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DISCRETE REPRESENTATIONS OF SECOND ORDER RANDOM FUNCTIONS. II

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O. I. PONOMARENKO

Abstract. This is a continuation of the author's 2012 paper. We study the discrete representations of a general basis type for a second order random function assuming scalar or vector values.

3. Discrete basis type representations

3.1. The representation

$$\xi(t) = \sum_{j\in J} \alpha_j(t) z_j, \qquad t \in T, \tag{3.1}$$

for a scalar second order random function $\xi(t)$ in the set T is called the general discrete representation if $\{z_j, j \in J\}$ is a system of random variables belonging to $L_2(\Omega)$ and $\mathsf{E}\, z_j \bar{z}_r = f_{jr}$, and $\{\alpha_j, j \in J\}$ is the corresponding set of complex-valued functions defined on the set T.

We assume that the sum on the right hand side of (3.1) contains at most a countable number of non-zero terms and that the series in (3.1) converges in a certain sense. If representation (3.1) is unique, then it is called a basis type representation. The uniqueness is understood in the sense that if a system $\{z_j, j \in J\}$ is given, then the coefficients $\alpha_j(t)$ are uniquely determined or, vice versa, if a system of functions $\{\alpha_j(t), j \in J\}$ is given, then the random variables z_j are uniquely determined. Representation (3.1) is called orthogonal if $\mathsf{E}\, z_j \bar{z}_r = \delta_{jr} f_j$ and $f_j > 0$; similarly, it is called orthonormal if additionally $f_j = 1$ for all $j, r \in J$.

Obviously, in $L_2(\xi)$ as in any other Hilbert space, there exists an orthonormal basis

$$\{z_j, j \in J\},$$

and thus every function $\xi(t), t \in T$, admits orthonormal basis representation (3.1), where $\alpha_j(t) = (\xi(t)|z_j)$ are Fourier coefficients for $\xi(t)$ with respect to the basis $\{z_j, j \in J\}$. Nevertheless, it is a complicated problem to construct such a basis $\{z_j, j \in J\}$ (except for some trivial cases), and thus one needs to develop an approach for obtaining representation (3.1) by using the correlation structure of the function $\xi(t)$ or, alternatively, by using the results of modern harmonic analysis.

Note that if (3.1) is a basis type representation and the series on the right hand side converges in the norm of $L_2(\Omega)$, then the system $\{z_j, j \in J\}$ is complete in $L_2(\xi)$ and is topologically free. Moreover, the covariance kernel $k(t, s)$ of the function $\xi(t)$ admits the

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discrete representation

$$k(t,s)=\sum_{j\in J}\sum_{r\in J}\alpha_j(t)\overline{\alpha_r(s)}f_{jr},\qquad t,s\in T, \tag{3.2}$$

where all square matrices $F=(f_{j_u j_v})_{u,v=1}^n$, $n\in\mathbf{N}$, $j_u\in J$, are non-negatively definite and the series on the right hand side of (3.2) converges. Conversely, if representation (3.2) holds for the correlation kernel k of a function $\xi(t)$, $t\in T$, then Cramér's theorem on the representation of random functions implies that representation (3.1) holds, too.

Example 3.1. Let $\xi(g)$, $g\in G$, be a square mean continuous random function defined on a compact topological group G. Then $\xi(g)$ admits a discrete representation with respect to the system of "elementary harmonics" in G; in other words, $\xi(g)$ is a harmonizable function in G.

Indeed, there exists at most a countable system of non-equivalent non-reducible finite-dimensional unitary representations of G (in other words, continuous homeomorphisms acting from G to the group of unitary matrices of finite sizes):

$$g\to U^{(\lambda)}(g)=\left\{u_{ij}^{(\lambda)}(g)\right\},\qquad 1\le i,j\le d_\lambda<\infty,\quad \lambda=1,2,\dots,$$

$$U^{(\lambda)}(gs)=U^{(\lambda)}(g)U^{(\lambda)}(s),\qquad U^{(\lambda)}(g^{-1})=\left\{U^{(\lambda)}(g)\right\}^{-1}=\left\{U^{(\lambda)}(g)\right\}^{*},$$

$$\int_G u_{ij}^{(\lambda)}(g)\overline{u_{rl}^{(\mu)}}(g)\,dg=\delta_{\lambda\mu}\delta_{ir}\delta_{jl}\cdot\frac{1}{d_\lambda},$$

where dg is the Haar measure in G normalized by unity (in other words, a two sided invariant measure in G). Moreover, the harmonics $u_{ij}^{(\lambda)}(a)$, $1\le i,j\le d_\lambda$, $\lambda=1,2\dots$, form a complete system in the Hilbert space $L_2(G)$ of functions whose absolute value is square integrable with respect to dg.

Then $\xi(g)$, $g\in G$, admits a non-orthogonal basis type representation

$$\xi(g)=\sum_\lambda\sum_{i,j=1}^{d_\lambda}u_{ij}^{(\lambda)}(g)z_{ij}^{(\lambda)},\qquad g\in G, \tag{3.3}$$

where $z_{ij}^{(\lambda)}=d_\lambda\int_G\overline{u_{ij}^{(\lambda)}(g)}\xi(g)\,dg\in L_2(\Omega)$ and the series on the right hand side of (3.3) converges in the square mean sense (note that the set of indices λ is countable).

If a function $\xi(g)$, $g\in G$, is left stationary, that is, if $\mathsf{E}\,\xi(g)=\text{const}$ and if its correlation kernel k is left invariant in the sense that $k(gt,gs)=k(t,s)=k(s^{-1}t)$, $g,t,s\in G$, then the orthogonality conditions hold for $z_{ij}^{(\lambda)}$, namely

$$\mathsf{E}\,z_{ij}^{(\lambda)}\overline{z_{rl}^{(\mu)}}=\delta_{\lambda\mu}\delta_{ir}f_{jl}^{(\lambda)}, \tag{3.4}$$

where $\{f_{jl}^{(\lambda)}\}=f^{(\lambda)}$ is a positive definite matrix and

$$\sum_\lambda\sum_{j=1}^{d_\lambda}f_{jj}^{(\lambda)}=\sum_\lambda\operatorname{tr}\left(f^{(\lambda)}\right)<\infty \tag{3.5}$$

(see [6]).

Thus the correlation function $\tilde{k}(g)$ of the random function $\xi(g)$ admits the following discrete representation:

$$\tilde{k}(g)=\sum_\lambda\sum_{j,l=1}^{d_\lambda}u_{jl}^{(\lambda)}(g)f_{jl}^{(\lambda)}. \tag{3.6}$$

3.2. Consider an extension of the results of Section 2 to the case where the correlation kernel k generates a general Hilbert–Schmidt integral operator.

Let T be a measurable space equipped with a σ-algebra $\mathcal{A}$ and positive measure μ, that is, $T \cong (T, \mathcal{A}, \mu)$. Assume that a second order random function $\xi(t)$, $t \in T$, is measurable in the strong sense as a vector $L_2(\Omega)$-valued function defined in T (in other words, $\xi(t)$ is a μ-almost sure limit of a sequence of simple measurable $L_2(\Omega)$-valued functions defined in T; see [11]). Then its correlation kernel $k(t,s)$ is a measurable function defined in $(T \times T, \mathcal{A} \otimes \mathcal{A}, \mu \times \mu) = (T \times T, \mu \times \mu)$.

We further assume that

$$\int_T \mathsf{E}\,|\xi(t)|^2\,\mu(dt) < \infty. \tag{3.7}$$

Thus the Bochner integrals

$$\int_T \psi(t)\xi(t)\,\mu(dt), \qquad \psi \in L_2(T,\mu),$$

are well defined in the strong topology of $L_2(\Omega)$.

Indeed, this property follows from the inequality

$$\begin{aligned}\int_T \|\psi(t)\xi(t)\|_{L_2(\Omega)}\,\mu(dt) &= \int_T |\psi(t)|\sqrt{\mathsf{E}\,|\xi(t)|^2}\,\mu(dt)\\ &\le \left(\int_T |\psi(t)|^2\,\mu(dt)\right)^{1/2}\left(\int_T \mathsf{E}\,|\xi(t)|^2\,\mu(dt)\right)^{1/2} < \infty\end{aligned}$$

(see [11]). Since

$$\begin{aligned}\int_T\int_T |k(t,s)|^2\,\mu(dt)\,\mu(ds) &= \int_T\int_T |\,\mathsf{E}\,\xi(t)\overline{\xi(s)}|^2\,\mu(dt)\,\mu(ds)\\ &\le \int_T\int_T \mathsf{E}\,|\xi(t)|^2\,\mathsf{E}\,|\xi(s)|^2\,\mu(dt)\,\mu(ds) = \left(\int_T \mathsf{E}\,|\xi(t)|^2\,\mu(dt)\right)^2,\end{aligned} \tag{3.8}$$

we conclude that $k(t,s) \in L_2(T \times T, \mu \times \mu)$.

Hence the integral operator K acting in the space $L_2(T,\mu)$ according to

$$(K\varphi)(t) = \int_T k(t,s)\varphi(s)\,\mu(ds), \qquad \varphi \in L_2(T,\mu),$$

is a general Hilbert–Schmidt integral operator such that

$$(K\psi)(t) = \sum_{j\in J} \lambda_j(\psi,\varphi_j)\varphi_j, \qquad \psi \in L_2(T,\mu), \tag{3.9}$$

where $\{\varphi_j(t), t \in J\}$ and $\{\lambda_j, j \in J\}$ are the orthonormal system of eigenfunctions and eigennumbers, $\lambda_j \neq 0$, of the operator K. Moreover, the series on the right hand side of (3.9) converges in the norm of the space $L_2(T,\mu)$ and the kernel $k(t,s)$ of the operator K admits the representation

$$k(t,s) = \sum_{j\in J} \lambda_j \varphi_j(t)\overline{\varphi_j(s)}, \tag{3.10}$$

where the series on the right hand side of (3.10) converges in the norm of the space $L_2(T \times T, \mu \times \mu)$ (see [6, 9, 16]).

We introduce the random variables $z_j \in L_2(\Omega)$ as follows:

$$z_j = \int_T \xi(t)\overline{\varphi_j(t)}\,\mu(dt), \qquad j \in J, \tag{3.11}$$

where the integrals are understood in the Bochner sense. By equality (3.10),

$$\begin{aligned}\mathsf{E}\, z_j \bar{z}_r &= \int_T\int_T (\mathsf{E}\,\xi(t)\overline{\xi(s)})\overline{\varphi_j(t)}\varphi_r(s)\,\mu(dt)\,\mu(ds)\\ &= \int_T\int_T k(t,s)\overline{\varphi_j(t)}\varphi_r(s)\,\mu(dt)\,\mu(ds) = \delta_{jr}\lambda_j,\end{aligned} \tag{3.12}$$

whence we conclude that z_j are orthogonal. Moreover,

$$\mathsf{E}\,\xi(t)\overline{z_j} = \int_T k(t,s)\varphi_j(s)\,\mu(ds) = \lambda_j\varphi_j(t). \tag{3.13}$$

We write the elements of the set J in descending order of their eigennumbers λ_j (the multiplicity of an eigennumber is taken into account). Now we consider the approximations of $\xi(t)$ by the sums

$$\sum_{j=1}^n \varphi_j(t) z_j.$$

It follows from (3.12) and (3.13) that the square error of this approximation is equal to

$$\begin{aligned}\sigma_n(t) = \mathsf{E}\,|\xi_n(t)|^2 = \mathsf{E}\left|\xi(t) - \sum_{j=1}^n \varphi_j(t) z_j\right|^2\\ = k(t,t) - 2\sum_{j=1}^n (\mathsf{E}\,\xi(t)\overline{z_j})\overline{\varphi_j(t)} + \sum_{j=1}^n \lambda_j|\varphi_j(t)|^2 = k(t,t) - \sum_{j=1}^n \lambda_j|\varphi_j(t)|^2.\end{aligned}$$

Taking into account the mode of the convergence of the series on the right hand side of (3.10), we conclude that

$$\sigma_n(t) \to 0 \quad \text{in } L_2(T,\mu) \tag{3.14}$$

as $n\to\infty$, since inequality (3.8) for $\xi_n(t)$ becomes an equality in the diagonal of the square $T\times T$.

Therefore we prove the following result.

Theorem 3.1. *Let $\xi(t)$, $t\in T$, be a measurable second order random function being measurable in the strong sense and defined in the measurable space $(T,\mathcal{A},\mu)$. Assume that condition* (3.7) *holds for $\xi(t)$.*

Then representation (3.10) *holds for the correlation kernel $k(t,s)$. The function $\xi(t)$ itself admits the following representation:*

$$\xi(t) = \sum_{j\in J}\varphi_j(t) z_j, \qquad t\in T, \tag{3.15}$$

where the random variables z_j are defined by equality (3.11) *and where the series on the right hand side of* (3.15) *converges in the sense of relation* (3.14).

3.3. Let T be an arbitrary non-empty set and let $\xi(t)$, $t\in T$, be a second order random function defined in T. Denote the correlation kernel of $\xi(t)$ by $k(t,s)$.

Theorem 3.2. *I. For a random function $\xi(t)$, $t\in T$, there exists a system of ω-independent complex-valued functions $\{f_j(t), j\in J\}$ defined in T and being unique up to an unitary transformation such that*

$$k(t,s) = \sum_{j\in J} f_j(t)\overline{f_j(s)}, \qquad t,s\in T, \tag{3.16}$$

$$\xi(t) = \sum_{j\in J} f_j(t) z_j, \qquad t\in T, \tag{3.17}$$

where z_j are orthonormal random variables belonging to $L_2(\Omega)$ such that, given an arbitrary pair t and s of T, there is at most countable number of non-zero terms in (3.16) *and* (3.17). *The series on the right hand side of* (3.16) *converges for all t and s, while the series on the right hand side of* (3.17) *converges in the square mean sense for each t.*

II. If T is a topological space and if a function $\xi(t)$ is continuous in T in the square mean sense, then the functions $f_j(t)$ are continuous in T.

III. If T is an interval of $\mathbf{R}$ and if a process $\xi(t)$, $t \in T$, is m times differentiable in T in the square mean sense, then the functions $f_j(t)$ are m times differentiable in T. Moreover, if the process $\xi(t)$ is analytical in T, then the functions $f_j(t)$ are analytical, too.

Proof. I. In fact, representation (3.16) is the Krein bilinear expansion of the positive definite kernel k (this result follows from Theorem 1 in [8]). Note that one can obtain such a representation by using the factorization expansion of the positive definite kernel k in an appropriate Hilbert space. The most known factorization spaces for k are the Kolmogorov space of second order random variables, the Krein space of aggregates in T, the Aronszajn–Parzen space of functions constructed from the reproducing kernel k, and space L_2 of the factorization representation of k in Karhunen's theorem [3, 8].

Representation (3.17) for the random function $\xi(t)$, $t \in T$, follows from representation (3.16) by using the Karhunen theorem with counting measure in J.

We deduce from expansion (3.17) that the functions $f_j(t)$ are of the form

$$f_j(t) = \mathsf{E}\,\xi(t)\overline{z_j}, \qquad j \in J,\ t \in T,$$

whence statements II and III of the theorem follow directly. □

3.4. The representation

$$\Xi_t = \sum_{j \in J} \alpha_j(t)\Phi_j, \qquad t \in T, \tag{3.18}$$

of a generalized second order random function Ξ_t, $t \in T$, assuming values in a Hilbert space H is called a basis type discrete representation if representation (3.18) is unique (in the sense that if Φ_j are given, then the functions $\alpha_j(t)$ are determined uniquely, and, conversely, if $\alpha_j(t)$ are given, then the elements Φ_j are determined uniquely), if there are at most a countable number of non-zero terms in (3.18), and if the series on the right hand side of (3.18) converges in a certain topology of the space $\mathcal{L}(H, L_2(\Omega))$, where Φ_j, $j \in J$, is a system of random elements in $\mathcal{L}(H, L_2(\Omega))$ and $\{\alpha_j(t), t \in J\}$ is the corresponding set of complex-valued functions.

If the elements Φ_j are orthogonal, that is, if $\Phi_j^*\Phi_r = 0$ for $j \neq r$, then representation (3.18) is called orthogonal.

If representation (3.18) holds, then the correlation kernel Γ of the function Ξ_t admits the following expansion:

$$R(t,s) = \sum_{j,r \in J} \alpha_j(t)\overline{\alpha_r(s)}F_{jr}, \qquad F_{jr} \in \mathcal{B}(H), \tag{3.19}$$

where $F_{jr} = \Phi_r^*\Phi_j$. Conversely, if expansion (3.19) holds, where F_{jr} is a positive definite $\mathcal{B}(H)$-valued kernel defined in $J \times J$, then Theorem 2 of [13] implies representation (3.18), where $\Phi_j \in \mathcal{L}(H, L_2(\Omega))$ and $F_{jr} = \Phi_r^*\Phi_j$.

Below are some examples of discrete representations for random functions Ξ_t being continuous in the strong topology of $\mathcal{L}(H, L_2(\Omega))$. The consideration below is based on some of the results of harmonic analysis.

Example 3.2. Let Ξ_t, $t \in \mathbf{R}^n$, be a continuous homogeneous random field in H, that is, $\mathsf{E}\,\Xi_t = \text{const}$ and $\Xi_s^*\Xi_t = R(t-s)$, $t, s \in \mathbf{R}^n$. Then Ξ_t and $R(t)$ admit the spectral representations

$$\Xi_t = \int_{\mathbf{R}^n} e^{i(\lambda|t)}\,\Phi(dt), \qquad R(t) = \int_{\mathbf{R}^n} e^{i(\lambda|t)}\,F(d\lambda), \tag{3.20}$$

where Φ and F are operator $\mathcal{L}(H, L_2(\Omega))$-valued and $\mathcal{B}(H)$-valued Radon measures in $\mathbf{R}^n$, respectively. Moreover, the measures Φ and F are such that $F(\triangle_1 \cap \triangle_2) = \Phi^*(\Delta_2)\Phi(\Delta_1)$ for Borel sets Δ_1 and Δ_2 of $\mathbf{R}^n$ (see [13]).

If the measure F is concentrated in the parallelepiped $\times_{r=1}^n[-\ell_r, \ell_r]$, $\ell_r > 0$, belonging to the space $\mathbf{R}^n$, then the field Ξ_t admits a non-orthogonal basis type representation of the following form:

$$\Xi_t = \sum_{j=(j_r)_{r=1}^n \in \mathbf{Z}^n} \prod_{r=1}^n \frac{\sin(\ell_r t_r - \pi j_r)}{(\ell_r t_r - \pi j_r)} \Phi_j^\ell, \qquad t \in \mathbf{R}^n, \tag{3.21}$$

where $\Phi_j^\ell = \Xi_{t_j^\ell} \in \mathcal{L}(H, L_2(\Omega))$, $t_j^\ell = (j_r\pi/\ell_r)_{r=1}^n \in \mathbf{R}^n$, and where the series on the right hand side of (3.21) converges in the strong topology of $\mathcal{L}(H, L_2(\Omega))$. Representation (3.21) is an analogue of the Kotelnikov–Shannon formula for scalar random fields. Representation (3.21) can be proved by expanding the function $g(\lambda) = e^{i(\lambda|t)}$ defined in $\times_{r=1}^n[-\ell_r, \ell_r]$ into the multiple Fourier series

$$e^{(\lambda|t)} = \sum_{j \in \mathbf{Z}^n} \prod_{r=1}^n \frac{\sin\,(\ell_r t_r - \pi j_r)}{(\ell_r t_r - \pi j_r)} e^{(\lambda|t_j^\ell)},$$

by applying representation (3.20), and by using the properties of the stochastic integral with respect to the measure Φ.

Example 3.3. Let Ξ_q, $q \in Q$, be a continuous homogeneous random function in H defined on a homogeneous topological space Q with a compact group of transitive transformations of G, that is, $\mathsf{E}\,\Xi_q = \text{const}$ and the correlation kernel R of the function $\Xi_q \in G$ is invariant, that is, $R(gp, gq) = R(p, q)$ for all $g \in G$ and $p, q \in Q$.

Then the random function Ξ_q admits the discrete expansion with respect to the system of spherical functions

$$\left\{\psi_{ij}^{(\lambda)}(q), i = 1, \ldots, d_\lambda, j = 1, \ldots, r_\lambda, \lambda = 1, 2, \ldots\right\}$$

on Q, namely

$$\Xi_q = \sum_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{r_\lambda} \psi_{ij}^{(\lambda)}(q)\Phi_{ij}^{(\lambda)}, \qquad q \in Q, \tag{3.22}$$

where the random elements $\Phi_{ij}^{(\lambda)} \in \mathcal{L}(H, L_2(\Omega))$ are defined by

$$\Phi_{ij}^{(\lambda)} = \left(\int_Q \left|\psi_{ij}^{(\lambda)}(q)\right|^2 dq\right)^{-1} \int_Q \overline{\psi_{ij}^{(\lambda)}(q)}\Xi_q\,dq,$$

where dq denotes a G-invariant measure on Q [17] and where the series on the right hand side of (3.22) converges in the strong topology of the space $\mathcal{L}(H, L_2(\Omega))$.

Example 3.4. Consider an important special case of a general scheme considered in Example 3.3 if G is a rotation group for the sphere $SO(3)$ centered at the point 0 and if the space Q coincides with the sphere S_2 in $\mathbf{R}^3$ centered at 0 and with the spherical coordinates (θ, ϕ). Then expansion (3.22) for a homogeneous field $\Xi_{\theta,\varphi}$ on S_2 can be rewritten as follows:

$$\Xi_{\theta,\phi} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta, \phi)\Phi_m^\ell, \tag{3.23}$$

where $\{Y_\ell^m(\theta,\phi); m = -\ell, \dots, \ell; \ell = 0, 1 \dots\}$ is the system of spherical harmonics in S_2, $\Phi_m^\ell \in \mathcal{L}(H, L_2(\Omega))$, and where $(\Phi_j^k)^* \Phi_m^\ell = \delta_{mj}\delta_{ik}F_m$ for $F_m \in \mathcal{B}_+(H)$ [17, 18].

A homogeneous field defined on the sphere S_{n-1} of the space $\mathbf{R}^n$ centered at the point 0 and with the group of rotations of the sphere $SO(n)$ can be represented in terms of the series with respect to hyper-spherical harmonics

$$Y_{\ell,m_1,\dots,m_{n-3},\pm m_{n-2}}(\theta_1, \dots, \theta_{n-2}, \phi), \qquad \ell = 0, 1, 2, \dots,$$
$$0 \le m_{n-2} \le m_{n-3} \le \dots \le m_1 \le \ell,$$

where $(\theta_1, \dots, \theta_{n-2}, \varphi)$ are spherical coordinates of a point of S_{n-1} and where

$$\Phi_{\ell,m_1,\dots,m_{n-3},\pm m_{n-2}} \in \mathcal{L}(H, L_2(\Omega))$$

are uncorrelated random coefficients whose correlation operators depend on ℓ only (see [16, 17]).

Example 3.5. Let Ξ_g, $g \in G$, be a continuous homogeneous random function that assumes values in H and is defined on a compact Abelian group G, that is, $\mathsf{E}\,\Xi_q = \text{const}$ and $\Xi_s^*\Xi_g = R(gs^{-1})$. Since the group of characters $\hat{G}$ of the group G is discrete, we obtain the expansion

$$\Xi_g = \sum_{\chi\in\hat{G}} \chi(g)\Phi_\chi, \qquad \chi(g) \in \hat{G}, \quad \Phi_\chi \in \mathcal{L}(H, L_2(\Omega)), \tag{3.24}$$

where $\Phi_\gamma^*\Phi_\chi = \delta_{\chi\gamma}F_\chi$ for $F_\chi \in \mathcal{B}_+(H)$, the series on the right hand side of (3.24) converges in the strong topology of the space $\mathcal{L}(H, L_2(\Omega))$, and where the series $\sum_{\chi\in\hat{G}} F_\chi$ converges to $R(e)$ (see [13]). Here the symbol e stands for the unit element of the group G.

Example 3.6. Let Ξ_t, $t \in T$, be a continuous stationary process defined on the torus $T = \{t \in \mathbf{C}\colon |t| = 1\}$ and let Ξ_t assume values in H. Then $\hat{T}$ is isomorphic to the additive group $\mathbf{Z} = \{0, \pm1, \pm2, \dots\}$, that is, $\hat{T} \cong \mathbf{Z}$. Thus the characters of the multiplicative group T are of the form $\chi_n(t) = t^n$, $t \in T$, $n \in Z$. Therefore

$$\Xi_t = \sum_{n\in\mathbf{Z}} t^n\Phi_n, \qquad \Phi_n \in \mathcal{L}(H, L_2(\Omega)), \quad t \in T, \tag{3.25}$$

where $\Phi_m^*\Phi_n = \delta_{nm}F_n$, $F_n \in \mathcal{B}_+(H)$, and where the series $\sum_{n\in\mathbf{Z}} F_n$ converges to $R(1) = \Xi_1^*\Xi_1$.

For the case of Example 3.5 with G being the m-dimensional torus T^m, a homogeneous random field Ξ_t, $t \in T^m$, in H admits the representation

$$\Xi_{t_1,\dots,t_m} = \sum_{(n_1,\dots,n_m)\in\mathbf{Z}^m} t_1^{n_1}t_2^{n_2}\cdots t_m^{n_m}\Phi_{n_1,\dots,n_m}, \qquad (t_1, \dots, t_m) \in T^m, \tag{3.26}$$

where $\Phi_{n_1,\dots,n_m}$ are orthogonal random elements of the space $\mathcal{L}(H, L_2(\Omega))$.

Example 3.7. Let G_n be a cyclic group of order n in Example 3.5 (the topology is discrete). Consider the simplest isomorphic realization of G_n as the set of integer numbers $\{0, 1, \ldots, n-1\}$ with the mod n addition operation. Then a stationary (or cyclic) random process Ξ_s, $s \in G_n$, in H admits the representation

$$\Xi_s = \sum_{\lambda=1}^{n} \exp\left\{\frac{i2\pi\lambda s}{n}\right\} \Phi_\lambda, \qquad \Phi_\lambda \in \mathcal{L}(H, L_2(\Omega)), \tag{3.27}$$

where $\Phi_\mu^* \Phi_\lambda = \delta_{\mu\lambda} F_\lambda \in \mathcal{B}_+(H)$.

If G is a direct product of the cyclic groups, $G = G_{n_1} \times G_{n_2} \times \cdots \times G_{n_m}$, then a random field Ξ_s, $s \in G$, considered in Example 3.5 is homogeneous (cyclic) and admits the following representation:

$$\Xi_{s_1,\ldots,s_m} = \sum_{\lambda_1=1}^{n_1} \cdots \sum_{\lambda_m=1}^{n_m} \prod_{r=1}^{m} e^{\frac{i2\pi\lambda_r s_r}{n_r}} \Phi_{\lambda_1,\ldots,\lambda_m}, \tag{3.28}$$

where $\Phi_{\lambda_1,\ldots,\lambda_m}$ are orthogonal random elements in $\mathcal{L}(H, L_2(\Omega))$.

Example 3.8. Let Ξ_t, $t \in S$, be a stationary random function in H defined on an involutive Abelian semigroup $(S, \circ, *)$, where the symbol $\circ$ stands for the operation in the semigroup and where $*$ denotes the involution (see [10]), that is, $\mathsf{E}\,\Xi_t = \text{const}$ and $\Xi_s^* \Xi_t = R(t \circ s^*)$. The function Ξ_t, $t \in S$, has a discrete spectrum in the semigroup S^* of semicharacters of the semigroup S if there exists at most a countable family of semicharacters $\{\chi_j, j \in J\}$, $\chi_j \in S^*$, such that

$$R(t) = \sum_{j \in J} \chi_j(t) F_j, \qquad F_j \in \mathcal{B}_+(H),\ t \in S,$$

where the series converges in the weak topology $\mathcal{B}(H)$. According to Theorem 2 of [13], the following discrete representation holds:

$$\Xi_t = \sum_{j \in J} \chi_j(t) \Phi_j, \qquad \Phi_j \in \mathcal{L}(H, L_2(\Omega)),\ t \in S,$$

where the series on the the right hand side converges in the strong topology of the space $\mathcal{L}(H, L_2(\Omega))$ and where $\Phi_r^* \Phi_j = \delta_{rj} F_j$.

Example 3.9. Let Ξ_t, $t \in [0, \ell]$, be a continuous second order process in the space H. Then its correlation kernel $R(t, s)$ admits a representation in terms of a double Fourier series, namely

$$R(t, s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \cos\frac{\pi m t}{\ell} \cos\frac{\pi n t}{\ell} F_{nm}, \qquad F_{nm} \in \mathcal{B}(H),$$

where F_{nm} is a positive definite kernel in the set $0, 1, 2, \ldots,$

$$F_{nm} = \frac{4}{\ell^2} a_{nm} \int_0^\ell \int_0^\ell \cos\frac{\pi n t}{\ell} \cos\frac{\pi m s}{\ell} R(t, s)\, dt\, ds,$$

$$a_{nm} = \begin{cases} \frac{1}{4}, & m = n = 0, \\ \frac{1}{2}, & m > 0,\ n = 0 \text{ or } m = 0,\ n > 0, \\ 1, & m > 0,\ n > 0, \end{cases}$$

and where the series on the right hand side converges in the weak topology of $\mathcal{B}(H)$. Then we obtain the representation

$$\Xi_t = \sum_{n=0}^{\infty} \cos\frac{\pi n t}{\ell}\Phi_n, \qquad \Phi_n \in \mathcal{L}(H, L_2(\Omega)),$$

where $\Phi_m^*\Phi_n = F_{nm}$ and the series converges in the strong topology of $\mathcal{L}(H, L_2(\Omega))$.

Example 3.10. Let $\mathcal{D}$ be a connected open domain of the complex plane $\mathbf{C}$ and let Ξ_ς, $\varsigma \in \mathcal{D}$, be a generalized second order random function in H being differentiable in $\mathcal{D}$ in the weak topology of $\mathcal{L}(H, L_2(\Omega))$. Then Ξ_ς is called holomorphic in $\mathcal{D}$. Note that the random function Ξ_ς is continuous and infinitely many times differentiable in $\mathcal{D}$ for the uniform topology $\mathcal{L}(H, L_2(\Omega))$. Moreover, the following equality holds for the derivatives $\Xi_\varsigma^{(n)}$, $n = 0, 1, 2, \dots$:

$$\Xi_\varsigma^{(n)} = \frac{n!}{2\pi i}\int_C \frac{\Xi_\tau\, d\tau}{(\tau-\varsigma)^{n+1}},$$

where C is the circumference centered at ς and belonging to $\mathcal{D}$.

If Ξ_ς is holomorphic in the circle $T_r(\varsigma_0) = \{\varsigma\colon |\varsigma-\varsigma_0| < r\}$ and if $\|\Xi_\varsigma\| \le M$, then the following Taylor expansion holds:

$$\Xi_\varsigma = \sum_{n=0}^{\infty}\frac{1}{n!}(\varsigma-\varsigma_0)^n\Xi_{\varsigma_0}^{(n)}, \qquad \varsigma \in T_r(\varsigma_0),$$

where the series converges in the uniform topology of the space $\mathcal{L}(H, L_2(\Omega))$ (see [11]).

Example 3.11. Let Ξ_t, $t \in \mathbf{R}^n$, be a continuous random field in $\mathbf{R}^n$ whose correlation kernel can be written as a function that depends on the sum of its arguments,

$$\Xi_s^*\Xi_t = R(t+s), \qquad t, s \in \mathbf{R}^n,$$

and let $\mathsf{E}\,\Xi_t = \text{const}$. Then the field Ξ_t is additive homogeneous and admits the following spectral representation:

$$\Xi_t = \int_{\mathbf{R}^n} e^{(\lambda|t)}\,\Phi(d\lambda), \qquad t \in \mathbf{R}^n,$$

where Φ is an orthogonal $\mathcal{L}(H, L_2(\Omega))$-valued measure on the σ-algebra of Borel sets of $\mathbf{R}^n$. The field Ξ_t has an analytic continuation $\tilde{\Xi}_\varsigma$, $\varsigma \in \mathbf{C}^n$, that can be represented in the form of a power series,

$$\tilde{\Xi}_{\varsigma_1,\dots,\varsigma_n} = \sum_{(m_1,\dots,m_n)\ge 0}\frac{\varsigma_1^{m_1}\varsigma_2^{m_2}\cdots\varsigma_n^{m_n}}{m_1!\,m_2!\,\cdots\,m_n!}\Phi_{m_1,\dots,m_n},$$

where

$$\Phi_{m_1,\dots,m_n} = \tilde{\Xi}_0^{(m_1,\dots,m_n)} = \int_{\mathbf{R}^n}\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n}\,\Phi(d\lambda_1\cdots d\lambda_n)$$

and where the series converges in the strong topology of $\mathcal{L}(H, L_2(\Omega))$ (see [19]).

3.5. Let T be an arbitrary non-empty set and let $k(t,s)$ be a complex positive definite kernel defined on $T \times T$.

Theorem 3.3. *If Ξ_t, $t \in T$, is a second order random function in H with the correlation kernel $R(t,s) = k(t,s)A$, $A \in \mathcal{B}_+(H)$, then there exists a system of complex-valued ω-independent functions $\{\alpha_j(t), j \in J\}$ being defined on T and being unique up to a unitary* *transformation such that*

$$R(t,s)=\sum_{j\in J}\alpha_j(t)\overline{\alpha_j(s)}A,\qquad t,s\in T, \tag{3.29}$$

$$\Xi_t=\sum_{j\in J}\alpha_j(t)\Phi_j,\qquad \Phi_j\in\mathcal{L}(H,L_2(\Omega)), \tag{3.30}$$

where at most a countable number of terms are non-zero and $\Phi_r^*\Phi_j=\delta_{jr}A$. *The series on the right hand side of* (3.29) *converges in the uniform topology of* $\mathcal{B}(H)$ *for all* $t,s\in T$, *while the series on the right hand side of* (3.30) *converges in the uniform topology of* $\mathcal{L}(H,L_2(\Omega))$ *for all* $t\in T$.

To prove Theorem 3.3 we note that expansion (3.29) follows from Krein's expansion for the kernel $k(t,s)$ mentioned in Section 3.3. Expansion (3.30) follows by an application of Theorem 2 in [13] to expansion (3.29). The convergence of the series in (3.30) in the corresponding sense follows from that of the series on the right hand side of (3.29).

Example 3.12. Let the kernel $k(t,s)$ of the random function Ξ_t, $t\in T$, in Theorem 3.3 admit the Karhunen factorization representation

$$k(t,s)=\int_\Lambda f(t,\lambda)\overline{f(s,\lambda)}\,\mu(d\lambda),$$

where $f(t,\cdot)\in L_2(\Lambda,\mathcal{B},\mu)$ and where $(\Lambda,\mathcal{B},\mu)$ is a measurable space with a positive measure μ. Denote by $\{g_j(\lambda),j\in J\}$ an orthonormal basis in $L_2(\Lambda,\mathcal{B},\mu)$. Then representations (3.29) and (3.30) hold with the functions

$$\alpha_j(t)=\int_\lambda f(t,\lambda)\overline{g_j(\lambda)}\,\mu(d\lambda).$$

Example 3.13. Let Ξ_t be a second order random field in H with n-dimensional orthogonal increments in the n-dimensional cube $M^n=[0,2\pi]^n$ of $\mathbf{R}^n$. Assume that the correlation kernel is such that

$$R(t,s)=\left(\prod_{r=1}^n\min(t_r,s_r)\right)A,\qquad A\in\mathcal{B}_+(H).$$

Denote by $\mathbb{1}_t(\lambda)$, $\lambda\in M^n$, the indicator function of the parallelepiped $\times_{r=1}^n[0,t_r]$, $t=(t_1,\dots,t_n)\in M^n$. Then $\mathbb{1}_t(\lambda)\in L_2(M^n)$ and

$$R(t,s)=(\mathbb{1}_t|\mathbb{1}_s)_{L_2(M^n)}A,$$

where M^n is viewed as a measurable space with the Lebesgue measure.

The functions

$$g_j(\lambda)=(2\pi)^{-n/2}e^{i(j|\lambda)},\qquad j\in\mathbf{Z}^n,\ \lambda\in M^n,$$

form an orthonormal system in $\mathcal{L}_2(M^n)$ and the multiple Fourier series

$$\sum_{j\in Z^n}\beta_j(t)g_j(\lambda)$$

for $\mathbb{1}_t(\lambda)$ converges to $\mathbb{1}_t$ in $L_2(M^n)$, where

$$\begin{aligned}\beta_j(t)&=(2\pi)^{-n/2}\int_{M^n}\mathbb{1}_t(\lambda)e^{-i(j|\lambda)}\,d\lambda\\&=(2\pi)^{-n/2}\prod_{r=1}^n\int_0^{t_r}e^{-ij_r\lambda_r}\,d\lambda_r=(2\pi)^{-n/2}\prod_{r=1}^n\frac{1-e^{-ij_rt_r}}{ij_r}.\end{aligned}$$

Then the Parseval equality implies that

$$R(t,s) = (2\pi)^{-n}\left[\sum_{j\in\mathbf{Z}^n}\prod_{n=1}^{n}\frac{(1-e^{-ij_rt_r})(1-e^{-ij_st_s})}{j_r^2}\right]A,$$

whence

$$\Xi_t = (2\pi)^{-n/2}\left[\sum_{j\in\mathbf{Z}^n}\prod_{r=1}^{n}\frac{1-e^{ij_rt_r}}{ij_r}\Phi_j\right], \qquad t=(t_1,\dots,t_n)\in M^n,$$

by Theorem 3.3, where $\Phi_s^*\Phi_j = \delta_{js}A$ for $j,s\in\mathbf{Z}^n$. If $n=1$, then the process has orthogonal increments.

Example 3.14. Let Ξ_t, $t\in[0,\ell]$, be a real-valued Gaussian process assuming values in H and let its correlation kernel be given by

$$R(t,s) = \left(\int_0^{t\wedge s} f(t,\lambda)f(s,\lambda)d\lambda\right)A, \qquad A\in\mathcal{B}_+(H),\ t,s\in[0,\ell],$$

where $t\wedge s = \min(t,s)$ and

$$f(t,\lambda) = \left(\frac{2h\Gamma\left(\frac{3}{2}-h\right)}{\Gamma\left(h+\frac{1}{2}\right)\Gamma(2-2h)}\right)^{1/2} \times\left[\left(\frac{1}{\lambda}\right)^{h-\frac{1}{2}}(t-\lambda)^{h-\frac{1}{2}} - \left(h-\frac{1}{2}\right)\lambda^{\frac{1}{2}-h}\int_\lambda^t u^{h-\frac{3}{2}}(u-\lambda)^{h-\frac{1}{2}}\,du\right],$$
$$h\in(0,1).$$

Assume that $\mathsf{E}\,\Xi_t = 0$. By analogy with the case of $\dim H = 1$, the process Ξ_t can naturally be called a generalized fractional Brownian motion in H with Hurst index h (see [4]). Thus if $g_j(t)$, $t\in[0,\ell]$, is an orthonormal basis in the space $L_2[0,\ell]$, then Ξ_t admits the representation

$$\Xi_t = \sum_{j\in J}\alpha_j(t)\Phi_j, \qquad \Phi_j\in\mathcal{L}(H,L_2(\Omega)), \quad \Phi_r^*\Phi_j = \delta_{rj}A,$$

where

$$\alpha_j(t) = \int_0^t f(t,\lambda)g_j(\lambda)\,d\lambda, \qquad j\in J.$$

In particular, if $h=\frac{1}{2}$, then the process Ξ_t is a generalized Brownian motion in H with $f(t,\lambda)\equiv 1$.

Example 3.15. Let Ξ_t, $t\in\mathbf{R}$, be a stochastic process in H whose correlation kernel is given by

$$R(t,s) = \left(\int_{-\infty}^{\infty} f(t,\lambda)\overline{f(s,\lambda)}\,\mu(d\lambda),\right)A, \qquad A\in\mathcal{B}_+(H),$$

where μ is a standard Gaussian measure on $\mathbf{R}$, that is, μ possesses a density

$$(2\pi)^{-1/2}\exp\left\{\frac{-\lambda^2}{2}\right\}$$

with respect to the Lebesgue measure in $\mathbf{R}$. Then Theorem 3.3 and Example 3.9 imply that the process Ξ_t admits the representation of the form (3.30), where $J=\{0,1,2,\dots\}$ and

$$\alpha_j(t) = \int_{-\infty}^{\infty} f(t,\lambda)\frac{H_j(\lambda)}{\sqrt{j!}}\,\mu(d\lambda), \qquad j\in J,$$

and where $H_j(\lambda)$ are Hermite polynomials, since the system

$$g_j(\lambda) = \frac{H_j(\lambda)}{\sqrt{j!}}$$

forms an orthonormal basis in $L_2(R, \mu)$ (see [18]).

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Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue, 4E, Kiev 03127, Ukraine

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