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RIEMANN INTEGRAL OF A RANDOM FUNCTION AND THE PARABOLIC EQUATION WITH A GENERAL STOCHASTIC MEASURE

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Abstract. For a stochastic parabolic equation driven by a general stochastic measure, the weak solution is obtained. The integral of a random function in the equation is considered as a limit in probability of Riemann integral sums. Basic properties of such integrals are studied in this paper.

1. Introduction

In this paper we consider the stochastic parabolic equation, which can be formally written as

$$dX(x,t) = AX(x,t)\,dt + f(x,t)\,d\mu(t), \qquad X(x,0) = \xi(x), \tag{1.1}$$

where $(x,t) \in \mathbb{R}^d \times [0,T]$, A is a second-order strongly elliptic differential operator, and μ is a general stochastic measure defined on the Borel σ-algebra of $[0,T]$. For μ we assume σ-additivity in probability only; assumptions for A, f and ξ are given in Section 6. Equation (1.1) is interpreted in the weak sense (see (6.1) below). We prove the existence and uniqueness of the solution.

The weak form of (1.1) includes the integral of a random function with respect to deterministic measure (Jordan content). We interpret this integral as a limit in the probability of Riemann integral sums. This definition of the integral allows us to interchange the order of integration with respect to deterministic and stochastic measures (Theorem 4.1) that is important for solving the equation. A large part of the paper is devoted to the study of this Riemann-type stochastic integral.

Parabolic stochastic partial differential equations (SPDEs) driven by the martingale measures had been introduced and discussed initially in [19]. This approach was later developed in [1, 3]. Parabolic SPDEs as equations in infinite dimensional space were studied in [4, 11]. In these and many other papers the stochastic noise has some distributional, integrability or martingale properties. In our paper, we consider a very general class of possible μ on $[0,T]$. On the other hand, the stochastic term in (1.1) is independent of u. One reason is that the appropriate definition of an integral of a random function with respect to μ does not exist.

Some motivating examples for studying SPDEs may be found in [4, Introduction] and [6, section 13.2]. For $A = \Delta$, (1.1) describes the evolution in time of the density X

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of some quantity for such a heat or chemical concentration in a system with random sources. In our model, the random influence can be rather general.

2. Preliminaries

Let $L_0 = L_0(\Omega, \mathcal{F}, \mathsf{P})$ be a set of all real-valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ (or equivalence classes of). Convergence in L_0 means the convergence in probability and is the convergence in the quasi-norm

$$\|\eta\| = \inf\{\delta\colon\ \mathsf{P}\{|\eta| > \delta\} \le \delta\}.$$

Note that $\|\eta_1 + \eta_2\| \le \|\eta_1\| + \|\eta_2\|$. The following inequality will be used in the sequel:

$$\left\|\sum_{k=1}^{l} c_k\xi_k\right\| \le 8 \max_{a_k=\pm 1}\left\|\sum_{k=1}^{l} a_k\xi_k\right\| \le 16 \max_{V}\left\|\sum_{k\in V} \xi_k\right\|, \qquad |c_k| \le 1,\ \xi_k \in L_0, \tag{2.1}$$

where the latter maximum is taken over all possible $V \subset \{1, \dots, l\}$ (see [16, Theorem 3]).

Let S be an arbitrary set and $\mathcal{B}$ be a σ-algebra of subsets of S.

Definition 2.1. Any σ-additive mapping $\mu\colon \mathcal{B} \to L_0$ is called a *stochastic measure.*

In other words, μ is a vector measure with values in L_0. We do not assume positivity or integrability for stochastic measures. In [7] such a μ is called a general stochastic measure. In the following, μ always denotes a stochastic measure.

Examples of stochastic measures are the following. Let $\mathsf{S} = [0, T] \subset \mathbb{R}_+$, $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0, T]$, and $Y(t)$ be a square integrable martingale. Then $\mu(\mathsf{A}) = \int_0^T \mathbf{1}_{\mathsf{A}}(t)\, dY(t)$ is a stochastic measure. If $W^H(t)$ is a fractional Brownian motion with Hurst index $H > \frac{1}{2}$ and $f\colon [0, T] \to \mathbb{R}$ is a bounded measurable function, then $\mu(\mathsf{A}) = \int_0^T f(t)\mathbf{1}_{\mathsf{A}}(t)\, dW^H(t)$ is also a stochastic measure, as follows from [8, Theorem 1.1]. Some other examples may be found in [7, subsection 7.2]. Theorem 8.3.1 in [7] states the conditions under which the increments of a real-valued Lévy process generates a stochastic measure.

For deterministic measurable functions $g\colon \mathsf{S} \to \mathbb{R}$, an integral of the form $\int_{\mathsf{S}} g\, d\mu$ is studied in [12] (see also [7, Chapter 7] and [2]). The construction of this integral is standard, uses an approximation by simple functions and is based on results of [15, 17, 18]. In particular, every bounded measurable g is integrable with respect to any μ. An analogue of the Lebesgue dominated convergence theorem holds for this integral (see [7, Proposition 7.1.1] or [12, Corollary 1.2]).

For equations with stochastic measures, weak solutions of some SPDEs were obtained in [13]. Regularity properties of a mild solution of the stochastic heat equation were considered in [14].

3. Riemann integral of a random function

Let $\mathsf{B} \subset \mathbb{R}^d$ be a Jordan measurable set and $\xi\colon \mathsf{B} \to L_0$ be a random function. We shall say that ξ has an integral on B if for any sequence of partitions

$$\mathsf{B} = \bigcup_{1\le k\le k_n} \mathsf{B}_{kn}, \quad n \ge 1, \qquad \max_k \operatorname{diam} \mathsf{B}_{kn} \to 0, \quad n \to \infty, \qquad x_{kn} \in \mathsf{B}_{kn},$$

the limit in probability

$$\operatorname{p}\lim_{n\to\infty} \sum_{1\le k\le k_n} \xi(x_{kn})\mathsf{m}(\mathsf{B}_{kn}) = \int_{\mathsf{B}} \xi(x)\, dx \tag{3.1}$$

exists. Here m denotes the Jordan content, and the sets B_{kn}, $1 \le k \le k_n$, are assumed to be Jordan measurable and have no common interior points. By the mixing of different sequences of partitions, we can prove that the limit is independent of the choice of the sequence. For deterministic ξ, our definition is equivalent to the definition of the standard Riemann integral in [9].

Lemma 3.1. *Let* ξ *be an integral on* $\mathsf{B} = \prod_{k=1}^{d}[a_k, b_k] \subset \mathbb{R}^d$. *Then the set of values* $\{\xi(x), x \in \mathsf{B}\}$ *is bounded in probability.*

Proof. Is analogous to the deterministic case. ☐

For some other $\mathsf{B} \subset \mathbb{R}^d$, limit (3.1) can exist for unbounded ξ (for instance, in the case $\mathsf{m}(\mathsf{B}) = 0$). We use the following

Definition 3.1. The random function ξ is called *integrable on* B if ξ has an integral on B and set of values $\{\xi(x), x \in \mathsf{B}\}$ is bounded in probability.

Let $\tilde{\mathsf{B}} \subset \mathbb{R}^d$ be an unbounded set for which there exists a sequence of Jordan measurable sets $\mathsf{B}^{(j)}$ such that

$$\mathsf{B}^{(j)} \uparrow \tilde{\mathsf{B}}, \quad \forall c > 0\ \exists j\colon\ \tilde{\mathsf{B}} \cap \{|x| \le c\} \subset \mathsf{B}^{(j)} \tag{3.2}$$

(we call $\mathsf{B}^{(j)}$ the exhaustive set). We shall say that ξ is integrable (in an improper sense) on $\tilde{\mathsf{B}}$ if ξ is integrable on each $\mathsf{B}^{(j)}$, and there exists the limit in probability

$$\operatorname{p}\lim_{j\to\infty} \int_{\mathsf{B}^{(j)}} \xi(x)\,dx = \int_{\tilde{\mathsf{B}}} \xi(x)\,dx$$

that is independent of the choice of $\mathsf{B}^{(j)}$.

All bounded subsets of $\mathbb{R}^d$ used in this paper are assumed to be Jordan measurable, and all unbounded sets are assumed to be approximable by Jordan measurable sets in the sense of (3.2). Sets in partitions are assumed to be non-overlapping.

Obviously, if ξ has the Riemann integrable paths, then ξ is integrable in our sense. Theorem 4.1 below gives other examples of integrable random functions.

Further, we establish basic properties of the integral.

Lemma 3.2. *Let* ξ *be integrable on* B. *Then* ξ *is integrable on each* $\mathsf{A} \subset \mathsf{B}$, *and for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $\mathsf{A} \subset \mathsf{B}$, $\mathsf{A} = \bigcup_{1\le k\le k_0} \mathsf{A}_k$, $x_k \in \mathsf{A}_k$, *and* $\operatorname{diam} \mathsf{A}_k < \delta$, *there holds*

$$\left\| \sum_{1\le k\le k_0} \xi(x_k)\mathsf{m}(\mathsf{A}_k) - \int_{\mathsf{A}} \xi(x)\,dx \right\| < \varepsilon.$$

Proof. Suppose the lemma were false. Then

$$\exists \varepsilon_0 > 0\ \forall \delta > 0\ \exists \mathsf{A} = \bigcup_{1\le k\le k_0} \mathsf{A}_k = \bigcup_{1\le i\le i_0} \mathsf{A}'_i,\ \operatorname{diam} \mathsf{A}_k, \operatorname{diam} \mathsf{A}'_i < \delta\colon$$

$$\left\| \sum_{1\le k\le k_0} \xi(x_k)\mathsf{m}(\mathsf{A}_k) - \sum_{1\le i\le i_0} \xi(x'_i)\mathsf{m}(\mathsf{A}'_i) \right\| \ge \varepsilon_0.$$

Take an arbitrary partition

$$\mathsf{B} \setminus \mathsf{A} = \bigcup_{1\le j\le j_0} \mathsf{C}_j, \qquad \operatorname{diam} \mathsf{C}_j < \delta,$$

and add

$$\sum_{1\le j\le j_0} \xi(x''_j)\mathsf{m}(\mathsf{C}_j), \qquad x''_j \in \mathsf{C}_j,$$

to each of the considered sums on A. Thus we can get two integral sums on B with arbitrarily small diameters such that the quasi-norm of their difference is greater than or equal to ε_0. This contradicts the integrability of ξ on B. ☐

Lemma 3.3. *Let ξ be integrable on $\tilde{\mathsf{B}}$ in the improper sense, $\tilde{\mathsf{A}} \subset \tilde{\mathsf{B}}$. Then ξ is integrable on $\tilde{\mathsf{A}}$ (if $\tilde{\mathsf{A}}$ is an unbounded set, the integral is meant in the improper sense).*

Proof. Take the exhaustive sets $\mathsf{B}^{(j)} \uparrow \tilde{\mathsf{B}}$, $\mathsf{A}^{(i)} \uparrow \tilde{\mathsf{A}}$. Then the sets

$$\big(\mathsf{B}^{(j)} \setminus \tilde{\mathsf{A}}\big) \cup \mathsf{A}^{(i)} \uparrow \tilde{\mathsf{B}}, \qquad i, j \to \infty$$

are exhaustive too, and

$$\int_{\tilde{\mathsf{B}}} \xi(x)\,dx = \operatorname{p}\lim_{i,j\to\infty} \left(\int_{\mathsf{B}^{(j)}\setminus\tilde{\mathsf{A}}} \xi(x)\,dx + \int_{\mathsf{A}^{(i)}} \xi(x)\,dx \right). \tag{3.3}$$

If $\operatorname{p}\lim_{i\to\infty} \int_{\mathsf{A}^{(i)}} \xi(x)\,dx$ does not exist, then we can choose $i, j \to \infty$ such that the limit in (3.3) does not exist. □

Lemma 3.4. *Let ξ be integrable on B. Then the set of values $\big\{\int_{\mathsf{A}} \xi(s)\,ds, \mathsf{A} \subset \mathsf{B}\big\}$ is bounded in probability.*

Proof. Suppose the lemma were false. Then

$$\exists \varepsilon_0 > 0,\ \mathsf{A}_n \subset \mathsf{B},\ n \ge 1\colon \qquad \left\| \frac{1}{n} \int_{\mathsf{A}_n} \xi(s)\,ds \right\| \ge \varepsilon_0.$$

By Lemma 3.2, we can choose a partition $\mathsf{B} = \bigcup_{1\le k\le k_0} \mathsf{B}_k$ fine enough such that all integral sums for partitions $\mathsf{A}_n = \bigcup_{1\le k\le k_0} (\mathsf{A}_n \cap \mathsf{B}_k)$ will be close enough to the integrals on A_n. Thus, for all n, $x_{kn} \in \mathsf{A}_n \cap \mathsf{B}_k$, we get

$$\left\| \frac{1}{n} \sum_{1\le k\le k_0} \xi(x_{kn}) \mathsf{m}(\mathsf{A}_n \cap \mathsf{B}_k) \right\| \ge \frac{\varepsilon_0}{2}.$$

Since the number of summands is fixed for all n, we arrive at a contradiction with the boundedness of ξ. □

Lemma 3.5. *Let ξ be integrable on B, and $f\colon \mathsf{B} \to \mathbb{R}$ be a deterministic uniformly continuous function on B. Then $f\xi$ is integrable on B.*

Proof. Consider the difference of two integral sums of $f\xi$:

$$\begin{aligned}
&\left\| \sum_{1\le k\le k_m} f(x_{km})\xi(x_{km})\mathsf{m}(\mathsf{B}_{km}) - \sum_{1\le i\le i_n} f(x_{in})\xi(x_{in})\mathsf{m}(\mathsf{B}_{in}) \right\| \\
&\quad = \left\| \sum_{1\le k\le k_m, 1\le i\le i_n} [f(x_{km})\xi(x_{km}) - f(x_{in})\xi(x_{in})]\mathsf{m}(\mathsf{B}_{km} \cap \mathsf{B}_{in}) \right\| \\
&\quad \le \left\| \sum_{1\le k\le k_m, 1\le i\le i_n} [\xi(x_{km}) - \xi(x_{in})] f(x_{in})\mathsf{m}(\mathsf{B}_{km} \cap \mathsf{B}_{in}) \right\| \\
&\qquad + \left\| \sum_{1\le k\le k_m, 1\le i\le i_n} [f(x_{km}) - f(x_{in})]\xi(x_{km})\mathsf{m}(\mathsf{B}_{km} \cap \mathsf{B}_{in}) \right\| \\
&\quad = S_1 + S_2.
\end{aligned}$$

From (2.1), for $|f(x)| \le C$ we get

$$S_1 \le 16 \max_V \left\| C \sum_{(k,i)\in V} [\xi(x_{km}) - \xi(x_{in})]\mathsf{m}(\mathsf{B}_{km} \cap \mathsf{B}_{in}) \right\|, \tag{3.4}$$

where the maximum is taken over all possible sets of pairs (k, i).

For example, consider

$$\begin{aligned}
&\sum_{(k,i)\in V} \xi(x_{km})\mathsf{m}(\mathsf{B}_{km}\cap \mathsf{B}_{in}) \\
&\quad = \sum_{1\le k\le k_m} \xi(x_{km})\Bigg[\sum_{i\colon (k,i)\in V} \mathsf{m}(\mathsf{B}_{km}\cap \mathsf{B}_{in}) + \mathsf{m}(\mathsf{B}_{km}\cap \mathsf{B}_{i'n})\mathbf{1}_{x_{km}\notin(\bigcup_{i\colon (k,i)\in V}\mathsf{B}_{in})}\Bigg] \\
&\quad\quad - \sum_{1\le k\le k_m} \xi(x_{km})\mathsf{m}(\mathsf{B}_{km}\cap \mathsf{B}_{i'n})\mathbf{1}_{x_{km}\notin(\bigcup_{i\colon (k,i)\in V}\mathsf{B}_{in})} \\
&\quad = I_1 - I_2.
\end{aligned}$$

Here $\mathsf{B}_{i'n}$ is one of the sets B_{in}, $1\le i\le i_n$, that contains x_{km}. (If x_{km} lies on the border of $\mathsf{B}_{i'n}$, we take it only once.) I_1 and I_2 are integral sums and, by Lemma 3.2, they approximate the integrals of ξ on respective sets. Therefore, for a diameter small enough, $I_1 - I_2$ will be close to the integral on $\bigcup_{(k,i)\in V}(\mathsf{B}_{km}\cap\mathsf{B}_{in})$. Similarly,

$$\sum_{(k,i)\in V} \xi(x_{in})\mathsf{m}(\mathsf{B}_{km}\cap \mathsf{B}_{in})$$

approximates the integral on the same set, and we make the right hand side of (3.4) arbitrarily small by choosing the diameter.

Further, for any $\alpha > 0$, for diameter small enough and $\mathsf{B}_{km}\cap\mathsf{B}_{in} = \varnothing$, we have $|f(x_{km}) - f(x_{in})| < \alpha$ in S_2. Inequality (2.1) implies

$$S_2 \le 16\max_V \Bigg\|\alpha \sum_{(k,i)\in V} \xi(x_{km})\mathsf{m}(\mathsf{B}_{km}\cap \mathsf{B}_{in})\Bigg\|.$$

As before, we can make the sum arbitrarily close to the integral on $\bigcup_{(k,i)\in V}(\mathsf{B}_{km}\cap\mathsf{B}_{in})$. From Lemma 3.4 it follows that $S_2 \to 0$ as $\alpha\to 0$. □

Lemma 3.6. *Let ξ be integrable on B, and $f\colon \mathsf{B}\to\mathbb{R}$ be a deterministic uniformly continuous function on B, $|f(x)|\le C$. Then*

$$\left\|\int_{\mathsf{B}} f(x)\xi(x)\,dx\right\| \le 16\sup_{\mathsf{A}\subset\mathsf{B}}\left\|C\int_{\mathsf{A}}\xi(x)\,dx\right\|.$$

Proof. The inequality for respective integral sums follows from (2.1). Further, we pass to the limit and apply Lemmas 3.2 and 3.5. □

Lemma 3.7. *Let ξ be integrable on B, $f_n\colon\mathsf{B}\to\mathbb{R}$, $n\ge 1$, be a deterministic uniformly continuous functions on B, and $\sup_{x\in\mathsf{B}}|f_n(x)|\to 0$, $n\to\infty$. Then*

$$\int_{\mathsf{B}} f_n(x)\xi(x)\,dx \xrightarrow{\mathsf{P}} 0, \qquad n\to\infty.$$

Proof. The statement follows from Lemmas 3.4 and 3.6. □

Lemma 3.8. *Let ξ be integrable on an unbounded set $\tilde{\mathsf{B}}$ in an improper sense, and $f\colon\tilde{\mathsf{B}}\to\mathbb{R}$ be a deterministic bounded uniformly continuous function on $\tilde{\mathsf{B}}$. Then $f\xi$ is integrable on $\tilde{\mathsf{B}}$ in an improper sense.*

Proof. For $\mathsf{B}^{(j)}\uparrow\tilde{\mathsf{B}}$ and $|f(x)|\le C$ Lemma 3.6 implies

$$\left\|\int_{\mathsf{B}^{(j)}\setminus\mathsf{B}^{(i)}} f(x)\xi(x)\,dx\right\| \le 16\sup_{\mathsf{A}\subset(\mathsf{B}^{(j)}\setminus\mathsf{B}^{(i)})}\left\|C\int_{\mathsf{A}}\xi(x)\,dx\right\|. \tag{3.5}$$

If the left hand side of (3.5) does not tend to 0 as $i,j\to\infty$, then we can construct a sequence of bounded sets $\mathsf{C}^{(j)}\uparrow\tilde{\mathsf{B}}$ such that the sequence $\int_{\mathsf{C}_j}\xi(x)\,dx$, $j\ge 1$, is non-fundamental. □

Lemma 3.9. *Let ξ be integrable on an unbounded set $\tilde{\mathsf{B}}$ in an improper sense, and $f_n\colon \tilde{\mathsf{B}} \to \mathbb{R}$ be a deterministic bounded uniformly continuous on $\tilde{\mathsf{B}}$ functions,*

$$\sup_{n\geq 1, x\in\tilde{\mathsf{B}}} |f_n(x)| = C < \infty, \qquad \sup_{x\in\mathsf{B}} |f_n(x)| \to 0, \quad n\to\infty,$$

for all bounded $\mathsf{B}\subset\tilde{\mathsf{B}}$. Then

$$\int_{\tilde{\mathsf{B}}} f_n(x)\xi(x)\,dx \xrightarrow{\mathsf{P}} 0, \qquad n\to\infty.$$

Proof. Suppose the lemma is false. Applying Lemma 3.7, one can find $\varepsilon_0 > 0$, a subsequence f_{n_j}, $j\geq 1$, and a bounded disjoint sets $\mathsf{B}_j \subset (\tilde{\mathsf{B}}\cap\{|x|\geq j\})$ such that

$$\left\|\int_{\mathsf{B}_j} f_{n_j}(x)\xi(x)\,dx\right\| > \varepsilon_0.$$

From Lemma 3.6 it follows that there exist bounded disjoint sets $\mathsf{A}_j \subset (\tilde{\mathsf{B}}\cap\{|x|\geq j\})$ such that $\left\|C\int_{\mathsf{A}_j}\xi(x)\,dx\right\| > (\varepsilon_0/16)$. This contradicts the integrability of ξ on $\tilde{\mathsf{B}}$. □

Note that the stochastic continuity of ξ does not imply the integrability.

Example 3.1. Consider $\mathsf{B}=[0,1]$ and $\xi_k(\omega) = 5^k\mathbf{1}_{F_k}$, $k\geq 1$, where $\mathsf{P}(F_k)=1/k$, with F_k independent. Set

$$\begin{cases} \xi(0)=0, & \\ \xi(x)=\xi_k, & 2^{-2k-1}\leq x\leq 2^{-2k}, \\ \xi(x) = 2^{2k+2}\left(\left(2^{-2k-1}-x\right)\xi_{k+1} + \left(x-2^{-2k-2}\right)\xi_k\right), & 2^{-2k-2}\leq x\leq 2^{-2k-1}. \end{cases}$$

Taking all possible finite unions $\mathsf{A}=\bigcup_k[2^{-2k-1}, 2^{-2k}]$, we see that the values $\int_{\mathsf{A}}\xi(x)\,dx$ are not bounded in probability. By Lemma 3.4, ξ is not integrable on $[0,1]$.

4. INTERCHANGE OF THE ORDER OF INTEGRATION

Theorem 4.1. *Let μ be a stochastic measure on $(\mathsf{S},\mathcal{B})$, and $\mathsf{B}\subset\mathbb{R}^d$ be a bounded set. Assume that $h(x,s)\colon \mathsf{B}\times\mathsf{S}\to\mathbb{R}$ is a measurable deterministic function which is Riemann integrable on B for each fixed s, and $|h(x,s)|\leq g(s)$, where $g\colon\mathsf{S}\to\mathbb{R}$ is integrable on S with respect to $d\mu(s)$. Then the random function $\xi(x)=\int_{\mathsf{S}} h(x,s)\,d\mu(s)$ is integrable on B, and*

$$\int_{\mathsf{B}} dx \int_{\mathsf{S}} h(x,s)\,d\mu(s) = \int_{\mathsf{S}} d\mu(s)\int_{\mathsf{B}} h(x,s)\,dx. \tag{4.1}$$

Proof. From the inequality $|h(x,s)|\leq g(s)$ and (2.1) it follows that the values of ξ are bounded in probability (see Lemma 1.1 and Theorem 1.3 in [12]). Integral sums of $\int_{\mathsf{B}}\xi(x)\,dx$ have the form

$$\sum_{1\leq k\leq k_n} \mathsf{m}(\mathsf{B}_{kn})\int_{\mathsf{S}} h(x_{kn},s)\,d\mu(s) = \int_{\mathsf{S}} g_n(s)\,d\mu(s),$$

$$g_n(s) = \sum_{1\leq k\leq k_n} h(x_{kn},s)\mathsf{m}(\mathsf{B}_{kn}) \to \int_{\mathsf{B}} h(x,s)\,dx.$$

The boundedness condition of h and the analogue of the Lebesgue theorem [7, Proposition 7.1.1] for the integral with respect to $d\mu(s)$ imply the statement. □

Corollary 4.1. *Let μ be a stochastic measure on $(\mathsf{S},\mathcal{B})$, and $\tilde{\mathsf{B}}\subset\mathbb{R}^d$ be an unbounded set. Assume that $h(x,s)\colon\tilde{\mathsf{B}}\times\mathsf{S}\to\mathbb{R}$ is a measurable deterministic function which is Riemann integrable on $\tilde{\mathsf{B}}$ in an improper sense for each fixed s, and $|h(x,s)|\le g(s)$,*

$$\int_{\tilde{\mathsf{B}}}|h(x,s)|\,dx=g_1(s),$$

where $g,g_1\colon\mathsf{S}\to\mathbb{R}$ are integrable on S with respect to $d\mu(s)$. Then the random function $\xi(x)=\int_{\mathsf{S}}h(x,s)\,d\mu(s)$ is integrable on $\tilde{\mathsf{B}}$ in an improper sense, and

$$\int_{\tilde{\mathsf{B}}}dx\int_{\mathsf{S}}h(x,s)\,d\mu(s)=\int_{\mathsf{S}}d\mu(s)\int_{\tilde{\mathsf{B}}}h(x,s)\,dx. \tag{4.2}$$

Proof. For bounded sets $\mathsf{B}^{(j)}\uparrow\tilde{\mathsf{B}}$, Theorem 4.1 implies

$$\int_{\mathsf{B}^{(j)}}dx\int_{\mathsf{S}}h(x,s)\,d\mu(s)=\int_{\mathsf{S}}d\mu(s)\int_{\mathsf{B}^{(j)}}h(x,s)\,dx.$$

Further, we use the analogue of the Lebesgue theorem and the integrability of g_1. □

Theorem 4.2. *Let $\mathsf{B}\subset\mathbb{R}^d$, $\mathsf{S}\subset\mathbb{R}^m$ be bounded sets, and the random function*

$$\xi(x,s)\colon\mathsf{B}\times\mathsf{S}\to L_0$$

be integrable on $\mathsf{B}\times\mathsf{S}$ with respect to $dx\times ds$ and be integrable on S with respect to ds for each fixed x. Then

$$\int_{\mathsf{B}\times\mathsf{S}}\xi(x,s)\,dx\times ds=\int_{\mathsf{B}}dx\int_{\mathsf{S}}\xi(x,s)\,ds. \tag{4.3}$$

Proof. The integral sums for the integral with respect to dx in (4.3) has the form

$$\sum_{1\le k\le k_0}\mathsf{m}(\mathsf{B}_k)\int_{\mathsf{S}}\xi(x_k,s)\,ds. \tag{4.4}$$

Each integral in (4.4) may be approximated by sums of the form $\sum_{1\le i\le i_0}\mathsf{m}(\mathsf{S}_i)\xi(x_k,s_i)$. Thus, the sums

$$\sum_{1\le k\le k_0}\sum_{1\le i\le i_0}\mathsf{m}(\mathsf{B}_k)\mathsf{m}(\mathsf{S}_i)\xi(x_k,s_i)$$

will approximate the right hand side of (4.4). But they are the integral sums for the integral with respect to $dx\times ds$ in (4.3), and will be close to the left hand side of (4.3) for sufficiently small diameters of $\mathsf{B}_k\times\mathsf{S}_i$. □

Corollary 4.2. *Let $\mathsf{S}\subset\mathbb{R}^m$ be a bounded set, and $\tilde{\mathsf{B}}\subset\mathbb{R}^d$ be an unbounded set. Assume that the random function $\xi(x,s)\colon\tilde{\mathsf{B}}\times\mathsf{S}\to L_0$ is integrable on $\tilde{\mathsf{B}}\times\mathsf{S}$ with respect to $dx\times ds$ in an improper sense, is integrable on $\tilde{\mathsf{B}}$ with respect to dx in an improper sense for each fixed s, and is integrable on S with respect to ds for each fixed x. Then*

$$\int_{\tilde{\mathsf{B}}\times\mathsf{S}}\xi(x,s)\,dx\times ds=\int_{\mathsf{S}}ds\int_{\tilde{\mathsf{B}}}\xi(x,s)\,dx=\int_{\tilde{\mathsf{B}}}dx\int_{\mathsf{S}}\xi(x,s)\,ds. \tag{4.5}$$

Proof. Consider exhaustive sets $\mathsf{B}^{(j)}\uparrow\tilde{\mathsf{B}}$. For the first of the repeated integrals (4.5), the integral sums have the form

$$\sum_{1\le i\le i_0}\mathsf{m}(\mathsf{S}_i)\int_{\tilde{\mathsf{B}}}\xi(x,s_i)\,dx. \tag{4.6}$$

The integrals in (4.6) can be approximated by $\int_{\mathsf{B}^{(j)}}\xi(x,s_i)\,dx$, and the last integral is the limit of the sums

$$\sum_{1\le k\le k_0}\mathsf{m}\left(\mathsf{B}_k^{(j)}\right)\xi\left(x_k^{(j)},s_i\right).$$

If the integral sums in (4.6) do not converge, then we can construct a non-convergent sequence of sums

$$\sum_{1\le i\le i_0}\sum_{1\le k\le k_0} \mathsf{m}(\mathsf{S}_i)\mathsf{m}\left(\mathsf{B}_k^{(j)}\right)\xi\left(x_k^{(j)}, s_i\right),$$

and this contradicts the integrability of ξ on $\mathsf{S}\times\tilde{\mathsf{B}}$.

Further, by Theorem 4.2, for each j we have

$$\int_{\mathsf{B}^{(j)}\times\mathsf{S}} \xi(x,s)\,dx\times ds = \int_{\mathsf{B}^{(j)}} dx\int_{\mathsf{S}}\xi(x,s)\,ds.$$

The left hand side has the limit in probability as $j\to\infty$. Hence, the right hand side has the limit, and the second equality of (4.5) holds. □

5. Integration by parts

To solve the parabolic stochastic equation, we need the following two lemmas.

Lemma 5.1. *Let a random function* $\xi(u)\colon [0,s]\to L_0$ *be integrable on* $[0,s]$. *Then* $\eta(u)=\int_0^u \xi(v)\,dv$ *is integrable on* $[0,s]$, *and*

$$\int_0^s du\int_0^u \xi(v)\,dv = \int_0^s (s-v)\xi(v)\,dv.$$

Proof. By Lemma 3.5 the function $(s-v)\xi(v)$ is integrable, and by Lemma 3.2 $\eta(u)$ is well defined. The integral sum of $\int_0^s \eta(u)\,du$ has the form

$$\sum_{1\le k\le k_0}\mathsf{m}(\mathsf{B}_k)\int_0^{u_k}\xi(v)\,dv,\qquad u_k\in\mathsf{B}_k. \tag{5.1}$$

We can take a new partition $[0,s]=\bigcup_{1\le i\le i_0}\mathsf{C}_i$ such that each integral $\int_0^{u_k}\xi(v)\,dv$ can be close enough to th integral sum with this partition (Lemma 3.2). Thus we can approximate (5.1) arbitrarily close by the sum

$$\sum_{1\le k\le k_0}\mathsf{m}(\mathsf{B}_k)\sum_{1\le i\le i_0}\mathsf{m}(\mathsf{C}_i\cap[0,u_k])\xi(v_i),\qquad v_i\in\mathsf{C}_i. \tag{5.2}$$

For $\int_0^s (s-v)\xi(v)\,dv$, take the integral sum

$$\sum_{1\le i\le i_0}\mathsf{m}(\mathsf{C}_i)(s-v_i)\xi(v_i). \tag{5.3}$$

The difference of (5.3) and (5.2) is equal to

$$\sum_{1\le i\le i_0}\xi(v_i)[\mathsf{m}(\mathsf{C}_i)(s-v_i)-\mathsf{m}(\mathsf{C}_i)\sum_{k\colon\, \mathsf{C}_i<\mathsf{B}_k}\mathsf{m}(\mathsf{B}_k)-\sum_{k\colon\, \mathsf{C}_i\cap\mathsf{B}_k\ne\varnothing}\mathsf{m}(\mathsf{B}_k)\mathsf{m}(\mathsf{C}_i\cap[0,u_k])]. \tag{5.4}$$

The notation $\mathsf{C}_i<\mathsf{B}_k$ means that $v<u$ for all $v\in\mathsf{C}_i$, $u\in\mathsf{B}_k$. We have

$$0\le(s-v_i)-\sum_{k:\mathsf{C}_i<\mathsf{B}_k}\mathsf{m}(\mathsf{B}_k)\le\max_i\operatorname{diam}\mathsf{C}_i+\max_k\operatorname{diam}\mathsf{B}_k.$$

The last sum of (5.4) is not greater than

$$\mathsf{m}(\mathsf{C}_i)\sum_{k\colon\, \mathsf{C}_i\cap\mathsf{B}_k\ne\varnothing}\mathsf{m}(\mathsf{B}_k)\le\mathsf{m}(\mathsf{C}_i)\left(\max_i\operatorname{diam}\mathsf{C}_i+2\max_k\operatorname{diam}\mathsf{B}_k\right).$$

Therefore, value (5.4) may be written in the form $\sum_{1\le i\le i_0}\xi(v_i)\mathsf{m}(\mathsf{C}_i)\alpha_i$, where $\alpha_i\to 0$ as $\operatorname{diam}\mathsf{C}_i,\operatorname{diam}\mathsf{B}_k\to 0$. From (2.1) we obtain

$$\left\|\sum_{1\le i\le i_0}\xi(v_i)\mathsf{m}(\mathsf{C}_i)\alpha_i\right\|\le 16\max_V\left\|\max_i|\alpha_i|\sum_{i\in V}\xi(v_i)\mathsf{m}(\mathsf{C}_i)\right\|. \tag{5.5}$$

The sums $\sum_{i\in V}\xi(v_i)\mathsf{m}(\mathsf{C}_i)$ are close to respective integrals for $\operatorname{diam}\mathsf{C}_i$ small enough (Lemma 3.2) and the values of integrals are bounded in probability (Lemma 3.4). Therefore, the left hand side of (5.5) tends to zero as $\max_i|\alpha_i|\to 0$. $\square$

Lemma 5.2. *Let a random function* $\xi(u)\colon[0,s]\to L_0$ *be integrable on* $[0,s]$, *and*

$$f\in\mathbb{C}^{(1)}([0,s])$$

be a deterministic function. Then

$$f(s)\int_0^s\xi(u)\,du=\int_0^s f(u)\xi(u)\,du+\int_0^s f'(u)\,du\int_0^u\xi(v)\,dv. \tag{5.6}$$

Proof. From Lemmas 3.5 and 5.1 it follows that the random functions

$$\zeta_1(u)=f(u)\xi(u),\qquad \zeta_2(u)=f'(u)\int_0^u\xi(v)\,dv$$

are integrable on $[0,s]$. First, let us show that for

$$0=u_0<u_1<\cdots<u_{k_0}=s,\qquad \alpha=\max_k|u_k-u_{k-1}|,$$

we have

$$\sum_{1\le k\le k_0}(f(u_k)-f(u_{k-1}))\int_0^{u_k}\xi(v)\,dv\xrightarrow{\mathsf{P}}\int_0^s f'(u)\,du\int_0^u\xi(v)\,dv,\qquad\alpha\to 0. \tag{5.7}$$

Applying the Lagrange formula and integrability of ζ_2, for some $\tilde u_k\in(u_{k-1},u_k)$ we obtain

$$\sum_{1\le k\le k_0}(f(u_k)-f(u_{k-1}))\int_0^{u_k}\xi(v)\,dv=\sum_{1\le k\le k_0}f'(\tilde u_k)(u_k-u_{k-1})\int_0^{u_k}\xi(v)\,dv$$

$$=\sum_{1\le k\le k_0}f'(\tilde u_k)(u_k-u_{k-1})\int_0^{\tilde u_k}\xi(v)\,dv+\sum_{1\le k\le k_0}f'(\tilde u_k)(u_k-u_{k-1})\int_{\tilde u_k}^{u_k}\xi(v)\,dv,$$

$$\sum_{1\le k\le k_0}f'(\tilde u_k)(u_k-u_{k-1})\int_0^{\tilde u_k}\xi(v)\,dv\xrightarrow{\mathsf{P}}\int_0^s f'(u)\,du\int_0^u\xi(v)\,dv,\qquad\alpha\to 0.$$

For $C_1=\max_u|f'(u)|$, from (2.1) we have

$$\left\|\sum_{1\le k\le k_0}f'(\tilde u_k)(u_k-u_{k-1})\int_{\tilde u_k}^{u_k}\xi(v)\,dv\right\|$$

$$\le 16\max_V\left\|C_1\alpha\sum_{k\in V}\int_{\tilde u_k}^{u_k}\xi(v)\,dv\right\|\le 16\sup_{\mathsf{A}}\left\|C_1\alpha\int_{\mathsf{A}}\xi(v)\,dv\right\|.$$

From Lemma 3.4 it follows that the last value tends to 0 as $\alpha\to 0$. Therefore, (5.7) is proved.

Integrability of ζ_1 implies

$$\sum_{1\le k\le k_0}f(u_{k-1})\int_{u_{k-1}}^{u_k}\xi(v)\,dv=\sum_{1\le k\le k_0}f(u_{k-1})\xi(u_{k-1})(u_k-u_{k-1})$$

$$+\sum_{1\le k\le k_0}f(u_{k-1})\int_{u_{k-1}}^{u_k}(\xi(v)-\xi(u_{k-1}))\,dv,$$

$$\sum_{1\le k\le k_0}f(u_{k-1})\xi(u_{k-1})(u_k-u_{k-1})\xrightarrow{\mathsf{P}}\int_0^s f(u)\xi(u)\,du,\qquad\alpha\to 0.$$

For $C_0 = \max_u |f(u)|$, from (2.1) we get

$$\begin{aligned}\Bigg\| \sum_{1\le k\le k_0} f(u_{k-1}) \int_{u_{k-1}}^{u_k} (\xi(v)-\xi(u_{k-1}))\, dv \Bigg\| &\\ \le 16 \max_V \Bigg\| C_0 \sum_{k\in V} \int_{u_{k-1}}^{u_k} (\xi(v)-\xi(u_{k-1}))\, dv \Bigg\| &\\ = 16 \max_V \Bigg\| C_0 \Bigg(\int_{\bigcup_{k\in V}[u_{k-1},u_k]} \xi(v)\, dv - \sum_{k\in V} \xi(u_{k-1})(u_k-u_{k-1}) \Bigg) \Bigg\| .&\end{aligned}$$

By Lemma 3.2, the last value tends to 0 as $\alpha\to 0$.

Further, we take the obvious equality

$$\begin{aligned}f(s)\int_0^s \xi(v)\, dv = &\sum_{1\le k\le k_0} (f(u_k)-f(u_{k-1})) \int_0^{u_k} \xi(v)\, dv \\ &+ \sum_{1\le k\le k_0} f(u_{k-1}) \int_{u_{k-1}}^{u_k} \xi(v)\, dv\end{aligned}$$

and pass to the limit as $\alpha\to 0$. □

6. Parabolic equation with a general stochastic measure

Consider the differential operator

$$Ag(x) = \sum_{1\le i,j\le d} a_{ij}(x) \frac{\partial^2 g(x)}{\partial x_i \partial x_j} + \sum_{1\le i\le d} b_i(x) \frac{\partial g(x)}{\partial x_i} + c(x)g(x),$$

where

$$g\colon \mathbb{R}^d \to \mathbb{R} \quad \text{and} \quad a_{ij}=a_{ji}.$$

Suppose that A is strongly elliptic in $\mathbb{R}^d$ (see (4.5) in [5]).

Assumption 1. *All functions*

$$a_{ij}, \qquad b_i, \qquad c, \qquad \frac{\partial a_{ij}}{\partial x_i}, \qquad \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}, \qquad \frac{\partial b_i}{\partial x_i}$$

are bounded and Hölder continuous in $\mathbb{R}^d$.

From now on let μ be a stochastic measure on Borel subsets of $[0,T]$.

We will study the equation

$$dX(x,t) = AX(x,t)\, dt + f(x,t)\, d\mu(t), \qquad X(x,0)=\xi(x), \tag{1.1}$$

where

$$X\colon \mathbb{R}^d\times[0,T]\to L_0$$

is an unknown random function.

We consider (1.1) in the weak sense, i.e.

$$\begin{aligned}\int_{\mathbb{R}^d} X(x,t)\varphi(x)\, dx = &\int_{\mathbb{R}^d} \xi(x)\varphi(x)\, dx + \int_{\mathbb{R}^d} A^*\varphi(x)\, dx \int_0^t X(x,s)\, ds \\ &+ \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} f(x,s)\varphi(x)\, dx\end{aligned} \tag{6.1}$$

for all test functions $\varphi\in\mathcal{S}(\mathbb{R}^d)$ (rapidly decreasing Schwartz functions from $\mathbb{C}^\infty(\mathbb{R}^d)$). For each fixed $t\in[0,T]$ equality (6.1) holds a.s. Integrals of random functions with respect to dx and ds are considered in the Riemann sense (see section 3), and A^* denotes the adjoint operator of A.

Assumption 2. *$\xi\colon \mathbb{R}^d \to L_0$ is such that $\xi(\cdot,\omega)$ is continuous and bounded in $\mathbb{R}^d$ for each fixed $\omega \in \Omega$.*

Assumption 3. *$f\colon \mathbb{R}^d \times [0,T] \to \mathbb{R}$ is Borel measurable,*

$$\sup_t |x|^{-k}|f(x,t)| \to 0, \qquad |x| \to \infty,$$

for some $k > 0$, and $f(x,\cdot)$ is continuous and bounded in $\mathbb{R}^d$ for each fixed $t \in [0,T]$.

By Theorem 1, §4 of [5], under Assumption 1, the equation $\partial g/\partial t = Ag$ has a fundamental solution $p(x,y,t-s)$ (recall that coefficients of A do not depend on t). The following estimate is well known:

$$|p(x,y,t)| \le C_1 t^{-d/2} \exp\left\{-C_2|x-y|^2/t\right\}$$

(see, for example, (4.16) in [5]). Consider the semigroup

$$S(t)g(x) = \int_{\mathbb{R}^d} p(x,y,t)g(y)\,dy, \qquad t > 0,\ S(0)g(x) = g(x).$$

Theorem 2, §4 of [5] implies that for any continuous bounded g,

$$S(t)g(x) = g(x) + A\int_0^t [S(s)g(x)]\,ds. \tag{6.2}$$

Theorem 6.1. *Suppose Assumptions* 1–3 *hold. Then the random function*

$$X(x,t) = S(t)\xi(x) + \int_{[0,t]} [S(t-s)f(x,s)]\,d\mu(s) \tag{6.3}$$

is the solution of (6.1).

In addition, suppose that the operator A is self-adjoint and that $X(x,t)$ satisfies (6.1), *is integrable on $\mathbb{R}^d \times [0,T]$ with respect to $dx \times dt$, is integrable on $\mathbb{R}^d$ with respect to dx for each fixed t, and is integrable on $[0,T]$ with respect to dt for each fixed x. Then $X(x,t)$ is given by* (6.3).

Proof. From (6.2) it follows that for $X_1(x,t) = S(t)\xi(x)$ and $f = 0$ equality (6.1) holds. For

$$X_2(x,t) = \int_{[0,t]} [S(t-s)f(x,s)]\,d\mu(s)$$

we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} A^*\varphi(x)\,dx \int_0^t X_2(s)\,ds + \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} f(x,s)\varphi(x)\,dx \\
&\quad = \int_{\mathbb{R}^d} A^*\varphi(x)\,dx \int_0^t ds \int_{[0,s]} [S(s-u)f(x,u)]\,d\mu(u) \\
&\qquad + \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} f(x,s)\varphi(x)\,dx \\
&\quad \overset{(4.1)}{=} \int_{\mathbb{R}^d} A^*\varphi(x)\,dx \int_{[0,t]} d\mu(u) \int_u^t [S(s-u)f(x,u)]\,ds \\
&\qquad + \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} f(x,s)\varphi(x)\,dx \\
&\quad \overset{(4.2)}{=} \int_{[0,t]} d\mu(u) \int_{\mathbb{R}^d} A^*\varphi(x)\,dx \int_u^t [S(s-u)f(x,u)]\,ds \\
&\qquad + \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} f(x,s)\varphi(x)\,dx \\
&\quad = \int_{[0,t]} d\mu(u) \int_{\mathbb{R}^d} \varphi(x)\,dx A \int_u^t [S(s-u)f(x,u)]\,ds \\
&\qquad + \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} f(x,s)\varphi(x)\,dx \\
&\quad \overset{(6.2)}{=} \int_{[0,t]} d\mu(u) \int_{\mathbb{R}^d} \varphi(x)\,dx([S(t-u)f(x,u)] - f(x,u)) \\
&\qquad + \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} f(x,s)\varphi(x)\,dx \\
&\quad = \int_{[0,t]} d\mu(u) \int_{\mathbb{R}^d} \varphi(x)[S(t-u)f(x,u)]\,dx \\
&\quad \overset{(4.2)}{=} \int_{\mathbb{R}^d} \varphi(x)\,dx \int_{[0,t]} [S(t-s)f(x,s)]\,d\mu(s) \\
&\quad = \int_{\mathbb{R}^d} X_2(x,t)\varphi(x)\,dx.
\end{aligned}
$$

Therefore (6.1) holds for $X = X_1 + X_2$.

Finally, we will prove the uniqueness of the solution. Section 4 implies that the random function X given by (6.3) is integrable. Thus, it is enough to prove that the equation

$$\int_{\mathbb{R}^d} X(x,t)\varphi(x)\,dx = \int_{\mathbb{R}^d} A^*\varphi(x)\,dx \int_0^t X(x,s)\,ds \tag{6.4}$$

has only the zero solution provided that $A = A^*$.

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $0 < s < t$ set

$$\psi_{t,s}(x) = S(t-s)\varphi(x).$$

Then

$$\psi_{t,s} \in \mathcal{S}\left(\mathbb{R}^d\right), \qquad A\psi_{t,s} + \frac{\partial}{\partial s}\psi_{t,s} = 0, \qquad \psi_{t,s} \to \varphi$$

uniformly on any bounded subset of $\mathbb{R}^d$ as $t \downarrow s$ (see (4.13) in [5]), and we get

$$\begin{aligned}\int_{\mathbb{R}^d} X(x,t)\psi_{t,s}(x)\,dx &\overset{(6.4)}{=} \int_{\mathbb{R}^d} A\psi_{t,s}(x)\,dx \int_0^s X(x,u)\,du \\ &\overset{(5.6)}{=} \int_{\mathbb{R}^d} dx \int_0^s A\psi_{t,u}(x)X(x,u)\,du + \int_{\mathbb{R}^d} dx \int_0^s A\frac{\partial}{\partial u}\psi_{t,u}(x)\,du \int_0^u X(x,v)\,dv \\ &\overset{(4.5)}{=} \int_{\mathbb{R}^d} dx \int_0^s A\psi_{t,u}(x)X(x,u)\,du + \int_0^s du \int_{\mathbb{R}^d} A\frac{\partial}{\partial u}\psi_{t,u}(x)\,dx \int_0^u X(x,v)\,dv \\ &\overset{(6.4)}{=} \int_{\mathbb{R}^d} dx \int_0^s A\psi_{t,u}(x)X(x,u)\,du + \int_0^s du \int_{\mathbb{R}^d} \frac{\partial}{\partial u}\psi_{t,u}(x)X(x,u)\,dx \\ &\overset{(4.5)}{=} \int_{\mathbb{R}^d} dx \int_0^s \Big(A\psi_{t,u}(x) + \frac{\partial}{\partial u}\psi_{t,u}(x)\Big)X(x,u)\,du = 0.\end{aligned}$$

Passing to the limit as $t \downarrow s$ and applying Lemma 3.9, we arrive at

$$\int_{\mathbb{R}^d} X(x,s)\varphi(x)\,dx = 0. \qquad \square$$

Example 6.1. Let the stochastic measure μ be generated by a continuous square integrable martingale Y, $\mu(\mathsf{A}) = \int_0^T \mathbf{1}_{\mathsf{A}}(t)\,dY(t)$, and λ be the Lebesgue measure on $\mathbb{R}^d$. Then

$$M_t(\mathsf{A}) = Y(t)\lambda(\mathsf{A}), \qquad 0 \le t \le T,\ \mathsf{A} \subset \mathbb{R}^d,$$

is a worthy martingale measure with the dominating measure

$$K(\mathsf{A} \times \mathsf{B} \times (0,t]) = \big|\langle Y\rangle_t\big|\lambda(\mathsf{A})\,\lambda(\mathsf{B})$$

(we use the terminology of [19]). In this case, (6.3) leads to

$$\begin{aligned}X(x,t) &= \int_{\mathbb{R}^d} p(x,y,t)\xi(y)\,dy + \int_{[0,t]} d\mu(s) \int_{\mathbb{R}^d} p(x,y,t-s)f(y,s)\,dy \\ &= \int_{\mathbb{R}^d} p(x,y,t)\xi(y)\,dy + \int_{[0,t]\times\mathbb{R}^d} p(x,y,t-s)f(y,s)\,M(dy\,ds).\end{aligned} \tag{6.5}$$

The results of [19, Chapter 2] imply that the integral with respect to $M(dy\,ds)$ is well defined and is the limit of integrals of elementary functions. For an elementary function, the equality of two stochastic integrals in (6.5) is obvious. Further, we can use the dominated convergence theorem for the integral with respect to $d\mu(s)$.

A similar solution of the parabolic SPDE with respect to general martingale measure is given in Example 9 and Remark 20 of [3].

Example 6.2. Assume that μ is generated by the real-valued Wiener process w and $\mathcal{J}$ denotes the set of Schwartz rapidly decreasing test functions in $\mathbb{R}^d$. Then the equation

$$\langle \mathcal{W}(t), \psi\rangle = w(t)\int_{\mathbb{R}^d} \psi(x)\,dx, \qquad \psi \in \mathcal{J},$$

defines the spatially homogeneous Wiener process with values in $\mathcal{J}'$ (we used the terminology of [10]). For this case, our equality (6.3) is a partial case of the mild solution (2.6) in [10].

Remark. In a similar manner, we can consider a more general equation

$$dX(x,t) = AX(x,t)\,dt + \sum_{1\le i\le j} f_i(x,t)\,d\mu_i(t), \qquad X(0) = \xi, \tag{6.6}$$

which includes the case

$$dX(x,t) = AX(x,t)\,dt + f_1(x,t)\,dt + f_2(x,t)\,d\mu(t), \qquad X(0) = \xi.$$

The solution of (6.6) is

$$X(x,t) = S(t)\xi(x) + \sum_{1\le i\le j} \int_{[0,t]} [S(t-s) f_i(x,s)]\, d\mu_i(s).$$

Under assumptions of Theorem 6.1, the solution of (6.6) is unique.

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