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AN ESTIMATE OF THE STABILITY FOR NONHOMOGENEOUS MARKOV CHAINS UNDER CLASSICAL MINORIZATION CONDITION

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Abstract. The stability of time inhomogeneous Markov chains is considered under the classical minorization condition. The key tool for the proofs is a modified coupling method for a pair of two (possibly time inhomogeneous) Markov chains.

1. Introduction

We consider the stability of a homogeneous Markov chain perturbed in such a way that it becomes time inhomogeneous. The stability is understood in the sense of the closeness of transition probabilities over n steps in the full variation norm.

The conditions used to obtain an estimate of the stability are expressed in terms of the closeness of transition probabilities over one step; in addition, we use an additional stochastic majorization condition for a renewal process at the coupling moment.

We follow a modified coupling method in our proofs. An application of this method for time nonhomogeneous Markov chains can be found in the paper [22]. Conditions for the existence of the coupling moment for homogeneous Markov chains are obtained in the paper [24].

2. Main result

We consider a measurable space $(E, \mathcal{E})$. Let μ be a measure in $(E, \mathcal{E})$. Introduce the full variation norm of a measure μ as follows:

$$\|\mu\|_{TV} = |\mu|(E) = \mu^+(E) + \mu^-(E), \tag{1}$$

where μ^+ and μ^- are the components of the measure μ in its Hahn decomposition.

Unless otherwise is indicated, we use the full variation norm throughout the paper.

We also introduce the f-norm. Let $f\colon E \to [1,\infty)$ be a certain function. Then

$$\|\mu\|_f = \sup_{|g|\le f} \int_E g(x)\,\mu(dx).$$

Let X be a homogeneous Markov chain with the transition probability P and let X' be a nonhomogeneous chain with the transition probabilities P_t for the step t. We assume that both P and P' admit the following decomposition:

$$P(x, A) = (1-\varepsilon)Q(x, A) + \varepsilon R(x, A),$$
$$P_t(x, A) = (1-\varepsilon)Q(x, A) + \varepsilon R_t(x, A),$$

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where Q is the "common part" of two transition probabilities and where R and R_t are two stochastic kernels.

Let both chains X and X' assume values in the phase space $(E, \mathcal{E})$.

Let P and Q be two transition kernels. We define the product of two kernels P and Q as follows:

$$PQ(x, dy) = \int_E P(x, dz)\, Q(z, dy).$$

Put

$$P^{t,n} = \prod_{i=t}^{t+n-1} P_i, \qquad n \geq 1,$$
$$P^{t,0} = I,$$

where I is the unit transition kernel, that is, $\mu I = \mu$ and $If = f$ for all measures μ and measurable functions f.

Consider the following condition.

(A1) *Minorization condition* (see [11, Section 5]):
there exist a set $C \in \mathcal{E}$, probability measure ν, and constant $\alpha > 0$ such that

$$\min\{P(x, \cdot), P_t(x, \cdot\} \geq \alpha\nu(\cdot) \tag{2}$$

for all $x \in C$.

Remark 2.1. Note that condition (A1) can be rewritten as follows:

$$\inf_{x \in C} Q(x, \cdot) \geq \alpha' \nu(\cdot)$$

for some constant α'. The constants α' and α in inequality (2) are related as follows: $\alpha' = \alpha/(1-\varepsilon)$. Indeed,

$$P(x, \cdot) \geq (1-\varepsilon) Q(x, \cdot) \geq \alpha'(1-\varepsilon)\nu(\cdot).$$

Now we introduce the following stochastic kernels:

$$P_\alpha(x, dy) = \frac{P(x, dy) - \alpha\nu(dy)}{1-\alpha}, \qquad P_{t,\alpha}(x, dy) = \frac{P_t(x, dy) - \alpha\nu(dy)}{1-\alpha}, \tag{3}$$

$$\begin{aligned} T_t(x, x'; dy, dy') = (1-\alpha) \mathbb{1}_{C\times C}(x, x') P_\alpha(x, dy) P_{t,\alpha}(x', dy') \\ + \mathbb{1}_{(C \times C)^c}(x, x') P(x, dy) P_t(x', dy'), \end{aligned} \tag{4}$$

$$T^{t,k} = \prod_{i=t}^{t+k-1} T_i.$$

Under the assumption that a chain X possesses a unique invariant measure π, we put

$$\lambda_t(dy, dy') = \int_E \pi(dx)\, R(x, dy) \times R_t(x, dy'), \tag{5}$$

$$m(x, x'; t) = \sum_{k \geq 0} T^{t,k}(x, x'; E, E). \tag{6}$$

Consider the nonhomogeneous sequence

$$s_n^{(t)}(x) = \alpha \int_{E \times E} R(x, dy)\, R_t(x, dy') T^{t+1, n-1}(y, y'; C, C), \qquad n \geq 1. \tag{7}$$

The number $s_n^{(t)}(x)$ can be viewed as the probability that the chain decoupled at the state x never couples again during time n.

Definition 2.1. A distribution $(\hat{g}_n)$ is called a *stochastic majorant* or *majorizing sequence* for a family of distributions $\left(g_n^{(t)}\right)$ if

$$\sum_{k>n} g_k^{(t)} \leq \sum_{k>n} \hat{g}_k$$

for all t and n.

Remark 2.2. Note that if the sequence $s_n^{(t)}(x)$ satisfies the uniform integrability condition

$$\sup_{t>0} \sum_{k>n} s_k^{(t)}(x) \to 0, \qquad n \to \infty, \tag{8}$$

then one can introduce the stochastic majorant

$$\hat{s}_n(x) = \sup_{t>0} \sum_{k \geq n} s_k^{(t)}(x) - \sup_{t>0} \sum_{k>n} s_k^{(t)}(x)$$

such that

$$\sum_{k>n} \hat{s}_k(x) \geq \sum_{k>n} s_k^{(t)}(x)$$

for all t and n.

We consider further conditions.

(M) *Majorization condition*:
Let, for every x, the sequence $s_n^{(t)}(x)$ be uniformly integrable with respect to n for the counting measure and let convergence (8) hold for some majorizing sequence $\hat{s}_n(x)$ and

$$\hat{m}(x) = \sum_{k \geq 0} k \hat{s}_k(x) < \infty \tag{9}$$

for all $x \in E$.

(M2) Let there exist two sequences $(\bar{s}_n, n \geq 0)$ and $(\bar{s}_n(x), n \geq 0)$ such that the mean values

$$\bar{m} = \sum_{n \geq 0} n \bar{s}_n, \qquad \bar{m}_x = \sum_{n \geq 0} n \bar{s}_n(x)$$

are finite. Put

$$\bar{s} = \sum_{n \geq 0} \bar{s}_n, \qquad \bar{s}(x) = \sum_{n \geq 0} \bar{s}_n(x).$$

Then

$$\begin{gathered} \iint \nu(dx) Q^{k-1}(x, dy) \hat{s}_n(y) \leq \bar{s}_n, \\ \int Q^{k-1}(x, dy) \hat{s}_n(y) \leq \bar{s}_n(x) \end{gathered} \tag{10}$$

for all $k > 1$ and $n \geq 0$, where $\hat{s}_n$ is a majorizing sequence introduced in condition (M).

Remark 2.3. If, for the transition probability Q, there exists a unique invariant measure π and if

$$\int \left(\sum_{k \geq 1} |\nu Q^k - \pi| \right)(dx) \hat{m}(x) < \infty,$$

$$\sum_{k \geq 0} k \int \pi(dx) \hat{s}_k(x) < \infty,$$

then the sequence

$$\int \Big(\sum_{k\ge 1} |\nu Q^k - \pi|\Big)(dx)\hat{s}_n(x) + \pi(dx)\hat{s}_n(x)$$

can be chosen as $\bar{s}_n$.

It is obvious that

$$\begin{aligned}\nu Q^{k-1}\hat{s}_n &= \int \left(\nu Q^{k-1} - \pi\right)(dx)\hat{s}_n(x) + \int \pi(dx)\hat{s}_n(x) \\ &\le \int \sum_{k\ge 1} \left(\left|\nu Q^{k-1} - \pi\right|\right)(dx)\hat{s}_n(x) + \int \pi(dx)\hat{s}_n(x) = \bar{s}_n.\end{aligned}$$

Moreover,

$$\bar{s} = \sum_{k\ge 1} \left\|\nu Q^k - \pi\right\| + 1 < \infty,$$

$$\bar{m} = \sum_{k\ge 0} k\bar{s}_k = \sum_{k\ge 1} \left\|\nu Q^k - \pi\right\|_{\hat{m}} + \int \pi(dx)\hat{m}(x) < \infty.$$

Remark 2.4. Generally speaking, the sequence $\bar{s}_n$ is not a probability distribution; however,

$$\sum_{n\ge 0} \bar{s}_n < \infty.$$

Theorem 2.1. *Let conditions* (A1), (M), *and* (M2) *hold. Then*

$$\sup_{n>0, A\in\mathcal{E}} \left|P^n(x,A) - P^{t,n}(x,A)\right| \le \varepsilon M(x,\varepsilon) \tag{11}$$

for all $\varepsilon < 1/4\bar{m}$, *where*

$$M(x,\varepsilon) = \bar{m}_x + \bar{m}(1 - 4\varepsilon\bar{m})^{-1/2}.$$

The numbers $\bar{m}$ *and* $\bar{m}_x$ *are defined in condition* (M2).

3. Some applications

Theorem 3.1. *Consider the chains corresponding to transition probabilities:*

$$P = Q = R = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \beta_1 & 1-\beta_1 & 0 & 0 & \dots \\ \beta_2 & 0 & 1-\beta_2 & 0 & \dots \\ \beta_3 & 0 & 0 & 1-\beta_3 & \dots \\ \dots \end{pmatrix},$$

$$R_t = \begin{pmatrix} 0 & a_1^{(t)} & a_2^{(t)} & a_3^{(t)} & \dots \\ b_1^{(t)} & 1-b_1^{(t)} & 0 & 0 & \dots \\ b_2^{(t)} & 0 & 1-b_2^{(t)} & 0 & \dots \\ b_3^{(t)} & 0 & 0 & 1-b_3^{(t)} & \dots \\ \dots \end{pmatrix},$$

$$P_t = \begin{pmatrix} 0 & \alpha_1^{(t)} & \alpha_2^{(t)} & \alpha_3^{(t)} & \dots \\ \beta_1^{(t)} & 1-\beta_1^{(t)} & 0 & 0 & \dots \\ \beta_2^{(t)} & 0 & 1-\beta_2^{(t)} & 0 & \dots \\ \beta_3^{(t)} & 0 & 0 & 1-\beta_3^{(t)} & \dots \\ \dots \end{pmatrix},$$

where $\beta_k^{(t)} = (1-\varepsilon)\beta_k + \varepsilon b_k^{(t)}$ *and* $0 \le b_k^{(t)} \le 1$.

Assume that

1) *there exists* $0 < \delta < 1$ *such that* $\beta_k > \delta$ *for all* t *and* i *for which* $b_i^{(t)} > \delta$ (*Kolmogorov's condition*);
2) *there exists* $k > 0$ *such that* $\inf_t\{\alpha_k^{(t)}, \alpha_k\} > 0$.

Then there exists ε_0 *such that*

$$\|P^n(x, A) - P^{t,n}(x, A)\| \le \varepsilon M(x)$$

for all x *and* $\varepsilon < \varepsilon_0$, *where* $M(x) = \sup_\varepsilon M(x, \varepsilon)$ *and where* $M(x, \varepsilon)$ *is defined in Theorem* 2.1.

The proof of Theorem 3.1 is given in Section 7.

4. The coupling procedure

Let

$$D = \{0, 1, 2\}. \tag{12}$$

Consider the space $Z = (E \times E \times D)$ and Markov chain $(Z_n, n \ge 0)$ defined on Z. The values $Z_n = \bigl(Z_n^{(1)}, Z_n^{(2)}, d_n\bigr)$ are defined recursively. First, let $d_0 \in \{0, 1\}$, $Z_0^{(1)} = X_0$, and $Z_0^{(2)} = X_0'$. In what follows, we assume that $d_0 = 0$ and $X_0 = X_0' = x$ for some $x \in E$ (in other words, $X_0 = X_0 \sim \nu(\cdot)$).

If Z_n is defined, then we are in a position to define Z_{n+1} as explained below.

If $d_n = 0$ and $(X_n, X_n') \in C \times C$, then one tosses a coin and gets 1 with probability α or 0 with probability $1 - \alpha$. If the result equals 1, then we put $d_{n+1} = 1$ and $Z_{n+1}^{(1)} = Z_{n+1}^{(2)} = X$, where $X \sim \nu(\cdot)$. Otherwise (if the result equals 0) we put $d_{n+1} = 0$ and $\bigl(Z_{n+1}^{(1)}, Z_{n+1}^{(2)}\bigr) \sim T\bigl(Z_n^{(1)}, Z_n^{(2)}, (\cdot, \cdot)\bigr)$.

If $d_n = 0$ and $(X_n, X_n') \notin C \times C$, then we put $d_{n+1} = 0$ and

$$\bigl(Z_{n+1}^{(1)}, Z_{n+1}^{(2)}\bigr) \sim P \times P_{n+1}(\cdot, \cdot).$$

If $d_n \in \{1, 2\}$, then one tosses a coin and gets 1 with probability ε or 0 with probability $1 - \varepsilon$. If the result equals 1, then $Z_{n+1} = (X, X', 0)$, where $X \sim R\bigl(Z_n^{(1)}, \cdot\bigr)$ and $X' \sim R_{n+1}\bigl(Z_n^{(2)}, \cdot\bigr)$. Otherwise (if the result equals 0) we put $Z_{n+1} = (X, X, 2)$, where $X \sim Q\bigl(Z_n^{(1)}\bigr)$.

By $\bar{\mathbb{P}}^{(t)}$ and $\bar{\mathbb{E}}^{(t)}$ we denote the probability and expectation, respectively, generated by the chain $(Z_{t+n}, n \ge 0)$.

The transition probability $\bar{P}_t$ for the chain Z_{t+n} is given by

$$\begin{gathered}
\bar{P}_t(x, x', 0; A \times A' \times \{0\}) = T_t(x, x'; A \times A'),\\
\bar{P}_t(x, x', 0; A \times A' \times \{1\}) = \mathbb{1}_{C\times C}(x, x')\alpha\nu(A \cap A'),\\
\bar{P}_t(x, x', 0; E \times E \times \{2\}) = 0,\\
\bar{P}_t(x, x', 1; A \times A' \times \{0\}) = \varepsilon\delta_x(x')R(x, A)R_t(x', A'),\\
\bar{P}_t(x, x', 1; A \times A' \times \{1\}) = 0,\\
\bar{P}_t(x, x', 1; A \times A' \times \{2\}) = (1 - \varepsilon)\delta_x(x')Q(x, A \cap A'),\\
\bar{P}_t(x, x', 2; A \times A' \times D) = \bar{P}_t(x, x', 1; A \times A' \times D).
\end{gathered}$$

Remark 4.1. Note that the state $\{d_n = 1\}$ is only attained from the state

$$(X_{n-1}, X_{n-1}', d_{n-1}) \in C \times C \times \{0\}.$$

Also

$$\bar{\mathbb{P}}_{\mu,1}^{(t)}\{d_n = 1, d_{n+k} = d\} = \bar{\mathbb{P}}_{\mu,1}^{(t)}\{d_n = 1\}\bar{\mathbb{P}}_{\nu,1}^{(t+n)}\{d_k = d\}$$

for an arbitrary measure μ on $\mathcal{E}\otimes\mathcal{E}$ and for all $d\in D$, since the transition probability does not depend on x, x' in this case.

Lemma 4.1. *For all* $t>0$ *and* $d\in D$,

$$P(x,A)=\bar{P}_t(x,x',d;A\times E\times D),$$
$$P_t(x',A')=\bar{P}_t(x,x',d;E\times A'\times D).$$

Proof. We prove Lemma 4.1 for $d\in\{0,1\}$ only, since the transition probabilities for $d=2$ coincide with those for $d=1$. We have

$$\begin{aligned}\bar{P}(x,x',0;A\times E\times D)&=\bar{P}(x,x',0;A\times E\times\{0\})+\bar{P}(x,x',0;A\times E\times\{1\})\\&=T_t(x,x';A\times E)+\alpha\nu(A\cap E)\\&=(1-\alpha)\mathbb{1}_{C\times C}(x,x')P_\alpha(x,dy)P_{t,\alpha}(x',dy')\\&\quad+\mathbb{1}_{(C\times C)^c}(x,x')P(x,dy)P_t(x',dy')+\alpha\nu(A)\\&=(1-\alpha)\frac{P(x,A)-\alpha\nu(A)}{1-\alpha}+\alpha\nu(A)=P(x,A),\\\bar{P}_t(x,x,1;A\times E\times D)&=\bar{P}_t(x,x,1;A\times E\times\{0\})+\bar{P}_t(x,x,1;A\times E\times\{2\})\\&=\alpha R(x,A)R_t(x,E)+(1-\alpha)Q(x,A)\\&=\alpha R(x,A)+(1-\alpha)Q(x,A)=P(x,A).\end{aligned}$$

Equalities for P_t are obtained similarly. □

Lemma 4.2. *For all* $t>0$, $n\geq 1$, $d\in D$, $x'\in E$, *and* $\phi\in\mathcal{E}_+$,

$$P^{t,n}\phi(x)=\int_{E\times E\times D}\bar{P}^{t,n}(x,x',d;dx_{n-1}\times dx'_{n-1}\times di)\phi(x_{n-1}).$$

Proof. We use the method of mathematical induction. Let $n=1$, $\phi\in f\mathcal{E}_+$, $d\in D$, and $x'\in E$. Then our aim is to prove that

$$P_t\phi(x)=\int_{E\times E\times D}\bar{P}_t(x,x',d;dx_1\times dx'_1\times di)\phi(x_1).$$

We will prove the latter equality for $\phi(x)=I_A(x)$, where $A\in\mathcal{E}$. Indeed,

$$\int_{E\times E\times D}\bar{P}_t(x,x',d;dx_1\times dx'_1\times di)I_A(x_1)=\bar{P}_t(x,x',1;A\times E\times D)=P_t(x,A)=P_t\phi(x).$$

Let Lemma 4.2 hold for some n. We are going to check it for $n+1$. We have

$$\begin{aligned}&\int_{E\times E\times D}\bar{P}^{t,n+1}(x,x',d;dx_n\times dx'_n\times di_n)\phi(x_n)\\&\quad=\int_{E\times E\times D}\bar{P}^{t,n}(x,x',d;dx_{n-1}\times dx'_{n-1}\times di_{n-1})\\&\qquad\times\int_{E\times E\times D}\bar{P}_{t+n}(x_{n-1},x'_{n-1},i_{n-1};dx_n\times dx'_n\times di_n)\phi(x_n)\\&\quad=\int_{E\times E\times D}\bar{P}^{t,n}(x,x',d;dx_{n-1}\times dx'_{n-1}\times di_{n-1})P_{t+n}\phi(x_{n-1}).\end{aligned}$$

Using the assumption of induction for the function $P_{t+n}\phi(x)$, we get

$$\begin{aligned}
&\int_{E\times E\times D} \bar{P}^{t,n+1}(x,x',d;dx_n\times dx'_n\times di_n)\phi(x_n)\\
&\quad= \int_{E\times E\times D} \bar{P}^{t,n}(x,x',d;dx_{n-1}\times dx'_{n-1}\times di_{n-1})P_{t+n}\phi(x_{n-1})\\
&\quad= P^{t,n}P_{t+n}\phi(x) = P^{t,n+1}\phi(x). \qquad \square
\end{aligned}$$

Lemma 4.3. *For all $t>0$,*

$$\sup_{n\geq 0}\|P^n(x,\cdot)-P^{t,n}(x,\cdot)\|_{TV}\leq \bar{P}^{t,n}(x,x,1;E\times E\times\{0\}).$$

Proof. First,

$$|P^n(x,A)-P^{t,n}(x,A)| = |\bar{P}^{t,n}(x,x,1;A\times E\times D-\bar{P}^{t,n}(x,x,1;E\times A\times D|;$$

$$\begin{aligned}
&\left|\bar{P}^{t,n}(x,x,1;A\times E\times\{1\}-\bar{P}^{t,n}(x,x,1;E\times A\times\{1\}\right|\\
&\quad=\left|\int_{A\times E\times\{1\}}\bar{P}_t(x,x,1;dx_1\times dx_2\times di)\cdots\int_E \alpha\nu(A\cap E)\right.\\
&\qquad\left.-\int_{E\times A\times\{1\}}\bar{P}_t(x,x,1;dx_1\times dx_2\times di)\cdots\int_E \alpha\nu(E\cap A)\right|\\
&\quad=0.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\bar{P}^{t,n}(x,x,1;A\times E\times\{2\}-\bar{P}^{t,n}(x,x,1;E\times A\times\{2\}\right|\\
&\quad=\left|\int_{A\times E\times\{2\}}\bar{P}_t(x,x,1;dx_1\times dx_2\times di)\cdots\int_E (1-\varepsilon)Q(x_{n-1},A\cap E)\right.\\
&\qquad\left.-\int_{E\times A\times\{2\}}\bar{P}_t(x,x,1;dx_1\times dx_2\times di)\cdots\int_E (1-\varepsilon)Q(x_{n-1},E\cap A)\right|\\
&\quad=0.
\end{aligned}$$

Thus

$$\begin{aligned}
|P^n(x,A)-P^{t,n}(x,A)| &= |\bar{P}^{t,n}(x,x,1;A\times E\times\{0\})-\bar{P}^{t,n}(x,x,1;E\times A\times\{0\})|\\
&\leq \max(\bar{P}^{t,n}(x,x,1;A\times E\times\{0\}),\bar{P}^{t,n}(x,x,1;E\times A\times\{0\})\\
&\leq \bar{P}^{t,n}(x,x,1;E\times E\times\{0\}). \qquad \square
\end{aligned}$$

We introduce the following stopping times with respect to the σ-algebra generated by the process Z_n with the initial state $(x,x,1)$ (that is, the chains start coupled together from the point x):

$$\begin{aligned}
&\tau_0\equiv 0, \qquad \nu_0\equiv 0,\\
&\tau_n=\inf(t>\nu_{k-1}\colon d_t=0), \qquad k\geq 1,\\
&\nu_k=\inf(t>\tau_k\colon d_t=1), \qquad k\geq 1.
\end{aligned} \tag{13}$$

Hence τ_k is the k^{th} decoupling moment, while ν_k is the k^{th} (after zero moment) coupling moment.

We also introduce the following random variables characterizing the intervals between the sequential coupling and decoupling moments:

$$\begin{aligned} &\xi_0 = \zeta_0 \equiv 0, \\ &\xi_k = \tau_k - \nu_{k-1}, \qquad k \ge 1, \\ &\zeta_k = \nu_k - \tau_k, \qquad k \ge 1. \end{aligned} \tag{14}$$

Thus ξ_k is the time between the $(k-1)^{\text{th}}$ coupling moment and the k^{th} decoupling moment, while ζ_k is the time between the k^{th} decoupling moment and the k^{th} coupling moment.

Note also that

$$\tau_k = \sum_{j=0}^{k} \xi_k + \sum_{j=0}^{k-1} \zeta_k,$$

$$\nu_k = \sum_{j=0}^{k} \xi_k + \sum_{j=0}^{k} \zeta_k.$$

5. Outlined proof of Theorem 2.1

Let

$$\begin{aligned} S_n^{(t)}(x) &= \sum_{k>n} s_n^{(t)}(x), \qquad n \ge -1, \\ \hat{S}_n(x) &= \sum_{k>n} \hat{s}_n(x), \qquad n \ge -1, \\ \bar{S}_n(x) &= \sum_{k>n} \bar{s}_n(x), \qquad n \ge -1, \end{aligned} \tag{15}$$

be the tails of the sequences $s_n^{(t)}(x)$, $\hat{s}_n(x)$, and $\bar{s}_n$, respectively.

Denote by $p_n = \varepsilon(1-\varepsilon)^{n-1}$, $n \ge 1$, $p_0 = 0$, the distribution of a geometrical random variable with parameter ε and let $(p_n^{\star k}, n \ge 0)$ be the k^{th} convolution of this distribution with itself. By Lemma 4.3,

$$\sup_{n\ge 0} \|P^n(x,\cdot) - P^{t,n}(x,\cdot)\|_{TV} \le \bar{P}^{t,n}(x,x,1; E\times E\times\{0\}) = \mathbb{P}_{x,x,1}^{(t)}\{d_n = 0\}.$$

The proof of Theorem 2.1 consists of the following steps.

1) We show that

$$\mathbb{P}_{x,x,1}^{(t)}\{d_n = 0\} = \mathbb{P}_{x,x,1}^{(t)}\{\tau_1 \le n, \nu_1 > n\} + \sum_{k=1}^{n} \mathbb{P}_{x,x,1}^{(t)}\{\nu_1 = k\}\mathbb{P}_{\nu,1}^{(t+k)}\{d_{n-k} = 0\} \tag{16}$$

(see Lemma 6.1).

Then we prove that $\mathbb{P}_{x,x,1}^{(t)}\{\tau_1 < n, \nu_1 > n\}$ and $\sup_{t,n} \mathbb{P}_{\nu,1}^{(t)}\{d_n = 0\}$ are of order ε. Thus, without loss of generality, we may assume that the chain $(Z_n, n \ge 0)$ starts from the distribution $(\nu, 1)$ (see Lemma 6.1).

2) Next we show that ξ_k does not depend on τ_i, ν_i, ξ_i, ζ_i, $i = 0, \ldots, k-1$; we also show that the distribution of ξ_k coincides with $(p_n, n \ge 0)$ and the distribution of $\sum_{i=1}^{k} \xi_i$ coincides with $(p_n^{\star k}, n \ge 0)$ and that the latter is the negative binomial distribution (see Lemma 6.6).

3) We write

$$\mathbb{P}_{\nu,1}^{(t)}\{d_n = 0\} = \sum_{k=1}^{[n/2]} \mathbb{P}_{\nu,1}^{(t)}\{\tau_k \le n < \nu_k\} = \sum_{k=1}^{[n/2]} \sum_{j=k}^{n} \mathbb{P}_{\nu,1}^{(t)}\{\zeta_k > n-j, \tau_k = j\}. \tag{17}$$

4) Then we deduce that

$$\mathbb{E}^{(t)}_{\nu,1}[f(X_{\tau_k-1}), \tau_k = j] = \mathbb{E}^{(t)}_{\nu,1}\left[\nu Q^{\xi_k-1} f, \tau_k = j\right] \tag{18}$$

for all measurable nonnegative functions $f\colon E \to \mathbb{R}$ (see Lemma 6.2).

5) After this, we conclude that

$$\mathbb{P}^{(t)}_{\nu,1}\{\zeta_k > n-j, \tau_k = j\} \leq \mathbb{E}^{(t)}_{\nu,1}[\hat{S}_{n-j}(X_{\tau_j-1}), \tau_k = j] \tag{19}$$

(see Lemma 6.3).

6) Then we establish

$$\mathbb{E}^{(t)}_{\nu,1}\left[\hat{S}_{n-j}(X_{\tau_j-1}), \tau_k = j\right] = \sum_{l=0}^{j} \mathbb{P}^{(t)}_{\nu,1}[\nu_{k-1} = l]\mathbb{E}^{(t+l)}_{\nu,1}\left[\nu Q^{\xi_k-1}\hat{S}_{n-j}, \xi_k = j-l\right] \tag{20}$$

(see Lemma 6.4).

7) Now we prove that

$$\mathbb{P}^{(t)}_{\nu,1}[\nu_{k-1} = l] \leq \left(p^{\star k-1} \star \bar{S}^{\star k-1}\right)_l, \tag{21}$$

$$\mathbb{P}^{(t)}_{\nu,1}\left[\xi_k = j-l, \nu Q^{\xi_k-1}\hat{S}_{n-j}\right] \leq \varepsilon(1-\varepsilon)^{j-l-1}\bar{S}_{n-j} \tag{22}$$

(see Lemma 6.5).

8) The results obtained in steps 5), 6), and 7) imply the following bound:

$$\mathbb{P}^{(t)}_{\nu,1}[\zeta_k > n-j, \tau_k = j] \leq \left(p^{\star k} \star \bar{S}^{\star k-1}\right)_j \bar{S}_{n-j}. \tag{23}$$

9) Using the exact form of $(p^{\star k}_n, n \geq 0)$, we derive from results obtained in steps 3) and 8) that

$$\mathbb{P}^{(t)}_{\nu,1}\{d_n = 0\} \leq \sum_{k=1}^{[n/2]} \left(p^{\star k} \star \bar{S}^{\star k-1} \star \bar{S}\right)_n \leq \varepsilon \sum_{k=1}^{[n/2]} \varepsilon^{k-1}\binom{2k-2}{k-1}(1 \star \bar{S}^{\star k})_n. \tag{24}$$

10) Combining the above results, we obtain

$$\begin{aligned}
\sup_{n\geq 0} \|P^n(x,\cdot) - P^{t,n}(x,\cdot)\|_{TV} &\leq \mathbb{P}^{(t)}_{x,x,1}\{d_n = 0\} \\
&= \mathbb{P}^{(t)}_{x,x,1}\{\tau_1 < n, \nu_1 > n\} + \sum_{k=1}^{n} \mathbb{P}^{(t)}_{x,x,1}\{\nu_1 = k\}\mathbb{P}^{(t+k)}_{\nu,1}\{d_{n-k} = 0\} \\
&\leq \varepsilon\left(\bar{m}_x + \bar{m}\sum_{k\geq 0}\varepsilon^k\bar{m}^k\binom{2k}{k}\right) = \varepsilon\left(\bar{m}_x + \bar{m}(1-4\varepsilon\bar{m})^{-1/2}\right).
\end{aligned}$$

6. Auxiliary results

Lemma 6.1.

$$\mathbb{P}^{(t)}_{x,x,1}\{\tau_1 \leq n, \nu_1 > n\} \leq \varepsilon\bar{m}, \tag{25}$$

$$\sum_{k=1}^{n} \mathbb{P}^{(t)}_{x,x,1}\{\nu_1 = k\}\mathbb{P}^{(t+k)}_{\nu,1}\{d_{n-k} = 0\} \leq \sup_{n,t} \mathbb{P}^{(t)}\{d_n = 0\}. \tag{26}$$

Proof. First we prove inequality (25):

$$\begin{aligned}
\mathbb{P}^{(t)}_{x,x,1}\{\tau_1 < n, \nu_1 > n\} &= \sum_{k=1}^{n} \mathbb{P}^{(t)}_{x,x,1}\{\tau_1 = k, \nu_1 > n-k\} = \sum_{k=1}^{n-1} \varepsilon(1-\varepsilon)^{k-1}\delta_x Q^{k-1} S^{t+k}_{n-k} \\
&\leq \varepsilon\sum_{k=1}^{n-1} \delta_x Q^{k-1}\hat{S}_{n-k} \leq \varepsilon\sum_{k=1}^{n-1} \bar{S}_{n-k} \leq \varepsilon\bar{m}.
\end{aligned}$$

Inequality (26) is obvious, since

$$\sum_{k=1}^{n} \mathbb{P}^{(t)}_{x,x,1}\{\nu_1 = k\}\mathbb{P}^{(t+k)}_{\nu,1}\{d_{n-k} = 0\} \le \sup_{n,t} \mathbb{P}^{(t)}\{d_n = 0\} \sum_{k=1}^{n} \mathbb{P}^{(t)}_{x,x,1}\{\nu_1 = k\}$$
$$= \sup_{n,t} \mathbb{P}^{(t)}\{d_n = 0\}\mathbb{P}^{(t)}_{x,x,1}\{\nu_1 \le n\} \le \sup_{n,t} \mathbb{P}^{(t)}\{d_n = 0\}. \qquad \square$$

Lemma 6.2. *For an arbitrary measurable nonnegative function $f\colon E \to \mathbb{R}$,*

$$\mathbb{E}^{(t)}_{\nu,1}[f(X_{\tau_k-1}), \tau_k = j] = \mathbb{E}^{(t)}_{\nu,1}\left[\nu Q^{\xi_k-1} f, \tau_k = j\right]. \tag{27}$$

Proof. Indeed,

$$\begin{aligned}\mathbb{E}^{(t)}_{\nu,1}[f(X_{\tau_k-1}), \tau_k = j] &= \sum_{i=1}^{j-1} \mathbb{E}^{(t)}_{\nu,1}[f(X_{\tau_k-1}), \nu_{k-1} + \xi_k = j, \nu_{k-1} = i] \\ &= \sum_{i=1}^{j-1} \mathbb{P}^{(t)}_{\nu,1}\{\nu_{k-1} = i\}\varepsilon(1-\varepsilon)^{j-i-1} Q^{j-i-1} f \\ &= \sum_{i=1}^{j-1} \mathbb{P}^{(t)}_{\nu,1}\{\nu_{k-1} = i\}\mathbb{P}^{(t)}_{\nu,1}\{\xi_k = j-i\}\nu Q^{j-i-1} f \\ &= \mathbb{E}^{(t)}_{\nu,1}\left[\nu Q^{\xi_k-1} f, \tau_k = j\right]. \qquad \square\end{aligned}$$

Lemma 6.3.

$$\mathbb{P}^{(t)}_{\nu,1}\{\tau_k = j, \zeta_k > n-j\} \le \mathbb{E}^{(t)}_{\nu,1}[\hat{S}_{n-j}(X_{\tau_k-1}), \tau_k = j].$$

Proof. First we show that

$$\mathbb{E}^{(t)}_{\nu,1}[\zeta_k > v, \xi_k = u \mid \nu_{k-1}] = \varepsilon(1-\varepsilon)^{u-1}\nu Q^{u-1} S_v^{t+\nu_{k-1}+u-1}. \tag{28}$$

The measurability is obvious. Then we obtain for all $i > 0$ that

$$\mathbb{E}^{(t)}_{\nu,1}[\xi_k = u, \zeta_k > v, \nu_{k-1} = i] = \varepsilon(1-\varepsilon)^{u-1}\nu Q^{u-1} S_v^{t+i+u-1}\mathbb{E}^{(t)}_{\nu,1}[\nu_{k-1} = i].$$

By condition (M),

$$\nu Q^{u-1} S_v^{t+\nu_{k-1}+u-1} \le \nu Q^{u-1}\hat{S}_v. \tag{29}$$

Using (28) and (29),

$$\begin{aligned}\mathbb{P}^{(t)}_{\nu,1}\{\tau_k = j, \zeta > n-j\} &= \sum_{l=1}^{j-1} \mathbb{E}^{(t)}_{\nu,1}[\nu_{k-1} = l, \xi_k = j-l, \zeta_k > n-j] \\ &= \sum_{l=1}^{j-1} \mathbb{E}^{(t)}_{\nu,1}[\mathsf{E}[\xi_k = j-l, \zeta_k > n-j \mid \nu_{k-1}], \nu_{k-1} = l] \\ &\le \sum_{l=1}^{j-1} \mathbb{E}^{(t)}_{\nu,1}[\varepsilon(1-\varepsilon)^{j-l}\nu Q^{j-l-1}\hat{S}_{n-j}, \nu_{k-1} = l] \\ &= \sum_{l=1}^{j-1} \mathbb{E}^{(t)}_{\nu,1}[\nu Q^{j-l-1}\hat{S}_{n-j}, \nu_{k-1} = l, \xi_k = j-l] \\ &= \sum_{l=1}^{j-1} \mathbb{E}^{(t)}_{\nu,1}[\hat{S}_{n-j}(X_{\tau_k-1}), \nu_{k-1} = l, \xi_k = j-l] \\ &= \mathbb{E}^{(t)}_{\nu,1}[\hat{S}_{n-j}(X_{\tau_k-1}), \tau_k = j],\end{aligned}$$

since the random variables ξ_k are independent. $\square$

Lemma 6.4.

$$\mathbb{E}^{(t)}_{\nu,1}[\hat{S}_{n-j}(X_{\tau_k-1}),\tau_k=j]=\sum_{l=0}^{j}\mathbb{P}^{(t)}_{\nu,1}[\nu_{k-1}=l]\mathbb{E}^{(t+l)}_{\nu,1}[\nu Q^{\xi_k-1}\hat{S}_{n-j},\xi_k=j-l].$$

Proof. By Lemma 6.2,

$$\begin{aligned}\mathbb{E}^{(t)}_{\nu,1}[\hat{S}_{n-j}(X_{\tau_j-1}),\tau_k=j]&=\mathbb{E}^{(t)}_{\nu,1}\big[\nu Q^{\xi_k-1}\hat{S}_{n-j},\tau_k=j\big]\\&=\sum_{l=0}^{j}\mathbb{E}^{(t)}_{\nu,1}\big[\nu Q^{\xi_k-1}\hat{S}_{n-j},\nu_{k-1}=l,\xi_k=j-l\big]\\&=\sum_{l=0}^{j}\mathbb{P}^{(t)}_{\nu,1}\{\nu_{k-1}=l\}\mathbb{E}^{(t+l)}_{\nu,1}\big[\nu Q^{\xi_k-1}\hat{S}_{n-j},\xi_k=j-l\big],\end{aligned}$$

whence Lemma 6.4 follows. □

Lemma 6.5.

$$\mathbb{P}^{(t)}_{\nu,1}[\nu_k=l]\le\left(p^{\star k}\star\bar{S}^{\star k}\right)_l, \tag{30}$$

$$\mathbb{P}^{(t)}_{\nu,1}\big[\xi_k=j-l,\nu Q^{\xi_k-1}\hat{S}_{n-j}\big]\le\varepsilon(1-\varepsilon)^{j-l-1}\bar{S}_{n-j}. \tag{31}$$

Proof. We prove inequality (30) by induction. For $k=1$ inequality (6.5) holds, since

$$\begin{aligned}\mathbb{P}^{(t)}_{\nu,1}[\nu_1=l]&=\sum_{j=1}^{l-1}\varepsilon(1-\varepsilon)^{j-1}\nu Q^{j-1}s^{(t+j)}_{l-j}\le\sum_{j=1}^{l-1}\varepsilon(1-\varepsilon)^{j-1}\nu Q^{j-1}S^{(t+j)}_{l-j}\\&\le\sum_{j=1}^{l-1}\varepsilon(1-\varepsilon)^{j-1}\nu Q^{j-1}\hat{S}_{l-j}\le\sum_{j=1}^{l-1}\varepsilon(1-\varepsilon)^{j-1}\bar{S}_{l-j}=(p\star\bar{S})_l.\end{aligned}$$

Let inequality (30) hold for some k, then our aim is to prove it for $k+1$. Indeed,

$$\begin{aligned}\mathbb{P}^{(t)}_{\nu,1}[\nu_{k+1}=l]&=\sum_{j=1}^{l-1}\mathbb{P}^{(t)}_{\nu,1}[\nu_k=j]\mathbb{P}^{(t+k)}_{\nu,1}\{\nu_{k+1}-\nu_k=l-j\}\\&\le\sum_{j=1}^{l-1}(p^{\star k}\star\bar{S})_j\sum_{i=1}^{l-j-1}\varepsilon(1-\varepsilon)^{i-1}\nu Q^{i-1}s^{(t+k+i)}_{l-j-i}\\&\le\sum_{j=1}^{l-1}(p^{\star k}\star\bar{S})_j\sum_{i=1}^{l-j-1}p_i\bar{S}_{l-j-i}=\left(p^{\star k+1}\star S^{\star k+1}\right)_l.\end{aligned}$$

Inequality (31) also holds, since

$$\begin{aligned}\mathbb{P}^{(t)}_{\nu,1}\big[\xi_k=j-l,\nu Q^{\xi_k-1}\hat{S}_{n-j}\big]&=\mathbb{E}^{(t)}_{\nu,1}\left[\varepsilon(1-\varepsilon)^{j-l-1}\nu Q^{j-l-1}S^{(t+\nu_{k-1}+j-l)}_{n-j}\right]\\&\le\varepsilon(1-\varepsilon)^{j-l-1}\nu Q^{j-l-1}\hat{S}_{n-j}\le\varepsilon(1-\varepsilon)^{j-l-1}\bar{S}_{n-j}.\end{aligned}$$ □

Lemma 6.6. *Random variables ξ_k do not depend on τ_i, ν_i, ξ_i, and ζ_i, $i=0,\dots,k-1$. The distribution of ξ_k coincides with $(p_n,n\ge0)$, while the distribution of $\sum_{i=1}^k\xi_i$ coincides with $(p^{\star k}_n,n\ge0)$.*

Proof. The random variable ξ_k is the time between the $(k-1)^{\text{th}}$ coupling moment and the k^{th} decoupling moment. By construction, the probability of transition from a state $d_i\in\{1,2\}$ to a state $d_{i+1}=2$ equals $1-\varepsilon$ and does not depend on i. Similarly, the probability of transition from a state $d_i\in\{1,2\}$ to a state $d_{i+1}=0$ equals ε. Thus the random variables ξ_k do not depend on the behavior of the process $(Z_n,n\ge0)$ until the moment ν_{k-1} and their common distribution is geometrical, that is, the distribution is $(p_k, k \geq 0)$.

Since the random variables ξ_k are jointly independent and identically distributed, the distribution of their sum is the k^{th} convolution of the distribution of an individual term with itself. □

7. Appendix

Theorem 7.1. *Let X and X' be two independent nonhomogeneous discrete Markov chains, let $\alpha > 0$ be a constant, and let $C = \{0\}$. Assume that ξ is a geometrical random variable with parameter α that does not depend on X and X'. In particular,* $\mathsf{P}\{\xi = 0\} = 0$ *and*

$$\mathsf{P}\{\xi = j\} = (1-\alpha)^{j-1}\alpha, \qquad j > 0.$$

Put $\tau_0 = 0$ and

$$\tau_k = \inf\{t \geq \tau_{k-1}, (X_t, X'_t) \in C \times C\}.$$

Let $\hat{S}_n$ be a majorant for τ_1; that is,

$$P_{ij}^{(t)}\{\tau_1 > n\} \leq \hat{S}_n(ij)$$

for all t and, moreover,

$$\mu = \sum_{n \geq 0} \hat{S}_n(00) < \infty.$$

Then there exists a majorant $\hat{S}'_n$ for the moment τ_ξ and for all $i, j \in E$. Moreover,

$$P_{ij}^{(t)}\{\tau_\xi > n\} \leq \hat{S}'_n(ij),$$
$$\sum_{n \geq 0} \hat{S}'_n(ij) \leq \mu/\alpha < \infty.$$

Proof. Let $Y_n = (X_n, X'_n)$ be the nonhomogeneous Markov chain and let

$$\mathcal{F}_n = \sigma[Y_k, k \leq n]$$

be its natural filtration. Then

$$\begin{aligned}
P_{xy}^{(t)}\{\tau_\xi > n\} &= \sum_{k \geq 1} P_{xy}\{\tau_{k-1} \leq n < \tau_k, \xi > k-1\} \\
&= \sum_{k \geq 1} \sum_{j=1}^{k} E_{xy}^{(t)}[\tau_{k-1} = j, \mathsf{E}[\tau_k - \tau_{k-1} > n - j \mid \mathcal{F}\tau_{k-1}]](1-\alpha)^{k-1} \\
&= \sum_{k \geq 1} \sum_{j=1}^{k} E_{xy}^{(t)}[\tau_{k-1} = j] E_{00}^{(t+j)}[\tau_1 > n - j](1-\alpha)^{k-1} \\
&\leq \sum_{k \geq 1} \sum_{j=1}^{k} E_{xy}^{(t)}[\tau_{k-1} = j] \hat{S}_{n-j}(00)(1-\alpha)^{k-1} \\
&= \sum_{j=1}^{n} \hat{S}_{n-j}(00) \sum_{k \geq 1} (1-\alpha)^{k-1} P_{xy}^{(t)}\{\tau_j = k\} = \hat{S}'_n(xy).
\end{aligned}$$

Now we show that $\hat{S}'_n(xy)$ is summable:

$$\begin{aligned}\sum_{n\ge 0}\hat{S}'_n(xy) &= \sum_{n\ge 0}\sum_{j=1}^{n}\bar{S}_{n-j}(00)\sum_{k\ge 1}(1-\alpha)^{k-1}P^{(t)}_{xy}\{\tau_j=k\}\\ &= \mu\sum_{j\ge 0}\sum_{k\ge 1}(1-\alpha)^{k-1}P^{(t)}_{xy}\{\tau_j=k\} = \mu\sum_{k\ge 1}(1-\alpha)^{k-1}\sum_{j\ge 0}P^{(t)}_{xy}\{\tau_j=k\}\\ &= \mu\sum_{k\ge 1}(1-\alpha)^{k-1}P^{(t)}_{xy}\{Y_k\in C\times C\}\le \mu/\alpha. \qquad\square\end{aligned}$$

Proof of Theorem 3.1. Minorization condition (A1) obviously holds in view of assumption 1) with parameter $\alpha=\inf_t\{\alpha_k^{(t)},\alpha_k\}$. Note, however, that the uniform minorization condition does not hold in this case. In order to apply Theorem 2.1 one needs to show that a majorant exists. We use Theorem 7.1 to show the existence of a majorant for the first passage time for the state $(0,0)$.

Denote by τ_k the moment of the k^{th} passage time to 0 for the first chain, by ν_k the number of entries to $\{0\}$ for the first chain over the time n, and by T the moment of the first entry of both chains to $\{0\}$. Then

$$P^{(t)}_{ij}\{T>n\} = P^{(t)}_{ij}\{X'_{\tau_1}\neq 0,\dots,X'_{\tau_{\nu_n}}\neq 0\}. \tag{32}$$

We are going to prove that there exists a number γ such that

$$P^{(t)}_{ij}\{X'_{i_1}\neq 0,\cdots,X'_{i_n}\neq 0\}\le(1-\gamma)^n$$

for an arbitrary set $(i_1,\dots,i_n)$ with the property that $i_{k+1}-i_k>1$.

First we show that

$$\inf_{n\ge 2,t}u_n^{(t)}\ge\gamma_0,$$
$$\inf_{n\ge 2,t}P^{t,n}(i,0)\ge\gamma_1,$$

where

$$u_n^{(t)}=P^{t,n}(0,0).$$

The first inequality is proved by induction. If $n=2$, then

$$u_2^{(t)}=\sum_{j\ge 1}\alpha_j^{(t)}\beta_j^{(t)}\ge\delta.$$

Let $u_k^{(t)}\ge\gamma_n$ for all t and all $k\le n$. Then

$$u_{n+1}^{(t)}=\sum_{k=0}^{n+1}g_k^{(t)}u_{n+1-k}^{(t+k)}\ge\gamma_n\left(\sum_{k=0}^{n-1}g_k^{(t)}+\gamma_{n+1}^{(t)}\right)=\gamma_n\big(1-G_{n+1}^{(t)}-g_n^{(t)}\big).$$

Let $\tilde{G}_n=G_{n+1}^{(t)}+g_n^{(t)}$, whence

$$u_{n+1}^{(t)}\ge\gamma_n(1-\tilde{G}_n)\ge\delta\prod_{k\ge 3}(1-\tilde{G}_k).$$

We are going to prove that $\tilde{G}_n$ is a decreasing sequence. This follows from the same property of the sequence $g_n^{(t)}$:

$$g_n^{(t)}=\sum_{j\ge 1}\alpha_j^{(t)}\prod_{i=1}^{n-2}\left(1-\beta_j^{t+i}\right)\beta_j^{t+n-1}\ge\sum_{j\ge 1}\alpha_j^{(t)}\prod_{i=1}^{n-1}\left(1-\beta_j^{t+i}\right)\beta_j^{t+n}=g_{n+1}^{(t)}.$$

Hence

$$0 \le \tilde{G}_n \le \tilde{G}_3 = g_3^{(t)} + \sum_{k>4} g_k^{(t)} = \sum_{k>2} g_k^{(t)} - g_3^{(t)} = 1 - g_2^{(t)} - g_3^{(t)},$$

$$\sum_{k\ge 3} \tilde{G}_k = \sum_{k\ge 3} g_k^{(t)} + \sum_{k\ge 3} G_k^{(t)} = \sum_{k\ge 2} G_k^{(t)} = m^{(t)} - G_0^{(t)} - G_1^{(t)} = m^{(t)} - 2.$$

Therefore

$$u_{n+1}^{(t)} \ge \delta \left(g_2^{(t)} + g_3^{(t)} \right)^{\frac{m^{(t)}-2}{1-g_1^{(t)}-g_2^{(t)}}} \ge \delta \left(\inf_t \left\{ g_2^{(t)} + g_3^{(t)} \right\} \right)^{\frac{\sup_t m^{(t)}-2}{1-\inf_t \{g_1^{(t)}+g_2^{(t)}\}}} = \gamma_0.$$

Next we prove that

$$\inf_{n\ge 2,t} P^{t,n}(i,0) \ge \gamma_1.$$

This follows from

$$\inf_{n\ge 2,t} P^{t,n}(i,0) = \sum_{k=0}^{n} P_0^{t,k}(i,0) u_{n-k}^{t+k} \ge \gamma_0 \sum_{k=0}^{n} \prod_{j=0}^{k-1} \left(1 - \beta_i^{(t+j)}\right) \beta_i^{t+k} \ge \gamma_0 \delta = \gamma_1.$$

Thus

$$P_{xy}^{(t)}\{T > n\} \le \sum_{k=1}^{n} \sum_{j=0}^{n} (1-\gamma)^k P_y\{\tau_k = j\} P_0\{\tau_{k+1} - \tau_k > n - j\} = E_{xy}[(1-\gamma)^{\nu_n}].$$

Finally we show that

$$\sum_{n\ge 0} E_{xy}[(1-\gamma)^{\nu_n}] < \infty.$$

Indeed,

$$\begin{aligned}\sum_{n\ge 0} E_{xy}[(1-\gamma)^{\nu_n}] &= \sum_{n\ge 0} \sum_{k=0}^{n} (1-\gamma)^k P_y\{\nu_n = k\} = \sum_{k\ge 0} (1-\gamma)^k \sum_{n\ge k} P_y\{\nu_n = k\} \\ &= \sum_{k\ge 0} (1-\gamma)^k \sum_{n\ge k} \sum_{j=0}^{n} P_y\{\tau_k = j\} P_0\{\tau_{k+1} - \tau_k > n - j\} \\ &\le \mu/\gamma. \qquad \square\end{aligned}$$

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