknown, but a family of admissible spectral densities is given, one applies the minimax method for solving the problems of estimation. The minimax method is to determine an estimator that minimizes the error of estimation for all densities from the given family. This method was proposed by Grenander [1] to solve the problem of extrapolation of stationary processes. Franke [2] and Franke and Poor [3] have studied the problems of minimax extrapolation and interpolation of stationary sequences in the framework of the theory of convex optimization. Moklyachuk in [7]–[10] studied

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the problems of extrapolation, interpolation, and filtration for stationary processes and sequences.

The problem of the optimal estimation of the functional

$$A_N\xi = \sum_{k=0}^{N} a(k)\xi(k)$$

that depends on unknown values of the stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ with stationary increments of order n from observations after this sequence at the moments $m \in \mathbb{Z} \setminus \{0, 1, \dots, N\}$ is studied in the current paper. We solve the problem of interpolation of stochastic sequences with stationary increments of order n if the spectral density of the sequence $\{\xi(m), m \in \mathbb{Z}\}$ is known. If the spectral density is unknown but a family of admissible spectral densities is given, we find the sets of least favorable spectral densities and formulas for evaluating the minimax spectral characteristics of the optimal linear estimate of the above functional.

2. Stationary increments. Spectral decomposition

Definition 2.1. The function

$$\xi^{(n)}(m,\mu) = (1 - B_\mu)^n \xi(m) = \sum_{l=0}^{n} (-1)^l C_n^l \xi(m - l\mu) \tag{1}$$

is called the stochastic increment of order n with step $\mu \in \mathbb{Z}$ for the stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ where B_μ is the shift operator corresponding to step μ defined by

$$B_\mu \xi(m) = \xi(m - \mu), \qquad m, \mu \in \mathbb{Z}.$$

It is clear that

$$\xi^{(n)}(m, -\mu) = (-1)^n \xi^{(n)}(m + n\mu, \mu), \tag{2}$$

$$\xi^{(n)}(m, k\mu) = \sum\nolimits_{l=0}^{(k-1)n} A_l \xi^{(n)}(m - l\mu, \mu), \qquad k \in \mathbb{N}, \tag{3}$$

where $\{A_l, l = 0, 1, 2, \dots, (k-1)n\}$ are the coefficients for the corresponding terms x^l in the decomposition of the polynomial $(1 + x + \dots + x^{k-1})^n$ with respect to the powers of the argument x.

Definition 2.2. The stochastic increment $\xi^{(n)}(m,\mu)$ of order n for a stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ is called stationary (in the wide sense) if the expectations

$$\mathsf{E}\,\xi^{(n)}(m_0, \mu) = c^{(n)}(\mu)$$

and

$$\mathsf{E}\,\xi^{(n)}(m_0 + m, \mu_1)\xi^{(n)}(m_0, \mu_2) = D^{(n)}(m, \mu_1, \mu_2)$$

exist for all integer numbers m_0, μ, m, μ_1, and μ_2, and, moreover, both expectations do not depend on m_0. The function $c^{(n)}(\mu)$ is called the mean value of the stationary increment of order n, while the function

$$D^{(n)}(m, \mu_1, \mu_2)$$

is called the structural function of the stationary increment of order n (or the structural function of order n for the stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$).

The stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ defining the stationary increment $\xi^{(n)}(m,\mu)$ of order n according to equality (1) is called a sequence with stationary increments of order n.

Theorem 2.1. *The mean value $c^{(n)}(\mu)$ and structural function $D^{(n)}(m,\mu_1,\mu_2)$ of a stochastic stationary increment $\xi^{(n)}(m,\mu)$ of order n can be represented in the form*

$$c^{(n)}(\mu) = c\mu^n, \tag{4}$$

$$D^{(n)}(m,\mu_1,\mu_2) = \int_{-\pi}^{\pi} e^{i\lambda m}(1-e^{-i\mu_1\lambda})^n(1-e^{i\mu_2\lambda})^n \frac{1}{\lambda^{2n}}\, dF(\lambda), \tag{5}$$

where c is a constant, the function $F(\lambda)$ is left continuous, bounded, and such that $F(-\pi)=0$. Moreover, the constant c and function $F(\lambda)$ are uniquely defined by the increment $\xi^{(n)}(m,\mu)$.

On the other hand, if a function $c^{(n)}(\mu)$ is of the form (4) with some constant c and if a function $D^{(n)}(m,\mu_1,\mu_2)$ is represented in the form of (5), where $F(\lambda)$ satisfies the above condition, then $c^{(n)}(\mu)$ is the mean value and $D^{(n)}(m,\mu_1,\mu_2)$ is the structural function of a certain stationary increment $\xi^{(n)}(m,\mu)$ of order n.

Using representation (5) of the structural function of a stationary increment $\xi^{(n)}(m,\mu)$ of order n together with Karhunen's theorem [4], we obtain a representation of a stationary increment $\xi^{(n)}(m,\mu)$ of order n written as follows:

$$\xi^{(n)}(m,\mu) = \int_{-\pi}^{\pi} e^{im\lambda}\left(1-e^{-i\mu\lambda}\right)^n \frac{1}{(i\lambda)^n}\, dZ(\lambda), \tag{6}$$

where $Z(\lambda)$ is a stochastic orthogonal measure on the interval $[-\pi,\pi)$ subordinated to the structural measure generated by the function $F(\lambda)$, namely

$$\mathsf{E}\, Z(A_1)\overline{Z(A_2)} = F(A_1\cap A_2) < \infty. \tag{7}$$

We use the spectral representation (6) to find the optimal linear estimator of unknown values of a stochastic sequence.

3. The problem of interpolation

Let a stochastic sequence $\{\xi(m), m\in\mathbb{Z}\}$ define the stationary increment $\xi^{(n)}(m,\mu)$ of order n with an absolutely continuous spectral function $F(\lambda)$ whose spectral density is $f(\lambda)$; that is,

$$F(\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\lambda} f(\lambda)\, d\lambda.$$

Assume that the observations after the sequence $\xi(m)$ at points $m\in\mathbb{Z}\setminus\{0,1,2,\dots,N\}$ are known. Consider the optimal (in the mean square sense) linear estimation of the functional

$$A_N\xi = \sum_{k=0}^{N} a(k)\xi(k)$$

that depends on the unknown sequence $\xi(m)$.

After some algebra, we obtain from (1) that

$$\xi(k) = \frac{1}{(1-B_\mu)^n}\xi^{(n)}(k,\mu) = \sum_{j=-\infty}^{k} d_\mu(k-j)\xi^{(n)}(j,\mu), \tag{8}$$

where $\{d_\mu(k)\colon k\ge 0\}$ are the coefficients for the terms x^k in the decomposition

$$\sum_{k=0}^{\infty} d_\mu(k)x^k = \left(\sum_{j=0}^{\infty} x^{\mu j}\right)^n.$$

Relations (8) and (1) imply that

$$\sum_{k=0}^{N} a(k)\xi(k) = -\sum_{i=-\mu n}^{-1} v(i)\xi(i) + \sum_{i=0}^{N}\left(\sum_{k=i}^{N} a(k)d_\mu(k-i)\right)\xi^{(n)}(i,\mu),$$

$$\sum_{k=0}^{N} b_\mu(k)\xi^{(n)}(k,\mu) = \sum_{i=-\mu n}^{-1}\xi(i)\sum_{l=\left[-\frac{i}{\mu}\right]'}^{n}(-1)^l C_n^l b_\mu(l\mu+i) + \sum_{i=0}^{N}\xi(i)\sum_{l=0}^{n}(-1)^l C_n^l b_\mu(l\mu+i),$$

where $[x]'$ is the least integer number being larger than or equal to x. The preceding two relations allow us to write the following representation of the functional $A_N\xi$ in the form of the difference of two functionals $A_N\xi = B_N\xi - V_N\xi$, where

$$B_N\xi = \sum_{k=0}^{N} b_\mu(k)\xi^{(n)}(k,\mu), \qquad V_N\xi = \sum_{k=-\mu n}^{-1} v(k)\xi(k),$$

$$v(k) = \sum_{l=\left[-\frac{k}{\mu}\right]'}^{n}(-1)^l C_n^l b_\mu(l\mu+k), \qquad k=-1,-2,\dots,-\mu n, \tag{9}$$

$$b_\mu(k) = \sum_{m=k}^{N} a(m)d_\mu(m-k) = \left(D_N^\mu a^{(1)}\right)_k, \qquad k=0,1,\dots,N. \tag{10}$$

Here $a^{(1)} = (a(0),a(1),a(2),\dots,a(N))$ and D_N^μ is an $(N+1)\times(N+1)$ matrix with the entries $D_{k,j}^\mu$ such that

$$D_{k,j}^\mu = \begin{cases} d_\mu(j-k), & \text{if } 0\le k\le j\le N,\\ 0, & \text{if either } j<k \text{ or } j,k>N.\end{cases}$$

Denote by $\widehat{A}_N\xi$ the mean square optimal linear estimator of the functional $A_N\xi$ constructed from observations after the stochastic sequence $\xi(m)$ at points belonging to the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$ and by $\widehat{B}_N\xi$ the mean square optimal linear estimator of the functional $B_N\xi$ constructed from observations after the stochastic increment $\xi^{(n)}(k,\mu)$ of order n at the points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N+\mu n\}$. Denote by $\Delta(f,\widehat{A}_N) = \mathsf{E}\,|A_N\xi-\widehat{A}_N\xi|^2$ the mean square error of the estimator $\widehat{A}_N\xi$ and by $\Delta(f,\widehat{B}_N)$ the mean square error of the estimator $\widehat{B}_N\xi$. Since the values of the sequence $\xi(m)$ are known at points $-1,-2,\dots,-\mu n$, we obtain

$$\widehat{A}_N\xi = \widehat{B}_N\xi - V_N\xi. \tag{11}$$

Thus

$$\begin{aligned}\Delta(f,\widehat{A}_N) &= \mathsf{E}\,|A_N\xi-\widehat{A}_N\xi|^2 = \mathsf{E}\,|A_N\xi+V_N\xi-\widehat{B}_N\xi|^2 = \mathsf{E}\,|B_N\xi-\widehat{B}_N\xi|^2\\ &= \Delta(f,\widehat{B}_N).\end{aligned}$$

We apply the method of orthogonal projections in Hilbert spaces introduced by Kolmogorov [5] to find the optimal (in the mean square sense) linear estimator of the functional $B_N\xi$.

Denote by $H^{0-}\big(\xi_\mu^{(n)}\big)$ the linear closed subspace generated by the random variables $\big\{\xi^{(n)}(k,\mu)\colon k\le -1\big\}$ in the space $H = L_2(\Omega,\mathcal{F},\mathsf{P})$ of second order random variables and by $H^{N+}\big(\xi_{-\mu}^{(n)}\big)$ the linear closed subspace in the space H generated by the random variables $\big\{\xi^{(n)}(k,-\mu)\colon k\ge N+1\big\}$. Since

$$e^{i\lambda k}\left(1-e^{i\lambda\mu}\right)^n = (-1)^n e^{i\lambda(k+\mu n)}\left(1-e^{-i\lambda\mu}\right)^n,$$

we get $\xi^{(n)}(k,-\mu)=(-1)^n\xi^{(n)}(k+\mu n,\mu)$. Hence $H^{N+}\big(\xi^{(n)}_{-\mu}\big)=H^{(N+\mu n)+}\big(\xi^{(n)}_{\mu}\big)$. Then we introduce the subspaces $L_2^{0-}(f)$ and $L_2^{N+}(f)$ in the space $L_2(f)$ generated by the functions

$$\left\{e^{i\lambda k}(1-e^{-i\lambda\mu})^n\frac{1}{(i\lambda)^n}:k\leq -1\right\}$$

and

$$\left\{e^{i\lambda k}(1-e^{-i\lambda\mu})^n\frac{1}{(i\lambda)^n}:k\geq N+1\right\},$$

respectively.

Equality (6) implies that there exists a one-to-one correspondence between elements $\xi^{(n)}(k,\mu)$ of the space H and elements $e^{i\lambda k}(1-e^{-i\lambda\mu})^n/(i\lambda)^n$ of the space $L_2(f)$.

We search for a linear estimator $\widehat{B}_N\xi$ for $B_N\xi$ represented in the form

$$\widehat{B}_N\xi=\int_{-\pi}^{\pi}h_\mu(\lambda)\,dZ(\lambda), \tag{12}$$

where $h_\mu(\lambda)$ is the spectral characteristic of the estimator.

The optimal estimator $\widehat{B}_N\xi$ is the projection of the element $B_N\xi$ of the space H to the subspace

$$H^{0-}\big(\xi^{(n)}_{\mu}\big)\oplus H^{N+}\big(\xi^{(n)}_{-\mu}\big)=H^{0-}\big(\xi^{(n)}_{\mu}\big)\oplus H^{(N+\mu n)+}\big(\xi^{(n)}_{\mu}\big).$$

The spectral characteristic $h_\mu(\lambda)$ of the optimal estimator is determined from the following two conditions:

1) $h_\mu(\lambda)\in L_2^{0-}(f)\oplus L_2^{(N+\mu n)+}(f)$;

2) $(B_N^\mu(e^{i\lambda})(1-e^{-i\lambda\mu})^n\frac{1}{(i\lambda)^n}-h_\mu(\lambda))\perp L_2^{0-}(f)\oplus L_2^{(N+\mu n)+}(f)$, where

$$B_N^\mu\big(e^{i\lambda}\big)=\sum_{k=0}^{N}b_\mu(k)e^{i\lambda k}.$$

Condition 2) implies that, for all $k\leq -1$ and for all $k\geq N+\mu n+1$, the function $h_\mu(\lambda)$ is such that

$$\int_{-\pi}^{\pi}\left(B_N^\mu(e^{i\lambda})\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}-h_\mu(\lambda)\right)e^{-i\lambda k}\left(1-e^{i\lambda\mu}\right)^n\frac{1}{(-i\lambda)^n}f(\lambda)\,d\lambda=0.$$

This implies that the spectral characteristic $h_\mu(\lambda)$ of the estimator $\widehat{B}_N\xi$ is of the following form:

$$h_\mu(\lambda)=B_N^\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}-\frac{(-i\lambda)^nC_N^\mu\left(e^{i\lambda}\right)}{(1-e^{i\lambda\mu})^nf(\lambda)},$$

$$C_N^\mu\left(e^{i\lambda}\right)=\sum_{k=0}^{N+\mu n}c_\mu(k)e^{i\lambda k},$$

where $c_\mu(k)$ are unknown coefficients. Our current goal is to determine these coefficients.

Condition 1) implies that the function $h_\mu(\lambda)$ is of the form

$$h_\mu(\lambda)=h(\lambda)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n},\qquad h(\lambda)=\sum_{k=-\infty}^{-1}s(k)e^{i\lambda k}+\sum_{k=N+\mu n+1}^{\infty}s(k)e^{i\lambda k}$$

and that $h_\mu(\lambda)$ satisfies the following relations:

$$\int_{-\pi}^{\pi} |h(\lambda)|^2 \left|1-e^{i\lambda\mu}\right|^2 \frac{f(\lambda)}{\lambda^{2n}}\, d\lambda < \infty,$$

$$\frac{(i\lambda)^n h_\mu(\lambda)}{(1-e^{-i\lambda\mu})^n} \in L_2^{0-} \oplus L_2^{(N+\mu n)+},$$

$$\int_{-\pi}^{\pi} \left(B_N^\mu\left(e^{i\lambda}\right) - \frac{\lambda^{2n} C_N^\mu\left(e^{i\lambda}\right)}{(1-e^{-i\lambda\mu})^n(1-e^{i\lambda\mu})^n f(\lambda)} \right) e^{-i\lambda l}\, d\lambda = 0, \tag{13}$$
$$l = 0,1,\dots,N+\mu n.$$

Assume that

$$\int_{-\pi}^{\pi} \frac{1}{f(\lambda)}\, d\lambda < \infty, \qquad \int_{-\pi}^{\pi} \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n} f(\lambda)}\, d\lambda < \infty. \tag{14}$$

Then the Fourier coefficients of the function $\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n} f(\lambda)}$ are given by

$$f_\mu(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda^{2n} e^{-i\lambda k}}{|1-e^{i\lambda\mu}|^{2n} f(\lambda)}\, d\lambda, \qquad |k| = 0,1,\dots,N+\mu n.$$

The spectral density $f(\lambda)$ is such that $f(\lambda) = f(-\lambda)$, whence we conclude that the Fourier coefficients $f_\mu(k)$ are such that

$$f_\mu(k) = f_\mu(-k), \qquad k = 0,1,\dots,N+\mu n.$$

Put $b_\mu(l) = 0$, $l = N+1, N+2, \dots, N+\mu n$. Then, from equality (13), we obtain the following system of equations:

$$b_\mu(l) = \sum_{k=0}^{N+\mu n} f_\mu(l-k) c_\mu(k), \qquad l = 0,1,\dots,N+\mu n. \tag{15}$$

The latter system of equations can be rewritten as

$$b_\mu = \mathbf{F} c_\mu, \tag{16}$$

where b_μ is the vector of dimension $(N+\mu n+1)$ formed by elements

$$b_\mu(l), \qquad l = 0,\dots,N+\mu n;$$

c_μ is the vector of dimension $(N+\mu n+1)$ formed by elements $c_\mu(l)$, $l = 0,\dots,N+\mu n$; and where $\mathbf{F}$ is the $(N+\mu n+1)\times(N+\mu n+1)$ matrix with entries $F_{l,k} = f_\mu(l-k)$, $l,k = 0,\dots,N+\mu n$.

Now we show that the matrix $\mathbf{F}$ possesses the inverse. Using the vector

$$\widetilde{b}_\mu = (\widetilde{b}_\mu(0), \widetilde{b}_\mu(1), \dots, \widetilde{b}_\mu(N+\mu n))$$

instead of b_μ in relation (16), we reduce the problem to that of the construction of the projection $\widehat{B}_{N+\mu n}$ of the element

$$B_{N+\mu n} = \sum_{k=0}^{N+\mu n} \widetilde{b}_\mu(k) \xi^{(n)}(k,\mu)$$

belonging to the space H to the subspace $H^{0-}\left(\xi_\mu^{(n)}\right) \oplus H^{(N+\mu n)+}\left(\xi_\mu^{(n)}\right)$. Since the space $H^{0-}\left(\xi_\mu^{(n)}\right) \oplus H^{(N+\mu n)+}\left(\xi_\mu^{(n)}\right)$ is closed and convex, the projection is uniquely defined for an arbitrary set of numbers $\widetilde{b}_\mu(0), \widetilde{b}_\mu(1), \dots, \widetilde{b}_\mu(N+\mu n)$. Since not all of them are equal to zero, system (16) has a unique solution for an arbitrary vector $\widetilde{b}_\mu \neq 0$. This implies that the matrix $\mathbf{F}$ is nonsingular and possesses the inverse $\mathbf{F}^{-1}$.

We derive from equality (16) that the unknown coefficients $c_\mu(k)$, $k=0,\dots,N+\mu n$, can be written as

$$c_\mu(k)=\left(\mathbf{F}^{-1}b_\mu\right)_k,$$

where $(\mathbf{F}^{-1}b_\mu)_k$ is the coordinate k of the vector $\mathbf{F}^{-1}b_\mu$. Therefore, the spectral characteristic $h_\mu(\lambda)$ of the optimal estimator $\widehat{B}_N\xi$ of the functional $B_N\xi$ is given by

$$h_\mu(\lambda)=B_N^\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}-\frac{(-i\lambda)^n}{(1-e^{i\lambda\mu})^n f(\lambda)}\sum_{k=0}^{N+\mu n}\left(\mathbf{F}^{-1}b_\mu\right)_k e^{i\lambda k}. \tag{17}$$

The mean square error of the estimator is evaluated according to

$$\Delta(f,\widehat{B}_N)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}f(\lambda)}\left|\sum_{k=0}^{N+\mu n}(\mathbf{F}^{-1}b_\mu)_k e^{i\lambda k}\right|^2 d\lambda=\langle\mathbf{F}^{-1}b_\mu,b_\mu\rangle. \tag{18}$$

Therefore the following result holds.

Theorem 3.1. *Let a stochastic sequence $\{\xi(m),m\in\mathbb{Z}\}$ determine a stationary increment $\xi^{(n)}(m,\mu)$ of order n for which the spectral function $F(\lambda)$ is absolutely continuous with the spectral density $f(\lambda)$ satisfying condition* (14). *Then the optimal linear estimator $\widehat{B}_N\xi$ of the functional $B_N\xi$ that depends on unknown elements $\xi^{(n)}(m,\mu)$, $m\in\{0,1,2,\dots,N\}$, $\mu>0$, and constructed from known observations after the sequence $\xi(m)$ at the points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$, is given by equality* (12). *The spectral characteristic $h_\mu(\lambda)$ of the optimal estimator $\widehat{B}_N\xi$ is evaluated by formula* (17), *while the mean square error $\Delta(f,\widehat{B}_N)$ is evaluated by formula* (18).

Theorem 3.1 allows one to construct an estimator of unknown value of the increment $\xi^{(n)}(m,\mu)$, $m\in\{0,1,\dots,N\}$, constructed from observations after the sequence $\xi(m)$ at points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$. Consider the vector b_μ whose coordinate m equals 1, while all other coordinates equal zero. Substituting the vector b_μ to equality (17), the spectral characteristic of the estimator

$$\widehat{\xi}^{(n)}(m,\mu)=\int_{-\pi}^{\pi}\varphi_m(\lambda,\mu)\,dZ(\lambda) \tag{19}$$

becomes of the following form:

$$\varphi_m(\lambda,\mu)=e^{i\lambda m}\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}-\frac{(-i\lambda)^n}{(1-e^{i\lambda\mu})^n f(\lambda)}\sum_{k=0}^{N+\mu n}\left(\mathbf{F}^{-1}e_m\right)_k e^{i\lambda k}. \tag{20}$$

The mean square error is evaluated by

$$\begin{aligned}\Delta\big(f,\xi^{(n)}(m,\mu)\big)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}f(\lambda)}\left|\sum_{k=0}^{N+\mu n}(\mathbf{F}^{-1}e_m)_k e^{i\lambda k}\right|^2 d\lambda\\&=\langle\mathbf{F}^{-1}e_m,e_m\rangle=(\mathbf{F}^{-1})_{m,m}.\end{aligned} \tag{21}$$

Therefore we proved the following result.

Corollary 3.1. *The optimal linear estimator $\widehat{\xi}^{(n)}(m,\mu)$ of the unknown values of the increment $\xi^{(n)}(m,\mu)$, $m=0,1,2,\dots,N$, $\mu>0$, constructed from known observations after the sequence $\xi(m)$ at the points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$, is evaluated by equality* (19). *The spectral characteristic $\varphi(\lambda,\mu)$ of the optimal estimator $\widehat{\xi}^{(n)}(m,\mu)$ is evaluated by equality* (20), *while the mean square error $\Delta\big(f,\xi^{(n)}(m,\mu)\big)$ is evaluated by formula* (21).

Using equality (11) and Theorem 3.1 we construct the estimator

$$\widehat{A}_N\xi = -\sum_{k=-\mu n}^{-1} v(k)\xi(k) + \int_{-\pi}^{\pi} h_\mu^{(a)}(\lambda)\, dZ(\lambda) \tag{22}$$

of the functional $A_N\xi$. Denote by $[D_N^\mu a^{(1)}]_{+\mu n}$ the vector constructed by adding μn zero coordinates to the vector $D_N^\mu a^{(1)}$. Substituting the vector b_μ whose coefficients are defined by (10) to relations (17) and (18), we deduce the following formulas for evaluating the spectral characteristic and mean square error of the estimator:

$$\begin{aligned} h_\mu^{(a)}(\lambda) = {} & A_N^\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n \frac{1}{(i\lambda)^n} \\ & - \frac{(-i\lambda)^n}{(1-e^{i\lambda\mu})^n f(\lambda)} \sum_{k=0}^{N+\mu n} \left(\mathbf{F}^{-1}\left[D_N^\mu a^{(1)}\right]_{+\mu n}\right)_k e^{i\lambda k}, \end{aligned} \tag{23}$$

where $A_N^\mu(e^{i\lambda}) = \sum_{k=0}^{N}\left(D_N^\mu a^{(1)}\right)_k e^{i\lambda k}$ and

$$\begin{aligned} \Delta(f, \widehat{A}_N) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n} f(\lambda)} \left|\sum_{k=0}^{N+\mu n}\left(\mathbf{F}^{-1}\left[D_N^\mu a^{(1)}\right]_{+\mu n}\right)_k e^{i\lambda k}\right|^2 d\lambda \\ &= \left\langle \mathbf{F}^{-1}\left[D_N^\mu a^{(1)}\right]_{+\mu n}, \left[D_N^\mu a^{(1)}\right]_{+\mu n}\right\rangle. \end{aligned} \tag{24}$$

Thus we proved the following assertion.

Theorem 3.2. *Let a stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ determine the stationary increment $\xi^{(n)}(m,\mu)$ of order n with the absolutely continuous spectral function $F(\lambda)$ such that its spectral density $f(\lambda)$ satisfies condition* (14). *Then the optimal linear estimator $\widehat{A}_N\xi$ of the functional $A_N\xi$ that depends on unknown elements $\xi(m)$, $m \in \{0,1,2,\dots,N\}$, $\mu > 0$, constructed from known observations after the sequence $\xi(m)$ at the points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$, is evaluated by equality* (22), *where the coefficients $v(k)$, $k=-1,-2,\dots,-\mu n$, are given by* (9). *The spectral characteristic $h_\mu^a(\lambda)$ of the optimal estimator $\widehat{A}_N\xi$ is evaluated by formula* (23), *while the mean square error $\Delta(f,\widehat{A}_N)$ is evaluated by equality* (24).

Example 3.1. Consider a stochastic sequence $\{\xi(m)\colon m \in \mathbb{Z}\}$ with stationary increments of the second order. Assume that the second order increments with step $\mu = 1$ form an autoregressive sequence of the first order with parameter ψ. We find the mean square optimal linear estimator of the value of the functional $A_1\xi = a\xi(0) + b\xi(1)$ that depends on unknown values of the sequence by using the observations at the moments $m \in \mathbb{Z}\setminus\{0,1\}$. Then the spectral density of the random sequence $\xi(m)$ is given by

$$f(\lambda) = \frac{\lambda^4}{|1-e^{i\lambda}|^4|1-\psi e^{i\lambda}|^2},$$

while the spectral characteristic of the optimal estimator of the functional $A_1\xi$ is evaluated by the formula

$$h_1^{(a)}(\lambda) = \Psi^{-1}\left(\Psi_{-1}e^{-i\lambda} + \Psi_4 e^{3i\lambda}\right)\left(1-e^{-i\lambda}\right)^2 \frac{1}{(i\lambda)^2},$$

where $\Psi = 1 + \psi^2 + \psi^4 + \psi^6 + \psi^8$,

$$\Psi_{-1} = a\psi\left(1+\psi^2+\psi^4+\psi^6\right) + b\psi\left(2+\psi+2\psi^2+\psi^3+2\psi^4+\psi^5+2\psi^6\right),$$

and $\Psi_4 = a\psi^3(1+\psi^2) + b\psi^2(1+2\psi^1+2\psi^2+2\psi^3+\psi^4)$. The optimal estimator of the functional is given by

$$\begin{aligned}\widehat{A}_1\xi &= -(a+2b)\xi(-2) + (2a+3b)\xi(-1) + \Psi^{-1}\Psi_{-1}\xi^{(2)}(-1,1) + \Psi^{-1}\Psi_4\xi^{(2)}(4,1)\\ &= \varphi_{-3}\xi(-3) + \varphi_{-2}\xi(-2) + \varphi_{-1}\xi(-1) + \varphi_2\xi(2) + \varphi_3\xi(3) + \varphi_4\xi(4),\end{aligned}$$

$$\varphi_{-3} = \Psi^{-1}\Psi_{-1}, \qquad \varphi_{-2} = -\left(a+2b+2\Psi^{-1}\Psi_{-1}\right), \qquad \varphi_{-1} = \left(2a+3b+\Psi^{-1}\Psi_{-1}\right),$$

$$\varphi_2 = \Psi^{-1}\Psi_4, \qquad \varphi_3 = -2\Psi^{-1}\Psi_4, \qquad \varphi_4 = \Psi^{-1}.$$

4. Minimax interpolation

The square mean error $\Delta\big(h_\mu^{(a)}(f);f\big) := \Delta(f,\widehat{A}_N)$ and spectral characteristic $h_\mu^{(a)}$ of the optimal linear estimator $\widehat{A}_N\xi$ of the functional $A_N\xi$ that depends on unknown values of a sequence $\xi(m)$ with stationary increments of order n are evaluated by equalities (23) and (24), respectively, if the spectral density $f(\lambda)$ of the sequence $\xi(m)$ is known. In the case where the spectral density is unknown but a set $\mathcal{D}$ of admissible densities is given, one can use the minimax approach to the problem of estimation of the functional. According to this approach, one finds an estimator that minimizes the mean square error for all spectral densities belonging to the class $\mathcal{D}$.

Definition 4.1. Given a family of spectral densities $\mathcal{D}$, the spectral density $f_0(\lambda) \in \mathcal{D}$ is called the least favorable in $\mathcal{D}$ for the problem of optimal linear interpolation of the functional $A_N\xi$ if

$$\Delta(f_0) = \Delta\big(h_\mu^{(a)}(f_0);f_0\big) = \max_{f\in\mathcal{D}} \Delta\big(h_\mu^{(a)}(f);f\big).$$

Definition 4.2. Given a family of spectral densities $\mathcal{D}$, the spectral characteristic $h^0(\lambda)$ of the optimal estimator of the functional $A_N\xi$ is called minimax (robust) spectral density if

$$h^0(\lambda) \in H_{\mathcal{D}} = \bigcap_{f\in\mathcal{D}} L_2^{0-}(f) \oplus L_2^{(N+\mu n)+}(f),$$

$$\min_{h\in H_{\mathcal{D}}} \max_{f\in\mathcal{D}} \Delta(h;f) = \sup_{f\in\mathcal{D}} \Delta(h^0;f).$$

Lemma 4.1. *The spectral density $f_0 \in \mathcal{D}$ that satisfies condition* (14) *is least favorable in the class $\mathcal{D}$ for the problem of optimal linear interpolation of the functional $A_N\xi$ if the matrix $\mathbf{F}^0$ whose entries equal the Fourier coefficients of the function*

$$f_\mu^0(\lambda) = \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n} f^0(\lambda)} \tag{25}$$

determines the solution of the extremum problem

$$\max_{f\in\mathcal{D}} \left\langle \mathbf{F}^{-1}\left[D_N^\mu a^{(1)}\right]_{+\mu n}, \left[D_N^\mu a^{(1)}\right]_{+\mu n} \right\rangle = \left\langle (\mathbf{F}^0)^{-1}\left[D_N^\mu a^{(1)}\right]_{+\mu n}, \left[D_N^\mu a^{(1)}\right]_{+\mu n} \right\rangle.$$

If $h_\mu^{(a)}(f_0) \in H_{\mathcal{D}}$, then the minimax spectral characteristic is equal to $h^0 = h_\mu^{(a)}(f_0)$.

The functions h^0 and f_0 form a saddle point of the function $\Delta(h;f)$ in the set $H_{\mathcal{D}} \times \mathcal{D}$. The saddle point inequalities

$$\Delta(h;f_0) \geq \Delta\left(h^0;f_0\right) \geq \Delta\left(h^0;f\right) \qquad \forall f\in\mathcal{D},\ \forall h\in H_{\mathcal{D}}$$

hold if $h^0 = h_\mu^{(a)}(f_0)$ and $h_\mu^{(a)}(f_0) \in H_{\mathcal{D}}$, where f_0 is the solution of the conditional extremum problem

$$\widetilde{\Delta}(f) = -\Delta\big(h_\mu^{(a)}(f_0); f\big) \to \inf, \qquad f \in \mathcal{D}, \tag{26}$$

$$\Delta\big(h_\mu^{(a)}(f_0); f\big) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n} f_0^2(\lambda)} \left| \sum_{k=0}^{N+\mu n} \Big((\mathbf{F}^0)^{-1} \big[D_N^\mu a^{(1)}\big]_{+\mu n} \Big)_k e^{i\lambda k} \right|^2 f(\lambda)\, d\lambda.$$

The latter problem is equivalent to the unconditional extremum problem

$$\Delta_{\mathcal{D}}(f) = \widetilde{\Delta}(f) + \delta(f|\mathcal{D}) \to \inf,$$

whose solution f_0 is determined by the condition $0 \in \partial\Delta_{\mathcal{D}}(f_0)$ (see [13]).

5. Least favorable densities in the class $\mathcal{D}_{0,n}^-$

Consider the following set of spectral densities:

$$\mathcal{D}_{0,n}^- = \left\{ f(\lambda)\colon \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^{2n}}{f(\lambda)}\, d\lambda \geq P \right\}.$$

Since $0 \in \partial\Delta_{\mathcal{D}}(f_0)$, we derive the relation

$$\left| \sum_{k=0}^{N+\mu n} \Big((\mathbf{F}^0)^{-1} \big[D_N^\mu a^{(1)}\big]_{+\mu n} \Big)_k e^{i\lambda k} \right|^2 = p_0^2 \left|1 - e^{i\lambda\mu}\right|^{2n} \tag{27}$$

that determines the least favorable spectral density.

Denote by p^μ the vector of dimension $N + \mu n + 1$ with the coordinates

$$p_\mu(\mu l) = (-1)^l p_0 C_n^l$$

for $l = 0, 1, \ldots, n$, and let $p(k) = 0$, $k \neq \mu l$, for $l = 0, 1, \ldots, n$. Then the Fourier coefficients $f_\mu^0(k) = f_\mu^0(-k)$, $k = 0, 1, \ldots, N + \mu n$, of the function (25) found from the equation

$$\mathbf{F}^0 p^\mu = \big[D_N^\mu a^{(1)}\big]_{+\mu n} \tag{28}$$

satisfy (27).

We study the latter equation in more detail. Note that the matrix $\mathbf{F}^0$ is symmetric and that the last N coordinates of the vector p^μ are equal to 0.

We introduce the following notation. Let $\mathbf{d}$ be some vector of dimension p. Given $p_1 < p$, denote by $[\mathbf{d}]_{p_1}$ the vector of dimension p_1 whose coordinates coincide with the first p_1 coordinates of the vector $\mathbf{d}$. Let $\mathbf{D}$ be a $(p \times q)$ matrix. Given $p_1 < p$ and $q_1 < q$, denote by $[\mathbf{D}]_{p_1,q_1}$ the $(p_1 \times q_1)$ matrix whose entries are equal to the corresponding entries of the matrix $\mathbf{D}$. Let $[\mathbf{D}]_{\cdot,q_1}$ denote the matrix formed by the first q_1 columns of the matrix $\mathbf{D}$.

Consider the system

$$\big[\mathbf{F}^0\big]_{\cdot,\mu n+1} [p^\mu]_{\mu n+1} = \big[D_N^\mu a^{(1)}\big]_{+\mu n},$$

which is equivalent to system (28). We split the matrix $[\mathbf{F}^0]_{\cdot,\mu n+1}$ into two blocks,

$$\mathbf{F}_{(1)}^0 = \big[\mathbf{F}^0\big]_{\mu n+1,\mu n+1}$$

and $\mathbf{F}_{(2)}^0$. The block $\mathbf{F}_{(1)}^0$ is a symmetric matrix, while $[p^0]_{\mu n+1}$ is a symmetric vector (perhaps, with a correction by the factor -1). Thus there are $K = [(\mu n + 1)/2]$ equations among the first $\mu n + 1$ equations of the system (28) whose left hand sides are the same. Since the coefficients $a(k)$, $k = 0, 1, \ldots, N$, are arbitrary, the system (28) is inconsistent in general. We introduce additional coefficients to avoid this problem. Put $a^{(2)} = (a(N+1),\dots,a(N+\mu n))$ and consider the functional

$$A_{N+\mu n}\xi = \sum_{k=0}^{N+\mu n} a(k)\xi(k),$$

where the coefficients $a(0), a(1),\dots,a(N)$ are given, while $a(N+1),\dots,a(N+\mu n)$ are arbitrary. Then

$$\begin{aligned} A_{N+\mu n}\xi &= \sum_{k=0}^{N+\mu n} b_\mu\big(k;a^{(2)}\big)\xi^{(n)}(k,\mu) - \sum_{k=-\mu n}^{-1} v\big(k,a^{(2)}\big)\xi(k) \\ &= B_{N+\mu n}\xi - \sum_{k=-\mu n}^{-1} v\big(k,a^{(2)}\big)\xi(k), \end{aligned}$$

$$A_N\xi = -\sum_{k=N+1}^{N+\mu n} a(k)\xi(k) - \sum_{k=-\mu n}^{-1} v\big(k,a^{(2)}\big)\xi(k) + B_{N+\mu n}\xi,$$

where the entries $v(k,a^{(2)})$, $k=-1,-2,\dots,-\mu n$, are defined by relation (9). Since $\{\xi(k)\colon k=-\mu n,\dots,-1\}$ and $\{\xi(k)\colon k=N+1,\dots,N+\mu n\}$ are known,

$$\Delta(f,A_N) = \mathsf{E}\,|A_N\xi - \widehat{A}_N\xi|^2 = \mathsf{E}\,|B_{N+\mu n}\xi - \widehat{B}_{N+\mu n}\xi|^2 = \Delta(f,B_{N+\mu n}\xi).$$

We consider the unconditional extremum problems

$$\Delta_{\mathcal{D}}(f) = \widetilde{\Delta}(f) + \delta(f|\mathcal{D}) \to \inf,$$

$$\begin{aligned} \widetilde{\Delta}(f) &= -\Delta\big(h_\mu^{(a)}(f_0);f\big) \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n} f_0^2(\lambda)} \left|\sum_{k=0}^{N+\mu n} \Big(\big(\mathbf{F}^0\big)^{-1} b_\mu\big(a^{(2)}\big)\Big)_k e^{i\lambda k}\right|^2 f(\lambda)\,d\lambda \end{aligned}$$

to determine the least favorable spectral density.

With $\mathcal{D} = \mathcal{D}_{0,n}^-$, the condition $0 \in \partial\Delta_{\mathcal{D}}(f_0)$ implies that

$$\left|\sum_{k=0}^{N+\mu n} \Big(\big(\mathbf{F}^0\big)^{-1} b_\mu\big(a^{(2)}\big)\Big)_k e^{i\lambda k}\right|^2 = p_0^2\left|1-e^{i\lambda\mu}\right|^{2n}. \tag{29}$$

Consider the system of equations

$$\mathbf{F}^0 p^\mu = b_\mu\big(a^{(2)}\big), \tag{30}$$

where $b_\mu(a^{(2)}) = (b_\mu(0,a^{(2)}),\dots,b_\mu(N+\mu n,a^{(2)}))$ is the vector of dimension $N+\mu n+1$. System (30) is consistent if

$$b_\mu\big(i;a^{(2)}\big) = (-1)^{\mu n} b_\mu\big(\mu n - i;a^{(2)}\big), \qquad i=0,1,\dots,K-1. \tag{31}$$

Hence the Fourier coefficients of the function (25) found from system (30)–(31) satisfy equation (29). Systems (30)–(31) contain $N+\mu n+1$ equations and $N+2\mu n+2$ unknowns: $\{f_\mu(k)\colon k=0,1,\dots,N+\mu n\}$, $a^{(2)} = (a(k)\colon k=N+1,\dots,N+\mu n)$, and p_0. The coefficient p_0 is determined by the equation

$$\sum_{m=-N-\mu n}^{N+\mu n} f_\mu^0(|m|)\int_{-\pi}^{\pi}\left|1-e^{i\lambda\mu}\right|^{2n} e^{i\lambda m}\,d\lambda = 2\pi P. \tag{32}$$

Thus the coefficients $\{f_\mu(k)\colon k=0,1,\dots,N+\mu n\}$ and

$$a^{(2)}=(a(k)\colon k=N+1,\dots,N+\mu n)$$

depend on the parameter $\alpha=(\alpha(1),\dots,\alpha(\mu n))$.

Let

$$\mathcal{B}=\{\alpha\in\mathbb{R}^{\mu n}\colon (f_\mu(0;\alpha)\dots,f_\mu(N+\mu n;\alpha)) \text{ is a strictly positive sequence}\}.$$

Generally speaking, the set $\mathcal{B}$ depends on the family $(a(0),a(1),\dots,a(N))$. The least favorable density is given by

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}f^0(\lambda)}=\sum_{m=-N-\mu n}^{N+\mu n}f^0_\mu(|m|;\alpha)e^{i\lambda m}=\left|\sum_{k=0}^{N+\mu n}\gamma_\mu(k;\alpha)e^{-i\lambda k}\right|^2,\qquad \alpha\in\mathcal{B}.$$

Thus the class of least favorable densities is

$$\mathcal{R}=\Bigg\{f^0(\lambda)\in\mathcal{D}^-_{0,n}\colon f^0(\lambda)=\lambda^{2n}\Bigg(|1-e^{i\lambda\mu}|^{2n}\sum_{m=-N-\mu n}^{N+\mu n}f^0_\mu(|m|;\alpha)e^{i\lambda m}\Bigg)^{-1},\ \alpha\in\mathcal{B}\Bigg\}. \tag{33}$$

The minimax estimator of the functional $A_N\xi$ is evaluated by

$$\widehat{A}_N\xi=-\sum_{k=N+1}^{N+\mu n}a(k;\alpha)\xi(k)-\sum_{k=-\mu n}^{-1}v\big(k,a^{(2)}(\alpha)\big)\xi(k)+\int_{-\pi}^{\pi}h^{(a)}_{\mu;\alpha}(\lambda)\,d\lambda,\qquad \alpha\in\mathcal{B}, \tag{34}$$

$$\begin{aligned}h^{(a)}_{\mu;\alpha}(\lambda)&=A_{N+\mu n}\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\\&-\frac{(-i\lambda)^n\sum_{k=0}^{N+\mu n}\left(\left(\mathbf{F}^0\right)^{-1}D^\mu_{N+\mu n}\mathbf{a}\right)_k e^{i\lambda k}}{(1-e^{i\lambda\mu})^n f(\lambda)},\end{aligned} \tag{35}$$

where

$$A_{N+\mu n}\left(e^{i\lambda}\right)=\sum_{k=0}^{N+\mu n}\left(D^\mu_{N+\mu n}\mathbf{a}\right)_k e^{i\lambda k},$$

$a^{(2)}(\alpha)=(a(k;\alpha)\colon k=N+1,\dots,N+\mu n)$, and where the vector $\mathbf{a}$ is a result of merging two vectors $a^{(1)}$ and $a^{(2)}(\alpha)$.

Theorem 5.1. *Let the coefficients* $\{f^0_\mu(k;\alpha)\colon k=0,\dots,N+\mu n,\alpha\in\mathcal{B}\}$ *be determined by systems* (30), (31), (32). *If the sequence* $(a(0),a(1),\dots,a(N))$ *is such that the set* $\mathcal{B}$ *is nonempty and if system* (31) *is consistent, then the set of least favorable densities in the class* $\mathcal{D}^-_{0,n}$ *for the construction of the linear estimator of the functional* $A_N\xi$ *from observations after the sequence at the moments* $\mathbb{Z}\setminus\{0,1,\dots,N\}$ *is given by* (33). *The minimax estimator of the functional* $A_N\xi$ *is evaluated by equality* (34), *while the minimax spectral characteristic is evaluated by formula* (35).

Example 5.1. We construct the minimax estimator of the functional

$$A_1\xi=\xi(0)+2\xi(1)$$

from observations after the process at the moments $\mathbb{Z}\setminus\{0,1\}$ where $\{\xi(m)\colon m\in\mathbb{Z}\}$ is a random sequence with stationary increments of the second order. Consider the increments corresponding to the step $\mu = 1$. Then the functional $A_1\xi$ can be written as follows:

$$A_1\xi = -a(2)\xi(2) - a(3)\xi(3) - v(-2)\xi(-2) - v(-1)\xi(-1) + \sum_{k=0}^{3} b_1(k)\xi^{(2)}(k,1),$$

where $b_1(0) = 5 + 3a(2) + 4a(3)$; $b_1(1) = 2 + 2a(2) + 3a(3)$; $b_1(2) = a(2) + 2a(3)$; $b_1(3) = a(3)$; $v(-1) = -4 - 3a(2) - 4a(3)$; $v(-2) = a(2) + 2a(3)$. Then the system for evaluating the least favorable spectral density is given by

$$\begin{cases} b_1(0) = p_0(f_1^0(0) - 2f_1^0(1) + f_1^0(2)), \\ b_1(1) = p_0(f_1^0(1) - 2f_1^0(0) + f_1^0(1)), \\ b_1(3) = p_0(f_1^0(3) - 2f_1^0(2) + f_1^0(1)), \\ b_1(0) = b_1(2). \end{cases}$$

The last equation of the system determines the following subspace of possible combinations of the coefficients $a^{(2)} = (a(2), a(3))$:

$$\mathcal{L} = \left\{(a(2), a(3)) \in \mathbb{R}^2 \colon 2a(2) + 2a(3) + 5 = 0\right\}.$$

Put $f_1^0(0) = \alpha(1)$ and $a(3) = \alpha(2)$. Then the above system yields

$$f_1^0(0) = \alpha(1), \qquad f_1^0(1) = \frac{-11 - 2\alpha(2)}{4p_0} + \alpha(1), \qquad f_1^0(2) = \frac{-21 - 4\alpha(2)}{2p_0} + \alpha(1),$$
$$f_1^0(3) = \frac{-83 - 18\alpha(2)}{4p_0} + \alpha(1).$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^4}{f^0(\lambda)}\, d\lambda = P,$$

we get $p_0 = P^{-1}$. The least favorable density is of the following form:

$$\begin{aligned} f^0(\lambda) = \lambda^4|1 - e^{i\lambda}|^{-4}\big(\alpha(1) &+ (2\alpha(1) - P\alpha(2) - 5.5P)\cos\lambda \\ &+ (2\alpha(1) - 4P\alpha(2) - 21\pi P)\cos 2\lambda \\ &+ (2\alpha(1) - 9P\alpha(2) - 41.5P)\cos 3\lambda\big)^{-1}, \\ \alpha = (\alpha(1), \alpha(2)) &\in \mathcal{B}. \end{aligned}$$

Consider, for example, the case where the coefficient $a(3)$ is equal to $-\frac{5}{4}$. Then $a(2) = -\frac{5}{4}$. In a particular case of $P = \frac{1}{4}$, the minimax spectral characteristic is evaluated by

$$\begin{aligned} h^{(a)}_{\mu;\alpha}(\lambda) = \Bigg(\frac{-121 + 2\alpha(1)}{8}e^{-3i\lambda} - \frac{-89 + \alpha(1)}{4}e^{-2i\lambda} + \frac{-5}{4}e^{-i\lambda} \\ - \frac{-89 + \alpha(1)}{4}e^{4i\lambda} + \frac{-121 + 2\alpha(1)}{8}e^{5i\lambda}\Bigg)\frac{(1 - e^{-i\lambda})^2}{(i\lambda)^2}. \end{aligned}$$

Therefore the mean square error is equal to $\Delta(h^{(a)}_{\mu;\alpha}, f^0) = 1/4$.

6. Least favorable densities in the class $\mathcal{D}^-_{M,n}$

Consider the set of spectral densities

$$\mathcal{D}^-_{M,n} = \left\{f(\lambda) \colon \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^{2n}}{f(\lambda)}\cos(m\lambda)\, d\lambda = r(m), m = 0, 1, \dots, M\right\},$$

where $r(m)$, $m=0,1,\dots,M$, is a strictly positive sequence. Since $0\in\partial\Delta_{\mathcal{D}}(f^0)$, we get the equation

$$\left|\sum_{k=0}^{N+\mu n} c(k)e^{i\lambda k}\right|^2 = \left|1-e^{i\lambda\mu}\right|^{2n}\sum_{m=0}^{M}\alpha_m\cos m\lambda = \left|\left(1-e^{i\lambda\mu}\right)^n\sum_{m=0}^{M}p(m)e^{i\lambda m}\right|^2, \tag{36}$$

where α_m, $m=0,1,\dots,M$, are the Lagrange multipliers and where

$$c(k),\qquad k=0,1,\dots,N+\mu n,$$

are solutions of the equation $\mathbf{F}^0c=b_\mu(a^{(2)})$.

Consider separately the following two cases, $M>N$ and $M\le N$.

First let $M>N$. We show that

$$\begin{aligned} f^0(\lambda) &= \left(\frac{\left|1-e^{i\lambda\mu}\right|^{2n}}{\lambda^{2n}}\sum_{k=-M-\mu n}^{M+\mu n} f^0_\mu(|k|)e^{i\lambda k}\right)^{-1}\\ &= \left|\frac{(i\lambda)^n}{(1-e^{-i\lambda\mu})^n\sum_{k=0}^{M+\mu n}\gamma_\mu(k)e^{-i\lambda k}}\right|^2 \end{aligned}\tag{37}$$

is the least favorable density.

The restrictions imposed on the moments of the function $\frac{\lambda^{2n}}{f^0(\lambda)}$ imply the following relations:

$$\frac{1}{2\pi}\sum_{k=-m-\mu n}^{m+\mu n} f^0_\mu(|k|)\int_{-\pi}^{\pi}\cos(\lambda m)\left|1-e^{I\lambda\mu}\right|^{2n}e^{I\lambda k}\,d\lambda = r(m),\qquad m=0,1,\dots,M.$$

Put

$$u_\mu(m,k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\lambda m)\left|1-e^{i\lambda\mu}\right|^{2n}e^{i\lambda k}\,d\lambda,\qquad k,m=0,1,\dots,M+\mu n.$$

Then

$$f^0_\mu(0)u_\mu(m,0)+2\sum_{k=1}^{M+\mu n} f^0_\mu(k)u_\mu(m,k)=2\pi r(m),\qquad m=0,1,\dots,M. \tag{38}$$

Equality (36) holds if

$$\sum_{k=0}^{N+\mu n} c(k)e^{i\lambda k} = \left(1-e^{i\lambda\mu}\right)^n\sum_{m=0}^{M}p(m)e^{i\lambda m} = \sum_{k=0}^{M+\mu n} w_\mu(k)e^{i\lambda k}, \tag{39}$$

where

$$w_\mu(k)=\sum_{l=\max\left\{0,\left[\frac{k-M}{\mu}\right]'\right\}}^{\min\left\{n,\left[\frac{k}{\mu}\right]\right\}}(-1)^lC_n^l p(k-\mu l).$$

Since $M>N$, relation (39) implies that $w_\mu(k)=c(k)$ for $k=0,1,\dots,N+\mu n$ and $w_\mu(k)=0$ for $k=N+\mu n+1,\dots,M+\mu n$, that is,

$$\mathbf{F}^0w=b_\mu\left(a^{(2)}\right),\qquad w=(w_\mu(0),\dots,w_\mu(N+\mu n)), \tag{40}$$

$$\sum_{l=\max\left\{0,\left[\frac{k-M}{\mu}\right]'\right\}}^{\min\left\{n,\left[\frac{k}{\mu}\right]\right\}}(-1)^lC_n^l p(k-\mu l)=0,\qquad k=N+\mu n+1,\dots,M+\mu n. \tag{41}$$

Solving system (40) we obtain the relations for the coefficients $b_\mu(i;a^{(2)})$ (similar to relations (31)) that define the linear subspace $\mathcal{L}_1$ of the space $\mathbb{R}^{\mu n}$ of parameters $a^{(2)}$.

Systems (38), (40), (41) contain $2M+\mu n+2$ equations for $2(M+\mu n+1)$ unknowns: $\{f_\mu(k)\colon k=0,1,\dots,M+\mu n\}$, $a^{(2)}=(a(k)\colon k=N+1,\dots,N+\mu n)$, and

$$\{p(k)\colon k=0,1,\dots,M\}.$$

Thus the solutions depend on the parameter $\alpha=(\alpha(1),\dots,\alpha(\mu n))$:

$$\{f_\mu(k;\alpha)\colon k=0,1,\dots,M+\mu n,\alpha\in\mathbb{R}^{\mu n}\}$$

and $a^{(2)}(\alpha)=(a(k;\alpha)\colon k=N+1,\dots,N+\mu n,\alpha\in\mathbb{R}^{\mu n})$.

Let $\mathcal{B}_1$ be the set of parameters $\alpha\in\mathbb{R}^{\mu n}$ such that the sequence of solutions

$$\{f_\mu(k;\alpha)\colon k=0,\dots,N+\mu n\}$$

of systems (38), (40), (41) is strictly positive.

Now we consider the case of $M\le N$. Let $w^N=(w_\mu(0),\dots,w_\mu(M+\mu n),0\dots,0)$ be a vector of dimension $N+\mu n+1$. Then (39) implies that

$$\mathbf{F}^0 w^N = b_\mu\big(a^{(2)}\big). \tag{42}$$

Using the restrictions imposed on the moments of the function

$$\frac{\lambda^{2n}}{f^0(\lambda)}=\big|1-e^{i\lambda\mu}\big|^{2n}\sum_{k=-N-\mu n}^{N+\mu n} f^0_\mu(|k|)e^{i\lambda k}$$

we obtain $M+1$ equations for the coefficients $p(m)$:

$$f^0_\mu(0)u_\mu(m,0)+2\sum_{k=1}^{N+\mu n} f^0_\mu(k)u_\mu(m,k)=2\pi r(m), \qquad m=0,1,\dots,M. \tag{43}$$

Solving system (42) we obtain relations for the coefficients $b_\mu(i;a^{(2)})$ (similar to relations (31)) that define the linear subspace $\mathcal{L}_2$ of the space $\mathbb{R}^{\mu n}$ of parameters $a^{(2)}$.

Systems (42), (43) contain $N+M+\mu n+2$ equations and $N+M+2\mu n+2$ unknowns: $\{f_\mu(k)\colon k=0,1,\dots,N+\mu n\}$, $a^{(2)}=(a(k)\colon k=N+1,\dots,N+\mu n)$, and

$$\{p(k)\colon k=0,1,\dots,M\}.$$

Thus the solutions depend on the parameter $\alpha=(\alpha(1),\dots,\alpha(\mu n))$:

$$\{f_\mu(k;\alpha)\colon k=0,1,\dots,N+\mu n,\alpha\in\mathbb{R}^{\mu n}\}$$

and $a^{(2)}(\alpha)=(a(k;\alpha)\colon k=N+1,\dots,N+\mu n,\alpha\in\mathbb{R}^{\mu n})$.

Let $\{f_\mu(k;\alpha):k=0,1,\dots,N+\mu n,\alpha\in\mathbb{R}^{\mu n}\}$ be the solutions of systems (42), (43). Let $\mathcal{B}_2$ be the set of parameters $\alpha\in\mathbb{R}^{\mu n}$ for which the sequence

$$\{f_\mu(k;\alpha)\colon k=0,1,\dots,N+\mu n\}$$

is strictly positive.

Then

$$\begin{aligned} f^0(\lambda;\alpha)&=\lambda^{2n}\left(\big|1-e^{i\lambda\mu}\big|^{2n}\sum_{k=-N-\mu n}^{N+\mu n} f^0_\mu(|k|;\alpha)e^{i\lambda k}\right)^{-1}\\ &=\left|\frac{(i\lambda)^n}{(1-e^{-i\lambda\mu})^n\sum_{k=0}^{N+\mu n}\gamma_\mu(k;\alpha)e^{-i\lambda k}}\right|^2, \qquad \alpha\in\mathcal{B}_2, \end{aligned} \tag{44}$$

is the least favorable density.

Therefore we proved the following result.

Theorem 6.1. *Let $M > N$ and let the coefficients*

$$\{f_\mu(k;\alpha)\colon k = 0, 1, \dots, M + \mu n, \alpha \in \mathcal{B}_1\}$$

be found from systems (38), (40), (41). *If a sequence $a(0), a(1), \dots, a(N)$ is such that the sets $\mathcal{B}_1$ and $\mathcal{L}_1$ are nonempty, then*

$$\mathcal{R}_1 = \Big\{ f^0(\lambda) \in \mathcal{D}^-_{M,n} \colon \quad f^0(\lambda) = \left(\frac{|1 - e^{i\lambda\mu}|^{2n}}{\lambda^{2n}} \sum_{m=-M-\mu n}^{M+\mu n} f^0_\mu(|m|;\alpha) e^{i\lambda m} \right)^{-1}, \ \alpha \in \mathcal{B}_1 \Big\} \tag{45}$$

is the set of least favorable densities in the class $\mathcal{D}^-_{M,n}$ used for the construction of the linear estimator of the functional $A_N\xi$ from observations after the process at the moments $\mathbb{Z} \setminus \{0, 1, \dots, N\}$. The optimal estimator of the functional $A_N\xi$ is evaluated by formula (34), *while the minimax spectral characteristic is evaluated by equality* (35).

Let $M \le N$ and let the coefficients $\{f_\mu(k;\alpha)\colon k = 0, 1, \dots, N + \mu n, \alpha \in \mathcal{B}_2\}$ be defined from systems of equations (42) *and* (43). *If a sequence $a(0), a(1), \dots, a(N)$ is such that the sets $\mathcal{B}_2$ and $\mathcal{L}_2$ are nonempty, then*

$$\mathcal{R}_2 = \Big\{ f^0(\lambda) \in \mathcal{D}^-_{M,n} \colon \quad f^0(\lambda) = \left(\frac{|1 - e^{i\lambda\mu}|^{2n}}{\lambda^{2n}} \sum_{m=-N-\mu n}^{N+\mu n} f^0_\mu(|m|;\alpha) e^{i\lambda m} \right)^{-1}, \ \alpha \in \mathcal{B}_2 \Big\} \tag{46}$$

is the set of least favorable densities in the class $\mathcal{D}^-_{M,n}$ for the construction of the linear estimator of the functional $A_N\xi$ from observations after the process at the moments

$$\mathbb{Z} \setminus \{0, 1, \dots, N\}.$$

The optimal estimator of the functional $A_N\xi$ is evaluated by formula (34), *while the minimax spectral characteristic is evaluated by equality* (35).

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