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# THE DYNAMICS OF THE MEAN MASS OF A SOLUTION OF THE STOCHASTIC POROUS MEDIA EQUATION

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S. A. MEL'NIK

Abstract. Conditions are found under which the mean mass of a non-trivial solution of the Cauchy problem for the stochastic porous media equation becomes infinitely large during a finite time.

## 1. Introduction

On a stochastic basis $(\Omega, \mathbf{F}, \mathsf{P}, \{\mathbf{F}_t\}_{t\ge 0})$, consider the problem

$$\begin{gathered} du(t,x) = a\Delta u^\sigma(t,x)\,dt + bu^\beta(t,x)\,dt + cu^\gamma(t,x)\,dw(t), \\ t\ge 0, \qquad x\in\mathbb{R}^n, \qquad u(0,x)=u_0(x). \end{gathered} \tag{1}$$

Here $a$, $b$, $c$, $\sigma$, $\beta$, and $\gamma$ are positive numbers, $w(t)$ a standard Wiener process subordinated to the filtration $\{\mathbf{F}_t\}_{t\ge 0}$, and $u_0(x)$ a non-negative non-random function.

The deterministic as well as stochastic porous media equations have been studied by many authors. For example V. Barbu, G. Da Prato, M. Röckner, and F.-Y. Wang in a series of papers studied conditions for the existence and uniqueness of a solution and the limit behavior of solutions in the case of a Lipschitz diffusion coefficient (see [1] and other papers of these authors).

In the deterministic case (corresponding to the case of $c=0$ in (1)), the limit behavior of the mass of a non-trivial solution of the porous media equation is well studied. It is shown in [2] that $\int_B u(t,x)\,dx$, called the mass of the solution in a compact set $B$, becomes infinite over a finite time if $1\le\beta\le\sigma+2/n$. The assumption $b>0$ is crucial in this case, since equation (1) describes a diffusion process in a media with a source. For the case of a stochastic equation (that is, in the case of $c\ne 0$), it is shown in [3] that

$$\mathsf{E}\int_{\mathbb{R}^n} u(t,x)\,dx \le \mathcal{K}e^{\mathcal{K}t}, \qquad \mathcal{K}=\text{const},$$

if $\sigma>1$ and $\beta=\gamma=1$, which implies that the mean mass of a solution of problem (1) in the whole space $\mathbb{R}^n$ remains bounded at each finite time $t\ge 0$. In the current paper, we obtain a condition under which the mean mass of a solution of problem (1) concentrated in a certain bounded part of the space $\mathbb{R}^n$ becomes infinite over a finite time. In other words, the latter property means that there are positive numbers $T$ and $R$ such that

$$\lim_{t\uparrow T}\mathsf{E}\int_{|x|\le R} u(t,x)\,dx = +\infty.$$

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## 2. Notation and definitions

Throughout this paper we use the following notation: $\mathsf{E}$ is the symbol of the mathematical expectation with respect to the measure $\mathsf{P}$; $\mathbb{N}$ means the set of positive integer numbers; $\mathbb{C} = \overset{\circ}{\mathbb{C}}{}^{\infty}(\mathbb{R}^n)$ is the space of infinitely many times differentiable functions with a compact support; $\mathcal{V}$ denotes the volume of a unit $n$-dimensional ball;

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \qquad \nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \qquad f_x = \frac{\partial f}{\partial x}.$$

The symbol $\mathcal{K}$ with or without indices stands for different positive constants that depend only on parameters $n$, $a$, $b$, $c$, $\sigma$, $\beta$, and $\gamma$.

For some combinations of parameters, solutions of problem (1) may exist only in a random time interval and evolve in the blow up mode. Moreover, a solution as a function of the spatial variable $x$ can be understood in both senses, either in the classical or generalized sense. Following [4], the one point compactification of the space $\mathbb{R}^1$ is understood as the phase space of a solution. Put

$$\tau_N = \inf \left\{ t \geq 0\colon \sup_{x \in \mathbb{R}^n} |u(t,x)| \geq N \right\}, \qquad N \in \mathbb{N}.$$

**Definition.** A stochastic process $u(t,x)$ with continuous realizations adapted to the filtration $\{\mathbf{F}_t\}_{t\geq 0}$ is called a solution of problem (1) if

1. for every non-random function $g \in \mathbb{C}$ and for all $t \geq 0$,

$$\begin{aligned} &\int u(t,x) g(x)\, dx - \int u_0(x) g(x)\, dx \\ &= c \int_0^t \int u^{\gamma}(s,x) g(x)\, dx\, dw(s) - a \int_0^t \int \nabla u^{\sigma}(s,x) \cdot \nabla g(x)\, dx\, ds \\ &+ b \int_0^t \int u^{\beta}(s,x) g(x)\, dx\, ds \end{aligned} \tag{2}$$

with probability one;

2. for all $N \in \mathbb{N}$ and $R > 0$,

$$\mathsf{E} \sup_{0 \leq t \leq \tau_N} \int_{|x|<R} u^2(t,x)\, dx < +\infty, \qquad \mathsf{E} \int_0^{\tau_N} \int_{|x|<R} |\nabla u^{\sigma}(s,x)|^2\, dx\, ds < +\infty,$$

$$\mathsf{E} \int_0^{\tau_N} \int_{|x|<R} u^{\beta}(s,x)\, dx\, ds < +\infty, \qquad \mathsf{E} \int_0^{\tau_N} \int_{|x|<R} u^{2\gamma}(s,x)\, dx\, ds < +\infty.$$

*Remark* 1. According to [5, Theorem 2], if

$$0 < \sigma \leq \beta < \left(1 + \frac{2}{n}\right) \sigma + \frac{2}{n}, \qquad \frac{1}{2}(\sigma+1) \leq \gamma < \left(\frac{1}{2} + \frac{1}{n}\right)(\sigma+1),$$

then a solution of problem (1) exists in the sense explained above. In what follows we assume that the latter conditions hold. According to [6], a solution is non-negative with probability one if the initial condition is non-negative. Throughout this paper we consider only non-negative and non-trivial solutions. In other words, we consider non-negative solutions that are strictly positive in a certain subset of the space $\mathbb{R}^n$ that has a positive volume. Therefore, following the classification proposed in [7], we consider stochastically strong but analytically weak solutions.

## 3. Main results

**Theorem 1.** *Assume that*

1. $a > 0$, $b > 0$, $\gamma > 0$;
2. $0 < \int u_0(x)\,dx < +\infty$;
3. $(1 - 2/n)_+ < \sigma$ *and* $\max\{1, \sigma\} < \beta < \sigma + 2/n$.

*Then there exist positive numbers* $T$ *and* $R$ *such that*

$$\lim_{t \uparrow T} \mathsf{E} \int_{|x| \le R} u(t,x)\,dx = +\infty.$$

**Theorem 2.** *Assume that*

1. $a > 0$, $b > 0$, *and* $\gamma > 0$;
2. $\mathcal{K}_0 < \int u_0(x)dx < +\infty$, *where* $\mathcal{K}_0$ *is defined in equality* (13);
3. $\beta = \sigma + \frac{2}{n}$ *and* $\sigma > 0$.

*Then there exist positive numbers* $T$ *and* $R$ *such that*

$$\lim_{t \uparrow T} \mathsf{E} \int_{|x| \le R} u(t,x)\,dx = +\infty.$$

**Theorem 3.** *Assume that*

1. $a > 0$, $c > 0$, *and* $b = 0$;
2. $0 < \int u_0(x)\,dx < +\infty$;
3. $(1 - 2/n)_+ < \sigma$ *and* $\max\{1, \sigma\} < \gamma < \sigma + 2/n$.

*Then there exist positive numbers* $T$ *and* $R$ *such that*

$$\lim_{t \uparrow T} \mathsf{E} \int_{|x| \le R} u^m(t,x)\,dx = +\infty$$

*for all* $m > 1$.

**Theorem 4.** *Assume that*

1. $a > 0$, $c > 0$, *and* $b = 0$;
2. $\mathcal{K}_0 < \int u_0(x)\,dx < +\infty$, *where* $\mathcal{K}_0$ *is defined by equality* (13) *with* $\gamma$ *instead of* $\beta$, *namely* $\mathcal{K}_0 = (\mathcal{K}_1(1)/\mathcal{K}_2)^{1/(\gamma-\sigma)}$;
3. $\gamma = \sigma + 2/n$ *and* $\sigma > 0$.

*Then there exist positive numbers* $T$ *and* $R$ *such that*

$$\lim_{t \uparrow T} \mathsf{E} \int_{|x| \le R} u^m(t,x)\,dx = +\infty$$

*for all* $m > 1$.

## 4. Auxiliary results

**Lemma 1.** *Let* $0 < \sigma < \beta$, $0 < \delta < R$, *and* $\mathcal{L} = \int_{-1}^{1} \exp(-1/(1-y^2))\,dy$. *Then the function*

$$H(x,R) = \begin{cases} 1, & |x| \le R - \delta, \\ \frac{1}{\mathcal{L}} \int_{-1}^{\frac{2}{\delta}(R-|x|)-1} \exp\left(-\frac{1}{1-y^2}\right) dy, & R - \delta < |x| < R, \\ 0, & |x| \ge R \end{cases}$$

*is such that*

1. $0 \le H(x,R) \le 1$;
2. $H_{x_i}(x,R)\big|_{|x|=R} = 0$ *and* $H_{x_i}(x,R)\big|_{|x|=R-\delta} = 0$, $i = 1, \dots, n$;

3. *if* $\delta = \varepsilon R$ *and* $\varepsilon \in (0;1)$, *then*

$$\int_{|x|\le R}\left(\frac{|\Delta H(x,R)|^{\beta}}{H^{\sigma}(x,R)}\right)^{\frac{1}{\beta-\sigma}} dx = \mathcal{H}(\varepsilon) R^{\frac{(n-2)\beta - n\sigma}{\beta-\sigma}},$$

*where*

$$\mathcal{H}(\varepsilon) = \frac{\pi^{\frac{n}{2}} 8^{\frac{\beta}{\beta-\sigma}}}{\varepsilon^{\frac{\beta+\sigma}{\beta-\sigma}}}\int_{-1}^{1}(1-0.5\varepsilon(r+1))^{n-1}\left[\frac{|r|^{\frac{\beta}{\sigma}} e^{-\frac{\beta}{\sigma(1-r^2)}}}{(1-r^2)^{\frac{2\beta}{\sigma}}\int_{-1}^{r} e^{-\frac{1}{1-y^2}}\,dy}\right]^{\frac{\sigma}{\beta-\sigma}} dr > \mathcal{H}(1) > 0.$$

*Proof.* Properties 1 and 2 follow directly from the definition of the function $H(x,R)$. Now we prove property 3. Note that

$$\Delta H(x,R) = \begin{cases} 0, & |x| \notin (R-\delta;R), \\ -\frac{8}{\mathcal{L}\delta^2}\frac{\frac{2R}{\delta}\left(1-\frac{|x|}{R}\right)-1}{\left[1-\left(\frac{2R}{\delta}\left(1-\frac{|x|}{R}\right)-1\right)^2\right]^2}\exp\left(-\frac{1}{1-\left(\frac{2R}{\delta}\left(1-\frac{|x|}{R}\right)-1\right)^2}\right), & |x| \in (R-\delta;R). \end{cases}$$

Passing to the polar coordinates we obtain from the latter equality that

$$\int_{|x|\le R}\left(\frac{|\Delta H(x,R)|^{\beta}}{H^{\sigma}(x,R)}\right)^{\frac{1}{\beta-\sigma}} dx$$
$$= \mathcal{K}\delta^{-\frac{\beta+\sigma}{\beta-\sigma}} R^{n-1}\int_{-1}^{1}\left(1-\frac{\delta}{2R}(r+1)\right)^{n-1}\left[\frac{|r|^{\frac{\beta}{\sigma}} e^{-\frac{\beta}{\sigma(1-r^2)}}}{(1-r^2)^{\frac{2\beta}{\sigma}}\int_{-1}^{r} e^{-\frac{1}{1-y^2}}\,dy}\right]^{\frac{\sigma}{\beta-\sigma}} dr.$$

The integrand may have singularities only at points $r=-1$ or $r=1$. We show that, in fact, there is no singularities at those points. Indeed,

$$\lim_{r\downarrow -1}\frac{e^{-\frac{\beta}{\sigma(1-r^2)}}}{(1-r^2)^{\frac{2\beta}{\sigma}}\int_{-1}^{r} e^{-\frac{1}{1-y^2}}\,dy} = \lim_{z\to+\infty}\frac{z^{\frac{2\beta}{\sigma}} e^{-\frac{\beta}{\sigma}z}}{\int_{-1}^{-\sqrt{1-\frac{1}{z}}} e^{-\frac{1}{1-y^2}}\,dy} = 0.$$

This proves that there is no singularity at the point $r=-1$.

Analogously,

$$\lim_{r\uparrow 1}\frac{e^{-\frac{\beta}{\sigma(1-r^2)}}}{(1-r^2)^2} = 0,$$

whence we conclude that there is no singularity at the point $r=1$, too.

Therefore the integral is finite and property 3 follows if $\delta=\varepsilon R$ and $\varepsilon\in(0;1)$. It is easy to see that $\mathcal{H}(1)$ is finite and positive. The lemma is proved. ☐

## 5. Proof of Theorems 1–4

5.1. **Proof of Theorem 1.** Put

$$J(t,R) = \mathsf{E}\int u(t,x)H(x,R)\,dx, \qquad I(t,R) = \mathsf{E}\int u^{\beta}(t,x)H(x,R)\,dx.$$

Equality (2) implies that

$$\text{(3)}\qquad \begin{aligned} &J_t(t,R) = a\,\mathsf{E}\int u^{\sigma}(t,x)\Delta H(x,R)\,dx + bI(t,R), \qquad t\ge 0,\\ &J(0,R) = \int u_0(x)H(x,R)\,dx. \end{aligned}$$

Next we estimate from below the first term on the right hand side of equality (3) with the help of the Hölder inequality and property 3 established in Lemma 1:

$$\mathsf{E}\int u^\sigma(t,x)\Delta H(x,R)\,dx \geq -\mathcal{H}^{\frac{\beta-\sigma}{\beta}}(\varepsilon)R^{\frac{(n-2)\beta-n\sigma}{\beta}}I^{\frac{\sigma}{\beta}}(t,R).$$

Then equality (3) implies

$$J_t(t,R) \geq I^{\frac{\sigma}{\beta}}(t,R)\left[-a\mathcal{H}^{\frac{\beta-\sigma}{\beta}}(\varepsilon)R^{\frac{(n-2)\beta-n\sigma}{\beta}} + bI^{\frac{\beta-\sigma}{\beta}}(t,R)\right]. \tag{4}$$

Now the Hölder inequality yields

$$J(t,R) \leq \mathcal{V}^{\frac{\beta-1}{\beta}}R^{\frac{n(\beta-1)}{\beta}}I^{\frac{1}{\beta}}(t,R)$$

and

$$I(t,R) \geq \mathcal{V}^{1-\beta}R^{n(1-\beta)}J^\beta(t,R). \tag{5}$$

Hence inequality (4) can be rewritten as follows:

$$J_t(t,R) \geq R^{\frac{(n-2)\beta-n\sigma}{\beta}}I^{\frac{\sigma}{\beta}}(t,R)\left[-\mathcal{K}_1 + \mathcal{K}_2R^{n\sigma+2-n\beta}J^{\beta-\sigma}(t,R)\right], \tag{6}$$

where

$$t \geq 0, \qquad \mathcal{K}_1 = a\mathcal{H}^{\frac{\beta-\sigma}{\beta}}(\varepsilon) > 0, \qquad \mathcal{K}_2 = b\mathcal{V}^{\frac{(1-\beta)(\beta-\sigma)}{\beta}} > 0.$$

Next we show that the exists a number $R_0 > 0$ such that if $R \geq R_0$, then inequality (6) implies that

$$J_t(t,R) \geq \mathcal{K}_3R^{n(1-\beta)}J^\beta(t,R), \tag{7}$$

where $\mathcal{K}_3 = \frac{1}{2}\mathcal{K}_2\mathcal{V}^{\frac{(1-\beta)\sigma}{\beta}} > 0$.

Since $J(t,R)$ is non-decreasing with respect to $R$ and since $\beta > \sigma$ and $n\sigma+2-n\beta > 0$ by assumption 3 of Theorem 1, the expression $R^{n\sigma+2-n\beta}J^{\beta-\sigma}(t,R)$ is unbounded as $R \to +\infty$. Choose $R_0$ such that

$$-\mathcal{K}_1 + \mathcal{K}_2R_0^{n\sigma+2-n\beta}J^{\beta-\sigma}(0,R_0) \geq \frac{1}{2}\mathcal{K}_2R_0^{n\sigma+2-n\beta}J^{\beta-\sigma}(0,R_0) > 0.$$

Note that such a number $R_0$ exists by condition 2 of Theorem 1. Applying bound (5) to inequality (6) we obtain

$$J_t(0,R_0) \geq \mathcal{K}_3R_0^{n(1-\beta)}J^\beta(0,R_0) > 0. \tag{8}$$

Since $J(t,R)$ is continuous with respect to $t$, there exists $\delta > 0$ such that

$$-\mathcal{K}_1 + \mathcal{K}_2R_0^{n\sigma+2-n\beta}J^{\beta-\sigma}(t,R_0) \geq \frac{1}{2}\mathcal{K}_2R_0^{n\sigma+2-n\beta}J^{\beta-\sigma}(t,R_0) > 0 \tag{9}$$

for $t \in [0;\delta]$. Since $J(t,R)$ is non-decreasing with respect to $R$, inequality (9) holds for $R \geq R_0$. This implies

$$J_t(t,R) \geq \mathcal{K}_3R^{n(1-\beta)}J^\beta(t,R) \tag{10}$$

by (6) and (5) for $t \in [0;\delta]$ and $R \geq R_0$.

It follows from (10) that $J(t,R_0)$ is increasing for $t \in [0;\delta]$ and thus

$$J(\delta,R_0) > J(0,R_0) > 0.$$

Substituting $t = \delta$ in bound (10) we get

$$J_t(\delta,R_0) \geq \mathcal{K}_3R_0^{n(1-\beta)}J^\beta(\delta,R_0) > 0.$$

Note that the constant $\mathcal{K}_3$ on the right hand side of the latter inequality is the same as that on the right hand side of inequality (8). Thus there exists a number $\delta_1 > 0$

such that inequality (9), and consequently inequality (10), holds for $t \in [0; \delta + \delta_1]$. Since inequalities (6) and (7) are invariant with respect to time shifts and since

$$J(\delta, R_0) > J(0, R_0),$$

we conclude that $\delta_1 \geq \delta$. Therefore inequality (7) is valid for $t \in [0; 2\delta]$. Repeating the reasoning we conclude that inequality (6) implies inequality (7) if $R \geq R_0$.

Dividing (7) by $J^\beta(t, R_0)$ and integrating with respect to $t$, we deduce that

$$J(t, R_0) \geq \left(J^{1-\beta}(0, R_0) - (1-\beta)\mathcal{K}_3 R_0^{n(1-\beta)} t\right)_+^{\frac{1}{1-\beta}}.$$

This implies that

$$\lim_{t \uparrow T} J(t, R_0) = +\infty,$$

where

$$T = \frac{R_0^{n(\beta-1)}}{(\beta-1)\mathcal{K}_3} J^{1-\beta}(0, R_0) < +\infty.$$

Theorem 1 is proved.

5.2. **Proof of Theorem 2.** To prove Theorem 2 we use the results obtained in the course of the proof of Theorem 1 among which inequality (6) is the key tool. Since $\beta = \sigma + \frac{2}{n}$ by condition 3 of Theorem 2, inequality (6) can be rewritten in the following form:

$$J_t(t, R) \geq R^{\frac{(n-2)\beta - n\sigma}{\beta}} I^{\frac{\sigma}{\beta}}(t, R) \left[-\mathcal{K}_1(\varepsilon) + \mathcal{K}_2 J^{\beta-\sigma}(t, R)\right], \qquad t \geq 0. \tag{11}$$

Recall that the constant $\mathcal{K}_1$ depends on $\varepsilon$, since $\mathcal{K}_1$ is expressed in terms of $\mathcal{H}(\varepsilon)$.

We show that, under condition 2 of Theorem 2, there are constants $\varepsilon \in (0; 1)$, $\varepsilon' \in (0; 1)$, and $R_0 > 0$ such that if $R \geq R_0$, then

$$-\mathcal{K}_1(\varepsilon) + \mathcal{K}_2 J^{\beta-\sigma}(0, R) \geq \varepsilon' \mathcal{K}_2 J^{\beta-\sigma}(0, R). \tag{12}$$

Inequality (12) is equivalent to the inequality

$$J(0, R) \geq \left(\frac{\mathcal{K}_1(\varepsilon)}{(1-\varepsilon')\mathcal{K}_2}\right)^{\frac{1}{\beta-\sigma}}.$$

The function $J^{\beta-\sigma}(0, R)$ is non-decreasing with respect to $R$, and thus

$$\lim_{R \to +\infty} J(0, R) = \int u_0(x)\, dx \in (0; +\infty).$$

Moreover,

$$\mathcal{K}_1(\varepsilon) > \mathcal{K}_1(1) \quad \text{and} \quad \frac{\mathcal{K}_1(\varepsilon)}{(1-\varepsilon')\mathcal{K}_2} > \frac{\mathcal{K}_1(1)}{\mathcal{K}_2} > 0,$$

whence

$$\lim_{\varepsilon \to 1, \varepsilon' \to 0} \frac{\mathcal{K}_1(\varepsilon)}{(1-\varepsilon')\mathcal{K}_2} = \frac{\mathcal{K}_1(1)}{\mathcal{K}_2}.$$

We introduce the constant $\mathcal{K}_0$ by

$$\mathcal{K}_0 = \left(\frac{\mathcal{K}_1(1)}{\mathcal{K}_2}\right)^{\frac{1}{\beta-\sigma}}. \tag{13}$$

According to condition 2 of Theorem 2, $\int u_0(x)\, dx > \mathcal{K}_0$. Hence there are some numbers $\varepsilon \in (0; 1)$, $\varepsilon' \in (0; 1)$, and $R_0 > 0$ such that inequality (12) is valid for $R \geq R_0$. Then (12) and (11) imply that

$$J_t(0, R) \geq \varepsilon' \mathcal{K}_2 R_0^{\frac{(n-2)\beta - n\sigma}{\beta}} I^{\frac{\sigma}{\beta}}(0, R_0) J^{\beta-\sigma}(0, R_0) > 0,$$

whence

$$J_t(0, R_0) \geq \mathcal{K}_4 R_0^{n(1-\beta)} J^\beta(0, R_0)$$

according to bound (5), where $\mathcal{K}_4$ is a positive constant. The latter inequality is analogous to inequality (8). Repeating the reasoning used in the proof of Theorem 1 starting with inequality (8), we complete the proof of Theorem 2.

5.3. **Proof of Theorems 3 and 4.** Assumption $b > 0$ plays an essential role in the proof of Theorems 1 and 2. In particular, the key inequalities (6) and (7) follow from this assumption. In contrast, $b = 0$ in Theorems 3 and 4. Nevertheless we show that the blow up mode occurs under the assumptions of Theorems 3 and 4, too.

First we prove Theorem 3. Put $\xi(t) = e^{-t/2+w(t)}$. The process $\xi(t)$ is a solution of the following stochastic differential equation:

$$\xi(t) = 1 + \int_0^t \xi(s)\, dw(s).$$

Moreover, $\mathsf{E}\,\xi(t) = 1$ and $\mathsf{E}\,\xi^l(t) = e^{l(l-1)t/2}$, $l \geq 0$. Applying Itô's formula to the process $u(t,x)H(x,R)\xi(t)$ and integrating the result we obtain

$$\begin{aligned}
\int u(t,x)H(x,R)\,dx\xi(t) = \int u_0(x)H(x,R)\,dx &+ a\int_0^t \int u^\sigma(s,x)\Delta H(x,R)\,dx\xi(s)\,ds \\
&+ c\int_0^t \int u^\gamma(s,x)H(x,R)\,dx\xi(s)\,ds \\
&+ c\int_0^t \int u^\gamma(s,x)H(x,R)\,dx\xi(s)\,dw(s) \\
&+ \int_0^t \int u(s,x)H(x,R)\,dx\xi(s)\,dw(s).
\end{aligned}$$

According to [4, Remark 2.2], the stochastic integrals in the latter equality are well defined and their mathematical expectations are equal to zero. Thus

$$J_t(t,R) = a\,\mathsf{E}\left(\int u^\sigma(t,x)\Delta H(x,R)\,dx\xi(t)\right) + cI(t,R), \tag{14}$$

where

$$J(t,R) = \mathsf{E}\left(\int u(t,x)H(x,R)\,dx\xi(t)\right), \qquad I(t,R) = \mathsf{E}\left(\int u^\gamma(t,x)H(x,R)\,dx\xi(t)\right).$$

As in the proof of Theorem 1, we use Lemma 1 and obtain the following bound:

$$\mathsf{E}\left(\int u^\sigma(t,x)\Delta H(x,R)\,dx\xi(t)\right) \geq -\mathcal{H}^{\frac{\gamma-\sigma}{\gamma}}(\varepsilon) R^{\frac{(n-2)\gamma-n\sigma}{\gamma}} I^{\frac{\sigma}{\gamma}}(t,R).$$

Then (14) implies the inequality

$$J_t(t,R) \geq I^{\frac{\sigma}{\gamma}}(t,R)\left[-a\mathcal{H}^{\frac{\gamma-\sigma}{\gamma}}(\varepsilon)R^{\frac{(n-2)\gamma-n\sigma}{\gamma}} + cI^{\frac{\gamma-\sigma}{\gamma}}(t,R)\right] \tag{15}$$

that coincides with inequality (4) where $\gamma$ substitutes $\beta$ and $c$ substitutes $b$. Since the assumptions imposed on $\gamma$ in Theorem 3 are the same as those imposed on $\beta$ in Theorem 1, one can use (15) and repeat the reasoning in the proof of Theorem 1 starting with inequality (4). This proves the inequality

$$\mathsf{E}\left(\int u(t,x)H(x,R_0)\,dx\xi(t)\right) \geq \left(J^{1-\gamma}(0,R_0) - (\gamma-1)\mathcal{K}R_0^{n(1-\gamma)}t\right)_+^{\frac{1}{1-\gamma}},$$

whence we conclude that

$$\lim_{t\uparrow T}\mathsf{E}\left(\int u(t,x)H(x,R_0)\,dx\xi(t)\right)=+\infty, \tag{16}$$

where

$$T=\frac{R_0^{n(\gamma-1)}J^{1-\gamma}(0,R_0)}{\mathcal{K}(\gamma-1)}\in(0;+\infty).$$

To complete the proof of Theorem 3 let $m>1$. Then the Hölder inequality implies that

$$\mathsf{E}\left(\int u(t,x)H(x,R_0)\,dx\xi(t)\right)\le\mathcal{V}^{\frac{m-1}{m}}R_0^{\frac{n(m-1)}{m}}e^{\frac{1}{2(m-1)}T}\left(\mathsf{E}\int u^m(t,x)H(x,R_0)\,dx\right)^{\frac{1}{m}}$$

for $t\in[0;T)$. As shown above, the left hand side of the latter inequality is unbounded as $t\uparrow T$. Hence

$$\lim_{t\uparrow T}\mathsf{E}\int u^m(t,x)H(x,R_0)\,dx=+\infty,\qquad\forall m>1,$$

and Theorem 3 is proved.

To prove Theorem 4 we use inequality (15) and repeat the same reasoning as in the proof of Theorem 2.

## 6. Concluding remarks

Theorems 1–4 contain only those assumptions imposed on the initial data of problem (1) that are needed in the proof of these theorems. It is clear that one should add some extra assumptions in Theorems 1–4 under which a solution of problem (1) exists. For example, one can add the assumptions mentioned in Remark 1. It is easy to see that the sets of assumptions in Theorems 1–4 and in Remark 1 do not contradict each other.

Theorems 1 and 2 provide conditions under which the mean mass of a solution explodes over a finite time. This result is explained by a deterministic source ($b>0$) in (1). If there is no deterministic source in (1), then we are not able to prove that equality (16) implies that the mean mass of a solution explodes. Theorems 3 and 4 provide conditions under which the integral moments of a solution explode if the order is higher than one. For example, the mean energy of a solution corresponds to the case of $m=2$.

It is worth mentioning that the assumption $c>0$ is not crucial at all. If $c<0$, then $c=-|c|$, and we introduce the new Wiener process $\tilde{w}(t)=-w(t)$ instead of the Wiener process $w(t)$. This transforms the initial equation to (1) with a positive coefficient in the stochastic term. Therefore the crucial assumption for Theorems 3 and 4 is that $c\neq0$.

## Bibliography


1. V. Barbu, G. Da Prato, and M. Röckner, *Existence of strong solutions of stochastic porous media equations*, Ann. Probab. **37** (2009), 428–452. MR2510012 (2011a:35592)
2. X. Liu and M. Wang, *The critical exponent of doubly singular parabolic equations*, J. Math. Anal. Appl. **257** (2001), 170–188. MR1824673 (2002c:35141)
3. S. A. Mel'nik, *The dynamics of solutions of the Cauchy problem for a stochastic equation of parabolic type with power non-linearities (the stochastic term is linear)*, Teor. Imovir. Mat. Stat. **64** (2001), 110–117; English transl. in Theory Probab. Math. Statist. **64** (2002), 129–137. MR1922959 (2003m:60056)
4. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, second edition, North-Holland Publishing Co., Amsterdam–Kodansha, Ltd., Tokyo, 1989. MR1011252 (90m:60069)
5. S. A. Mel'nik, *The existence of solutions of parabolic-type stochastic equations with power non-linearities*, Teor. Imovir. Mat. Stat. **53** (1995), 103–108; English transl. in Theory Probab. Math. Statist. **53** (1996), 113–118. MR1449111 (98c:60069)

6. S. A. Mel'nik, *Maximum principle for stochastic combustion equations*, Dep. UKRINTEI, 434-Uk 921 06.04.92, 1992, 1–5. (Russian)
7. C. Prevot and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Springer-Verlag, Berlin, 2007. MR2329435 (2009a:60069)

Department of Probability Theory and Mathematical Statistics, Institute of Applied Mathematics and Mechanics, National Academy of Science of Ukraine, R. Luxemburg Street, 74, Donetsk, 83114, Ukraine

*E-mail address*: melnik@iamm.ac.donetsk.ua

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