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THE UNIQUENESS OF THE QUASI-LIKELIHOOD ESTIMATOR IN THE POISSON MODEL WITH AN ERROR IN THE REGRESSOR

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Abstract. The Poisson regression Berkson type model with a Gaussian error in the regressor is studied. Simple score and quasi-likelihood estimators of the regression parameters are considered. Sufficient conditions for the strong consistency of the estimators and sufficient conditions for the uniqueness of a solution of estimating equations are found. The proof of the uniqueness does not require that the parameter set be bounded.

1. Introduction

We consider the log-linear Poisson regression model. Given an observation X, the output Y assumes nonnegative integer values with the conditional probabilities

$$\mathsf{P}[Y = k \mid X] = e^{-\lambda}\frac{\lambda^k}{k!},$$

where $\lambda = e^{\beta_0+\beta_1 X}$ and $\beta = (\beta_0, \beta_1)$ is the regression parameter. Here X is the true value of the regressor. We consider the Berkson model for an error in the regressor X, namely

$$X = X_0 + U^{\mathrm{Berk}}.$$

We assume that the error is Gaussian, $U^{\mathrm{Berk}} \sim N(0, \tau^2)$, and that it does not depend on X_0. We also assume that the error is indifferent in the sense that

$$\mathsf{P}\left[Y{=}k \mid X, U^{\mathrm{Berk}}\right] = \mathsf{P}[Y{=}k \mid X].$$

We study the structural model, where the observed values of the regressor X_{0i} are random and (X_{0i}, X_i, Y_i), $i{=}1, 2, \dots$, is a sequence of independent identically distributed realizations. We construct estimators of the parameter β under the assumption that the parameter τ^2 is known.

The Poisson regression model with the classical errors in the regressor is considered in the papers [2, 5]. The quasi-likelihood method (also known as the quasi-score method) is considered in the paper [8]. The quasi-likelihood estimator is constructed in [7] for various classical regression models with normally distributed regressor and error of observation; these results are also useful in the so-called Berkson model.

If the regressor and error have a normal distribution in the model with the classical error, then the model reduces to the Berkson model [1, §2.2.3]. The consistency of the quasi-likelihood estimator in generalized linear models with such errors is proved in [3]. We drop the assumption that the set of parameters is bounded; the consistency of the quasi-score estimator and uniqueness of a solution of the estimating equation under the

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assumption of the boundedness of the parametric set is proved in [6] for the classical model.

In the current paper, we consider the structural Berkson model. We change the assumption that the variable X_0 has a normal distribution and use the more general conditions (14) and (24). We also find sufficient conditions that solutions of equations determining the simple score and quasi-likelihood estimators are eventually unique. We also find sufficient conditions for the strong consistency of these estimators.

2. The classical model of the Poisson regression

2.1. **Statistical model.** Values of the regressor are denoted by X_i, while those of the output are denoted by Y_i. As a rule, we consider the functional model, where the values of the regressor X_i are nonrandom numbers. The distribution of the output Y_i (the conditional distribution of Y_i given X_i if X_i is random) is Poissonian,

$$Y_i \mid X_i \sim \mathrm{Poi}\left(e^{\beta_0+\beta_1 X_i}\right),$$
$$\mathsf{P}[Y_i{=}k \mid X_i] = \frac{1}{k!}\exp\left\{-e^{\beta_0+\beta_1 X} + k(\beta_0+\beta_1 X_i)\right\},$$

where $\beta = (\beta_0, \beta_1)$ is the regression parameter.

The observations are assumed to be independent. If X_i are nonrandom (in this case, the model is called functional), then Y_i are independent random variables that have Poisson distributions with, in general, different parameters. If X_i are independent and identically distributed random variables (in this case, the model is called structural), then the pairs (X_i, Y_i) are independent and identically distributed random vectors.

2.2. **Maximal likelihood estimator.** Given the observations

$$(X_i, Y_i), \qquad i = 1, 2, \ldots, N,$$

the estimator of the parameter β is defined as a point of minimum of the functional

$$Q_{\text{no error}}(b) = \sum_{i=1}^{N}\left(e^{b_0+b_1 X_i} - (b_0+b_1 X_i)Y_i\right) \tag{1}$$

or as a solution of the system of equations

$$\begin{cases} \sum_{i=1}^{N} e^{b_0+b_1 X_i} = \sum_{i=1}^{N} Y_i, \\ \sum_{i=1}^{N} X_i e^{b_0+b_1 X_i} = \sum_{i=1}^{N} X_i Y_i. \end{cases} \tag{2}$$

Since the function $Q_{\text{no error}}(b)$ is convex, the set of points of the global minimum in $\mathbb{R}^2$ of the function defined by (1) coincides with the set of solutions of the system of equations (2).

Lemma 2.1 (Uniqueness of a solution)**.** *A solution of equation* (2) *is unique if and only if*

$$\begin{cases} \exists i{\le}N\ \exists j{\le}N\colon \quad X_i < X_j,\ Y_i > 0, \\ \exists i{\le}N\ \exists j{\le}N\colon \quad X_i < X_j,\ Y_j > 0. \end{cases} \tag{3}$$

Moreover, if all X_i, $i{=}1,\ldots,N$, are identical and there is a nonzero output Y_i, then equation (2) *has infinitely many solutions. The equation has no solutions at all in all other cases.*

Proof. If $Y_1 = Y_2 = \cdots = Y_N = 0$, then the system of equations (2) does not have solutions. Consider the case where there exists a nonzero output, that is, $Y_i > 0$ for some i. Put

$$\overline{X}_Y = \sum_{i=1}^{N} X_i Y_i \Big/ \sum_{i=1}^{N} Y_i.$$

Exclude the variable b_0 from the system of equations (2):

$$\frac{\sum_{i=1}^{N} X_i e^{b_1 X_i}}{\sum_{i=1}^{N} e^{b_1 X_i}} = \overline{X}_Y.$$

The left hand side of the equation is a nondecreasing function (or a constant, or a strictly monotone function) of the argument b_1 that approaches

$$\min_{i=1,\dots,N} X_i \quad \text{as } b_1 \to -\infty \qquad \text{and} \qquad \max_{i=1,\dots,N} X_i \quad \text{as } b_1 \to +\infty.$$

The equation possesses exactly one solution if and only if

$$\min_{i=1,\dots,N} X_i < \overline{X}_Y < \max_{i=1,\dots,N} X_i,$$

which is equivalent to condition (3). The equation possesses infinitely many solutions if and only if

$$\min_{i=1,\dots,N} X_i = \overline{X}_Y = \max_{i=1,\dots,N} X_i.$$

In all other cases, the equation does not have solutions. □

If at least two observations are such that the corresponding Y are positive and X are different, then system (2) possesses exactly one solution.

3. The Poisson errors-in-variables model. The Berkson model

3.1. **Statistical model.** We consider the Berkson model for the scalar log-linear Poisson regression.

3.1.1. *A single observation.* Let X_0 be an observed value of the regressor, X be the true value of the regressor, and Y be the value of the output (being a nonnegative integer number).

A feature of the Berkson model is that the true value of the regressor equals the observed value of the regressor with the error

$$X = X_0 + U^{\text{Berk}}.$$

We also assume that the error is homoscedastic and normally distributed, that is,

$$X \mid X_0 \sim N\left(X_0, \tau^2\right).$$

The parameter τ^2 is assumed to be known.

The distribution of Y depends on the true value of the regressor,

$$Y \mid (X, X_0) \sim \text{Poi}\left(e^{\beta_0 + \beta_1 X}\right),$$

$$\mathsf{P}[Y{=}k \mid X, X_0] = \frac{1}{k!} \exp\left\{-e^{\beta_0+\beta_1 X} + k(\beta_0 + \beta_1 X)\right\},$$

where $\beta = (\beta_0, \beta_1)$ is the regression parameter.

The assumption that the distribution of $Y \mid (X, X_0)$ does not depend on X_0 explicitly, that is,

$$Y \mid (X, X_0) \sim Y \mid X,$$

is called the indifferentiability condition.

3.1.2. *Functional and structural model.* In all cases under consideration we assume that the realizations (X_{0i}, X_i, Y_i) (as random three dimensional vectors) are independent. Observed are the values (X_{0i}, Y_i), $i=1,2,\dots,N$.

Functional model. In this model, X_{0i} are nonrandom numbers and the pairs (X_i, Y_i) are independent but not necessarily identically distributed.

Structural model. For this model, X_{0i} are independent identically distributed random variables. Then the realizations (X_{0i}, X_i, Y_i) also are independent identically distributed random triples.

3.2. **Estimators of the regression parameter.**

3.2.1. *A naive estimator.* When constructing a naive estimator, we ignore the error and substitute X_0 instead of X in the equation for the maximal likelihood estimator in the case of the model without errors (2). Then the estimator is defined as a point of minimum of the function

$$Q_{\text{naive}}(b) = \sum_{i=1}^{N} \left(e^{b_0+b_1X_{0i}} - (b_0+b_1X_{0i})Y_i\right) \tag{4}$$

or as a solution of the equation

$$\begin{cases} \sum_{i=1}^{N} e^{b_0+b_1X_{0i}} = \sum_{i=1}^{N} Y_i, \\ \sum_{i=1}^{N} X_{0i}e^{b_0+b_1X_{0i}} = \sum_{i=1}^{N} X_{0i}Y_i. \end{cases} \tag{5}$$

3.2.2. *The simple score estimator.* We consider the regression of Y with respect to X_0. The conditional expectation equals

$$\mathsf{E}[Y|X_0] = \exp\left\{\beta_0 + \beta_1X_0 + \tfrac{1}{2}\beta_1^2\tau^2\right\}, \tag{6}$$

whence $\mathsf{E}\big[\big(\exp\{\beta_0+\beta_1X_0+\frac{1}{2}\beta_1^2\tau^2\} - Y\big)\cdot a(X_0)\big] = 0$ for Borel functions $a(\cdot)$ such that the latter expectation exists.

The following system of equations for the estimator of the parameter β is unbiased:

$$\begin{cases} \sum_{i=1}^{N} e^{b_0+b_1X_{0i}+\frac{1}{2}b_1^2\tau^2} = \sum_{i=1}^{N} Y_i, \\ \sum_{i=1}^{N} X_{0i}e^{b_0+b_1X_{0i}+\frac{1}{2}b_1^2\tau^2} = \sum_{i=1}^{N} X_{0i}Y_i. \end{cases} \tag{7}$$

The property of a system of equations to be unbiased means that both sides of the equations have the same expectation if the true value of the parameter β is used instead of b. Note that the system of equations (7) in the structural model is unbiased if

$$\mathsf{E}\,|X_0|\,e^{\beta_1X_0} < \infty.$$

The estimator is defined as a point of minimum of the function

$$Q_{\text{SS}}(b) = \sum_{i=1}^{N} \left(e^{b_0+b_1X_{0i}+\frac{1}{2}b_1^2\tau^2} - \left(b_0+b_1X_{0i}+\tfrac{1}{2}b_1^2\tau^2\right)Y_i\right), \tag{8}$$

$$Q_{\text{SS}}(b_0,b_1) = Q_{\text{naive}}\left(b_0+\tfrac{1}{2}b_1^2\tau^2, b_1\right). \tag{9}$$

Relation (9) between target functions has its counterpart describing a relationship between the naive estimator and simple score estimator written as follows:

$$\hat\beta_{1,\text{SS}} = \hat\beta_{1,\text{naive}}, \tag{10}$$

$$\hat\beta_{0,\text{SS}} = \hat\beta_{0,\text{naive}} - \tfrac{1}{2}\hat\beta_{1,\text{naive}}^2\tau^2. \tag{11}$$

3.2.3. *The quasi-likelihood estimator.* This method of estimation is also called the quasi-score method or weighted least squares method. The system of equations for the estimator is given by

$$\sum_{i=1}^{N} \frac{\mathsf{E}_b[Y_i \mid X_{0i}] - Y_i}{\mathrm{Var}_b[Y_i \mid X_{0i}]} \cdot \frac{\partial\, \mathsf{E}_b[Y_i \mid X_{0i}]}{\partial b} = 0$$

(b is the unknown parameter). Since

$$\mathrm{Var}_b[Y \mid X_0] = e^{b_0+b_1X_0+\frac{1}{2}b_1^2\tau^2} + e^{2b_0+2b_1X_0+2b_1^2\tau^2} - e^{2b_0+2b_1X_0+b_1^2\tau^2}$$

(also see equality (6)), the system of equations for the estimator is written as follows:

$$\begin{cases} \sum_{i=1}^{N} \frac{\exp\left\{b_0+b_1X_{0i}+\frac{1}{2}b_1^2\tau^2\right\}-Y_i}{1+\left(e^{b_1^2\tau^2}-1\right)\exp\left\{b_0+b_1X_{0i}+\frac{1}{2}b_1^2\tau^2\right\}} = 0, \\ \sum_{i=1}^{N} \frac{\left(\exp\left\{b_0+b_1X_{0i}+\frac{1}{2}b_1^2\tau^2\right\}-Y_i\right)\left(X_{0i}+b_1\tau^2\right)}{1+\left(\exp\left\{b_1^2\tau^2\right\}-1\right)\exp\left\{b_0+b_1X_{0i}+\frac{1}{2}b_1^2\tau^2\right\}} = 0. \end{cases} \tag{12}$$

Note that the factor $(X_{0i} + b_1\tau^2)$ in the second equation can be changed by X_{0i}, and this results in an equivalent system of equations.

3.2.4. *The maximal likelihood estimator.* The conditional probability equals

$$\mathsf{P}[Y{=}k \mid X_0] = \frac{1}{k!} f\left(k, \beta_0 + \beta_1 X_0, \beta_1^2\tau^2\right),$$

where $f(k, \mu, \sigma^2) = \mathsf{E}\exp\{-e^{\zeta} + k\zeta\}$, $\zeta \sim N(\mu, \sigma^2)$.

The estimator is defined as a point of maximum of the functional

$$\sum_{i=1}^{N} \ln f\left(Y_i, b_0 + b_1X_{0i}, b_1^2\tau^2\right).$$

An expression for the solution of this problem involves special functions.

3.3. **Uniqueness of a solution and consistency of the simple score estimator in the structural model.** We say that a random event A_N depending on an index N *eventually* holds if the probability of

$$\varliminf_{N\to\infty} A_N = \left\{\exists N_0 \in \mathbb{N}\colon \text{the random event } A_N \text{ occurs for all } N > N_0\right\}$$

equals one.

Theorem 3.1. *If the distribution of X_0 is not concentrated at a single point and is such that*

$$\exists\varepsilon{>}0\colon\ \mathsf{E}\left[e^{(\beta_1+\varepsilon)X_0} + e^{(\beta_1-\varepsilon)X_0}\right] < \infty,$$

then the system of equations (7) *for the simple score estimator eventually has a unique solution in $\mathbb{R}^2$. This solution is a strongly consistent estimator of the parameter β.*

Proof. The limit value of the normalized target function for the naive estimator is denoted by

$$q^*_{\text{naive}}(b) = \frac{1}{N}\,\mathsf{E}\,Q(b) = \mathsf{E}\left[e^{b_0+b_1X_0} - (b_0 + b_1X_0)Y\right].$$

If

$$|b_1 - \beta_1| \le \varepsilon, \tag{13}$$

then the expectation exists. If b is such that (13) holds, then

$$\frac{1}{N}\,\mathsf{E}[Q(b)\mid X,X_0]=e^{b_0+b_1X_0}-(b_0+b_1X_0)e^{\beta_0+\beta_1X},$$
$$\frac{1}{N}\,\mathsf{E}[Q(b)\mid X_0]=e^{b_0+b_1X_0}-(b_0+b_1X_0)e^{\beta_0+\beta_1X_0+\frac{1}{2}\beta_1^2\tau^2},$$
$$q^*_{\text{naive}}(b)=\mathsf{E}\left[e^{b_0+b_1X_0}-(b_0+b_1X_0)e^{\beta_0+\beta_1X_0+\frac{1}{2}\beta_1^2\tau^2}\right].$$

The function $q^*_{\text{naive}}(b)$ is convex with respect to the vector variable b (in fact, $q^*_{\text{naive}}(b)$ is strictly convex, since the distribution of X_0 is not concentrated at a single point). Since

$$\begin{aligned}&e^{b_0+b_1X_0}-(b_0+b_1X_0)e^{\beta_0+\beta_1X_0+\frac{1}{2}\beta_1^2\tau^2}\\&\quad=e^{\beta_0+\beta_1X_0}\left(e^{b_0-\beta_0-\frac{1}{2}\beta_1^2\tau^2+(b_1-\beta_1)X_0}-b_0-b_1X_0\right)\\&\quad\geq e^{\beta_0+\beta_1X_0}\left(1-\beta_0-\frac{1}{2}\beta_1^2\tau^2-\beta_1X_0\right),\end{aligned}$$

we get

$$q^*_{\text{naive}}(b)\geq\mathsf{E}\left[e^{\beta_0+\beta_1X_0}\left(1-\beta_0-\tfrac{1}{2}\beta_1^2\tau^2-\beta_1X_0\right)\right].$$

The equality is attained if and only if

$$\mathsf{P}\left[b_0-\beta_0-\tfrac{1}{2}\beta_1^2\tau^2+(b_1-\beta_1)X_0=0\right]=1.$$

Thus the function $q^*_{\text{naive}}(b)$ attains its minimum at the point

$$(b_0,b_1)=\left(\beta_0+\frac{1}{2}\beta_1^2\beta_1^2,\beta_1\right).$$

By the strong law of large numbers,

$$\mathsf{P}\left(\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}Q_{\text{naive}}(b)=q^*_{\text{naive}}(b)\right)=1$$

for all $b\in[\beta_0-\varepsilon,\beta_0+\varepsilon]\times[\beta_1-\varepsilon,\beta_1+\varepsilon]$, whence we conclude that

$$\lim_{N\to\infty}\sup_{\substack{|b_0-\beta_0|\leq\varepsilon\\|b_1-\beta_1|\leq\varepsilon}}\left|\frac{1}{N}Q_{\text{naive}}(b)-q^*_{\text{naive}}(b)\right|=0$$

almost surely by Theorem 10.8 in [4]. Hence the function $Q_{\text{naive}}(b)$ eventually attains the minimum and, for any point of minimum denoted by $\hat\beta_{\text{naive}}$, the almost sure convergence holds:

$$\lim_{N\to\infty}\hat\beta_{0,\text{naive}}=\beta_0+\tfrac{1}{2}\beta_1^2\tau^2,$$
$$\lim_{N\to\infty}\hat\beta_{1,\text{naive}}=\beta_1.$$

A solution is unique, since the function $Q_{\text{naive}}(b)$ is analytic and convex. According to relations (10)–(11), this implies that

$$\lim_{N\to\infty}\hat\beta_{\text{SS}}=\beta$$

almost surely. □

3.4. **Uniqueness of a solution and consistency of the quasi-likelihood estimator in the structural model.**

Theorem 3.2. *If the distribution of X_0 is not concentrated at a single point and is such that*

$$\mathsf{E}\left[e^{(-\beta_1-2\varepsilon)X_0}+e^{(-\beta_1+2\varepsilon)X_0}+e^{(2\beta_1-2\varepsilon)X_0}+e^{(2\beta_1+2\varepsilon)X_0}\right]<\infty, \tag{14}$$

then the system of equations (12) *for the quasi-likelihood estimator eventually has a solution in $\mathbb{R}^2$. The estimator defined as a solution of the system of equations* (12) *and denoted by $\hat\beta_{\mathrm{QL}}$ is a strongly consistent estimator of the parameter β, that is,*

$$\lim_{N\to\infty}\hat\beta_{\mathrm{QL}}=\beta$$

almost surely.

To prove Theorem 3.2 we consider the family of estimators depending on a parameter $\alpha\in[0,1]$ and defined as solutions of the following system of equations:

$$\begin{cases}\sum_{i=1}^N \frac{\exp\{b_0+b_1X_{0i}\}-Y_i}{\alpha+(1-\alpha)\exp\{b_0+b_1X_{0i}\}}=0,\\ \sum_{i=1}^N \frac{(\exp\{b_0+b_1X_{0i}\}-Y_i)X_{0i}}{\alpha+(1-\alpha)\exp\{b_0+b_1X_{0i}\}}=0.\end{cases} \tag{15}$$

These estimators can also be defined as the points of minimum of the functional

$$Q_{\text{nai-}\alpha}(b)=\sum_{i=1}^N q_{\text{nai-}\alpha}(X_{0i},Y_i;b),$$

where

$$\begin{aligned}&q_{\text{nai-}\alpha}(x_0,y;b)\\ &\quad=\frac{1}{1-\alpha}\ln\left(\alpha+(1-\alpha)e^{b_0+b_1x_0}\right)+\frac{y}{\alpha}\ln\left(\alpha e^{-b_0-b_1x_0}+1-\alpha\right),\qquad 0<\alpha<1,\\ &\qquad q_{\text{nai-}\alpha}(x_0,y;b)=b_0+b_1x_0+ye^{-b_0-b_1x_0}-y,\qquad \alpha=0,\\ &\qquad q_{\text{nai-}\alpha}(x_0,y;b)=e^{b_0+b_1x_0}-1-(b_0+b_1x_0)y,\qquad \alpha=1.\end{aligned}$$

Since the functional $Q_{\text{nai-}\alpha}(b)$ is convex for all $\alpha\in[0,1]$, the set of its solutions coincides with the set of solutions of the extremum equation (15).

First we study the asymptotic properties of the estimators for a fixed α.

Lemma 3.1. *Necessary and sufficient conditions for the existence and uniqueness of a solution of equation* (15) *are given in the following table:*

α	*Condition that equation* (15) *has a unique solution*	*Condition that equation* (15) *has infinitely many solutions*
$0<\alpha\le1$	$\exists i\le N,\ \exists j\le N,\ \exists k\le N,\ \exists l\le N$: $X_{0i}<X_{0j},\ X_{0k}<X_{0l},\ Y_i>0,\ Y_l>0$	$\forall i\le N$: $X_{0i}=X_{01}$; $\exists j\le N$: $Y_j>0$
$\alpha=0$	$\exists i_1\le N,\ \exists i_2\le N$: $Y_{i_1}>0, Y_{i_2}>0$, $X_{0i_1}<\overline{X_0}<X_{0i_2}$	$\forall i\le N$: $(X_{0i}-\overline{X_0})Y_i=0$; $\exists i\le N$: $Y_j>0$

Here

$$\overline{X_0}=\frac{1}{N}\sum_{i=1}^N X_{0i}$$

is the average of observed values of the regressor.

If conditions written in the above table do not hold, then equation (15) *does not have solutions.*

Proof of Lemma 3.1. a) The statement of Lemma 3.1 follows from Lemma 2.1 if $\alpha = 1$.

b) If x_0, y, and α, $0 < \alpha \le 1$, are fixed, then

$$\inf_{b\in\mathbb{R}^2} q_{\text{nai-}\alpha}(x_0, y; b) > -\infty;$$

$$q_{\text{nai-}\alpha}\left(x_0, y; b^{(n)}\right) \to +\infty \Leftrightarrow \max\left(b_0^{(n)} + b_1^{(n)} x_0, -y\left(b_0^{(n)} + b_1^{(n)} x_0\right)\right) \to +\infty.$$

The function $Q_{\text{nai-}\alpha}(b)$ is convex and analytical. Thus the set of points of minimum is convex.

Moreover, the set of points of minimum is a singleton if the function is bounded. This implies that the common for all α, $0 < \alpha \le 1$, a necessary and sufficient condition for the existence and uniqueness of a solution is written as follows:

$$\lim_{\|b\|\to\infty} \max_{i=1,\dots,N} \max(b_0 + b_1 X_{0i}, -Y_i\left(b_0 + b_1 X_{0i}\right)) = +\infty. \tag{16}$$

Equation (15) has a unique solution if and only if the function $Q_{\text{nai-}\alpha}(b)$ has a single point of minimum. This is equivalent to

$$\lim_{\|b\|\to\infty} Q_{\text{nai-}\alpha}(b) = +\infty, \qquad \lim_{\|b\|\to\infty} \max_i q_{\text{nai-}\alpha}(X_{0i}, Y_i; b) = +\infty,$$

or to condition (16).

Therefore the condition for the existence and uniqueness of a solution of the system of equations (15) for $0 < \alpha < 1$ is the same as that for $\alpha = 1$. The proof of the cases where (15) has infinitely many solutions or does not have solutions is omitted here.

c) The case of $\alpha = 0$. The system of equations (15) reduces to the following one:

$$\begin{cases} N = \sum_{i=1}^N e^{-b_0 - b_1 X_{0i}} Y_i, \\ \sum_{i=1}^N X_{0i} = \sum_{i=1}^N e^{-b_0 - b_1 X_{0i}} X_{0i} Y_i. \end{cases}$$

The latter system does not have solutions if $Y_1 = Y_2 = \dots = Y_N = 0$. Otherwise we exclude the unknown variable b_0:

$$\overline{X_0} = \frac{\sum_{i=1}^N X_{0i} Y_i e^{-b_1 X_{0i}}}{\sum_{i=1}^N Y_i e^{-b_1 X_{0i}}}.$$

The right hand side of the equation decreases from $\max_{Y_i>0} X_{0i}$ to $\min_{Y_i>0} X_{0i}$ (it is either a constant or strictly monotone function with respect to b_1). A solution exists and is unique if and only if $\min_{Y_i>0} X_{0i} < \overline{X_0} < \max_{Y_i>0} X_{0i}$. The equation has infinitely many solutions if $\min_{Y_i>0} X_{0i} = \overline{X_0} = \max_{Y_i>0} X_{0i}$. In the rest of the cases, there are no solutions for this equation. □

Let

$$s_{\text{nai-}\alpha}(x_0, y; b) = \frac{\partial q_{\text{nai-}\alpha}(x_0, y; b)}{\partial b} = \frac{e^{b_0 + b_1 x_0} - y}{\alpha + (1-\alpha) e^{b_0 + b_1 x_0}} \begin{pmatrix} 1 \\ x_0 \end{pmatrix}$$

be the elementary estimating function assuming values in $\mathbb{R}^2$. Using this notation, the system of equations for the estimator (15) is rewritten as follows:

$$\sum_{i=1}^N s_{\text{nai-}\alpha}(X_{0i}, Y_i; b) = 0.$$

Lemma 3.2. *Let $\varepsilon > 0$ and let the distribution of X_0 be such that*

$$\mathsf{E}\left[e^{(\beta_1+\varepsilon) X_0} + e^{(\beta_1-\varepsilon) X_0}\right] < \infty. \tag{17}$$

Let Θ be a compact set in $\mathbb{R}^2$ belonging to the strip $\{(b_0, b_1)\colon |b_1 - \beta_1| < \varepsilon\}$. Then the random functions $(b, \alpha) \mapsto q_{nai\text{-}\alpha}(X_0, Y; b)$ and $(b, \alpha) \mapsto s_{nai\text{-}\alpha}(X_0, Y; b)$ are bounded in the set $\Theta \times [0, 1]$ by a random variable with finite expectation, that is,

$$\mathsf{E} \max_{b\in\Theta} \max_{\alpha\in[0,1]} |q_{nai\text{-}\alpha}(X_0, Y; b)| < \infty,$$
$$\mathsf{E} \max_{b\in\Theta} \max_{\alpha\in[0,1]} \|s_{nai\text{-}\alpha}(X_0, Y; b)\| < \infty.$$

If condition (17) *is changed by a stronger assumption* (14)*, then the derivative $(b, \alpha) \mapsto \frac{\partial}{\partial\alpha} q_{nai\text{-}\alpha}(X_0, Y; b)$ is bounded from above in the set $\Theta \times (0, 1)$ by an integrable random variable, that is,*

$$\mathsf{E} \sup_{b\in\Theta} \sup_{\alpha\in(0,1)} \left| \frac{\partial q_{nai\text{-}\alpha}(X_0, Y; b)}{\partial\alpha} \right| < \infty.$$

Here (X_0, Y) is an observation in the Berkson model of the Poisson regression and $\beta = (\beta_0, \beta_1)$ is the true value of the regression parameter.

Proof. We start with a bound proved in [6] (see the proof of Lemma 2.3 therein on pp. 63–65): for all $b \in \mathbb{R}^2$ and for all $\alpha \in [0, 1]$,

$$(1 - y)(b_0 + b_1 X_0) \le q_{\text{nai-}\alpha}(X_0, Y; b) \le e^{b_0+b_1X_0} - 1 + \left(e^{-b_0-b_1X_0} - 1\right) Y,$$
$$\left|s^{(0)}_{\text{nai-}\alpha}(X_0, Y; b)\right| \le \max\left\{\left|e^{b_0+b_1X_0} - Y\right|; \left|1 - Ye^{-b_0-b_1X_0}\right|\right\},$$
$$\left|s^{(1)}_{\text{nai-}\alpha}(X_0, Y; b)\right| \le |X_0| \max\left\{\left|e^{b_0+b_1X_0} - Y\right|; \left|1 - Ye^{-b_0-b_1X_0}\right|\right\}.$$

Moreover,

$$\frac{\partial q_{\text{nai-}\alpha}(X_0, Y; b)}{\partial\alpha} \ge y \min\left\{1 - e^{b_0+b_1X_0} + b_0 + b_1X_0; -\frac{(e^{-b_0-b_1X_0} - 1)^2}{2}\right\},$$
$$\frac{\partial q_{\text{nai-}\alpha}(X_0, Y; b)}{\partial\alpha} \le \max\left\{b_0 + b_1X_0 - 1 + e^{-b_0-b_1X_0};\ \frac{(e^{b_0+b_1X_0} - 1)^2}{2}\right\}$$

for all $b \in \mathbb{R}^2$ and for all $\alpha \in (0, 1)$. Here

$$s^{(j)}_{\text{nai-}\alpha}(X_0, Y; b) = \frac{\partial}{\partial b^{(j)}} q_{\text{nai-}\alpha}(X_0, Y; b)$$

is the jth coordinate of the vector $s_{\text{nai-}\alpha}(X_0, Y; b)$, j=0, 1.

By condition (17), the bounds for $q_{\text{nai-}\alpha}(X_0, Y; b)$ and $s^{(j)}_{\text{nai-}\alpha}(X_0, Y; b)$ are bounded functions of the argument b in the set Θ, and the extrema of the functions have finite expectations. If condition (14) holds, then the bounds for $\frac{\partial}{\partial\alpha} q_{\text{nai-}\alpha}(X_0, Y; b)$ are also bounded in the set Θ by a random variable with finite expectation. □

Theorem 3.3. *Let $\alpha \in [0, 1]$ be a fixed number. If the distribution of X_0 is not concentrated at a single point and if condition* (17) *holds for some $\varepsilon > 0$, then the system of equations* (15) *eventually has a unique solution in $\mathbb{R}^2$; moreover, this solution approaches $(\beta_0 + \frac{1}{2}\beta_1^2\tau^2, \beta_1)$ almost surely.*

Proof. Put $\beta^* = (\beta_0 + \frac{1}{2}\beta_1^2\tau^2, \beta_1)$ and

$$\Theta = \left[\beta_0 + \tfrac{1}{2}\beta_1^2\tau^2 - \tfrac{1}{2}\varepsilon, \beta_0 + \tfrac{1}{2}\beta_1^2\tau^2 + \tfrac{1}{2}\varepsilon\right] \times \left[\beta_1 - \tfrac{1}{2}\varepsilon, \beta_1 + \tfrac{1}{2}\varepsilon\right].$$

By Lemma 3.2, the expectation $\mathsf{E}\,|q_{\text{nai-}\alpha}(X_0, Y; b)|$ exists for all $b \in \Theta$. This implies the strong law of large numbers,

$$\forall b \in \Theta\colon \quad \mathsf{P}\left(\lim_{N\to\infty} \frac{1}{N} Q_{\text{nai-}\alpha}(b) = \mathsf{E}\, q_{\text{nai-}\alpha}(X_0, Y; b)\right) = 1.$$

Since the functions $Q_{\text{nai-}\alpha}(b)$ are convex, the convergence is uniform, that is,

$$\mathsf{P}\left(\lim_{N\to\infty}\sup_{b\in\Theta}\left|\frac{1}{N}Q_{\text{nai-}\alpha}(b)-\mathsf{E}\,q_{\text{nai-}\alpha}(X_0,Y;b)\right|=0\right)=1. \tag{18}$$

Our current aim is to show that the limit function attains its minimum in the set Θ at the point $(\beta_0+\frac{1}{2}\beta_1^2\tau^2,\beta_1)$. Indeed,

$$\mathsf{E}\,q_{\text{nai-}\alpha}(X_0,Y;b)=\min_{\Theta}\mathsf{E}\,q_{\text{nai-}\alpha}(X_0,Y;\cdot)\Leftrightarrow b=\beta^* \tag{19}$$

for all $b\in\Theta$.

Now we prove equivalence (19) in the general case of $0<\alpha<1$. It can be checked that the function

$$g\mapsto\frac{\ln(\alpha+(1-\alpha)e^g)}{1-\alpha}+\frac{e^{g_0}}{\alpha}\ln(\alpha e^{-g}+1-\alpha)$$

attains its minimum at the point $g=g_0$. Thus all the points of minimum of the function $\mathsf{E}[q_{\text{nai-}\alpha}(X_0,Y;b)\mid X_0]$ with respect to the argument b constitute the line

$$\left\{(b_0,b_1)\colon b_0+b_1X_0=\beta_0+\beta_1X_0+\tfrac{1}{2}\beta_1^2\tau^2\right\},$$

where

$$\begin{aligned}&\mathsf{E}[q_{\text{nai-}\alpha}(X_0,Y;b)\mid X_0]\\&\quad=\frac{\ln\left(\alpha+(1-\alpha)e^{b_0+b_1X_0}\right)}{1-\alpha}+\frac{\exp\left\{\beta_0+\beta_1X_0+\frac{1}{2}\beta_1^2\tau^2\right\}}{\alpha}\ln\left(e^{-b_0-b_1X_0}+1-\alpha\right).\end{aligned}$$

Thus the function

$$\mathsf{E}\,q_{\text{nai-}\alpha}(X_0,Y;b)$$

attains its minimum at the point $\beta^*=(\beta_0+\frac{1}{2}\beta_1^2\tau^2,\beta_1)$. Since the distribution of X_0 is not concentrated at a single point, the set of points of minimum

$$\left\{(b_0,b_1)\colon\ \mathsf{P}\left(b_0+b_1X_0=\beta_0+\beta_1X_0+\tfrac{1}{2}\beta_1^2\tau^2\right)=1\right\}$$

is not a singleton. Property (19) for $0<\alpha<1$ is proved.

The uniqueness of a point of minimum of (19) for the case of $\alpha=1$ is already shown in the proof of Theorem 3.1. The proof of equality (19) for $\alpha=0$ is similar to that for $\alpha=1$.

Since the minimum of the limit function (19) is unique and $Q_{\text{nai-}\alpha}(b)$ is convex in the whole plane $\mathbb{R}^2$, we obtain in view of the convergence in (18) that $\min_{b\in\Theta}Q_{\text{nai-}\alpha}(b)=\min_{b\in\mathbb{R}^2}Q_{\text{nai-}\alpha}(b)$ eventually and that

$$\max_{Q_{\text{nai-}\alpha}(b)=\min_{\Theta}Q_{\text{nai-}\alpha}(\cdot)}\|b-\beta^*\|\to 0 \tag{20}$$

almost surely. The uniqueness of the point of minimum follows from convergence (20), since the function $Q_{\text{nai-}\alpha}(b)$ is convex and analytical. Thus the function $Q_{\text{nai-}\alpha}(b)$ eventually has a unique point of minimum, and moreover this point approaches β^* and is a unique solution of the system of equations (15). □

Below is a corollary of Lemma 3.1 and Theorem 3.3.

Corollary. *If the distribution of X_0 is not concentrated at a single point and condition* (17) *holds for some $\varepsilon>0$, then the system of equations* (15) *eventually has a unique solution for all $\alpha\in[0,1]$.*

If the system of equations (15) has a unique solution for some parameter $\alpha\in[0,1]$, then we denote this solution by $\hat\beta_{\text{nai-}\alpha}$.

A fixed sample is considered in the following result (in particular, the size of the sample is constant).

Lemma 3.3. *The estimator $\hat{\beta}_{nai\text{-}\alpha}$ is continuous as a function of α at those points $\alpha \in [0,1]$ (if they exist) where the system of equations* (15) *has a unique solution.*

Proof. The method of the proof is similar to that of Theorem 3.3. Let α_0 be such that the system of equations (15) for $\alpha = \alpha_0$ has a unique solution. Denote this solution by $\hat{\beta}_{\text{nai-}\alpha_0}$. Put

$$\Theta = \left[\hat{\beta}_{0,\text{nai-}\alpha_0} - 1, \hat{\beta}_{0,\text{nai-}\alpha_0} + 1\right] \times \left[\hat{\beta}_{1,\text{nai-}\alpha_0} - 1, \hat{\beta}_{1,\text{nai-}\alpha_0} + 1\right].$$

The function $Q_{\text{nai-}\alpha}(b)$ is convex and analytical with respect to b and continuous with respect to all arguments (b, α). Thus

$$\forall b \in \mathbb{R}^2: \qquad \lim_{\alpha \to \alpha_0} Q_{\text{nai-}\alpha}(b) = Q_{\text{nai-}\alpha_0}(b).$$

The limit function $Q_{\text{nai-}\alpha_0}(b)$ attains its minimum at exactly one point, $\hat{\beta}_{\text{nai-}\alpha_0}$. Using a proof similar to that of Theorem 3.3, we establish that the function $Q_{\text{nai-}\alpha}(b)$ has a unique point of minimum (which also is a unique solution of the system of equations (15) denoted by $\hat{\beta}_{\text{nai-}\alpha}$) for all α belonging to some neighborhood $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \cap [0,1]$ of α_0. Moreover, this solution approaches $\hat{\beta}_{\text{nai-}\alpha_0}$:

$$\lim_{\alpha \to \alpha_0} \hat{\beta}_{\text{nai-}\alpha} = \hat{\beta}_{\text{nai-}\alpha_0}. \qquad \square$$

Corollary. *If the distribution of X_0 is not concentrated at a single point and condition* (17) *holds for some $\varepsilon > 0$, then the estimator $\hat{\beta}_{nai\text{-}\alpha}$ eventually is continuous in $[0,1]$ as a function of the argument α.*

Lemma 3.4. *If the distribution of X_0 is not concentrated at a single point and condition* (14) *holds for some $\varepsilon > 0$, then the convergence is uniform as the size of the sample increases, that is,*

$$\mathsf{P}\left(\sup_{\alpha \in [0,1]} \left\|\hat{\beta}_{nai\text{-}\alpha} - \beta^*\right\| \to 0\right) = 1,$$

where $\beta^* = (\beta_0 + \frac{1}{2}\beta_1^2\tau^2,\ \beta_1)$.

Proof. Put

$$\Theta = \left[\beta_0 + \tfrac{1}{2}\beta_1^2\tau^2 - \tfrac{1}{2}\varepsilon, \beta_0 + \tfrac{1}{2}\beta_1^2\tau^2 + \tfrac{1}{2}\varepsilon\right] \times \left[\beta_1 - \tfrac{1}{2}\varepsilon, \beta_1 + \tfrac{1}{2}\varepsilon\right].$$

By Lemma 3.2, the function $q_{\text{nai-}\alpha}(X_0, Y; b)$ and its derivatives with respect to b and α are bounded in $\Theta \times [0,1]$ and in $\Theta \times (0,1)$, respectively. This implies that the function $\mathsf{E}\, q_{\text{nai-}\alpha}(X_0, Y; b)$ is continuous in $\Theta \times [0,1]$ and that the convergence is uniform, that is,

$$\mathsf{P}\left(\lim_{N \to \infty} \max_{b \in \Theta} \max_{\alpha \in [0,1]} \left|\frac{1}{N} Q_{\text{nai-}\alpha}(b) - \mathsf{E}\, q_{\text{nai-}\alpha}(X_0, Y; b)\right| = 0\right) = 1. \tag{21}$$

It remains to show that

$$\max_{\alpha \in [0,1]} \left\|\hat{\beta}_{\text{nai-}\alpha} - \beta^*\right\| < \varepsilon_1 \tag{22}$$

eventually for all $\varepsilon_1 > 0$. Choose $0 < \varepsilon_1 \le \frac{1}{2}\varepsilon$.

Since the function $\mathsf{E}\, q_{\text{nai-}\alpha}(X_0, Y; b)$ for a fixed $\alpha \in [0,1]$ attains its minimum at $b = \beta^*$, we conclude that

$$\min_{\alpha \in [0,1]} \min_{\|b - \beta^*\| = \varepsilon_1} \left(\mathsf{E}\, q_{\text{nai-}\alpha}(X_0, Y; b) - \mathsf{E}\, q_{\text{nai-}\alpha}(X_0, Y; \beta^*)\right) > 0.$$

In view of (21), we eventually obtain

$$\min_{\alpha \in [0,1]} \min_{\|b - \beta^*\| = \varepsilon_1} \frac{1}{N}\left(Q_{\text{nai-}\alpha}(b) - Q_{\text{nai-}\alpha}(\beta^*)\right) > 0.$$

By the convexity, this implies that all points of minimum of the functions $Q_{\text{nai-}\alpha}(b)$ eventually belong to the circle $\{b\colon \|b-\beta\| < \varepsilon_1\}$ for all α. The function $Q_{\text{nai-}\alpha}(b)$ is anlytical, whence we conclude that the point of minimum (for all α) is unique. If the point of minimum is unique we denote it by $\hat\beta_{\text{nai-}\alpha}$. This means that (22) holds eventually. □

Proof of Theorem 3.2. *Existence of a solution.* Introducing an auxiliary variable $\alpha \in [0,1]$, we rewrite the system of equations (12) as follows:

$$\begin{cases} \sum_{i=1}^N s_{\text{nai-}\alpha}(b_0 + \frac{1}{2}b_1^2\tau^2, b_1) = 0, \\ \alpha = \exp\{-b_1^2\tau^2\}. \end{cases}$$

If α is fixed, then the first equation of the latter system has a unique solution b, and the system is equivalent to the following one:

$$\begin{cases} \left(b_0 + \frac{1}{2}b_1^2\tau^2, b_1\right) = \hat\beta_{\text{nai-}\alpha}, \\ \alpha - \exp\{-b_1^2\tau^2\} = 0. \end{cases} \tag{23}$$

The equation $\alpha - \exp\{-\hat\beta_{1,\text{nai-}\alpha}^2\tau^2\} = 0$ has a solution in $[0,1]$, since the left hand side is continuous and assumes values of different signs at the points $\alpha = 0$ and $\alpha = 1$. Thus system (12) has solutions.

Consistency. The first equation of system (23) implies that

$$\left(\hat\beta_{0,\text{QL}} + \tfrac{1}{2}\hat\beta_{1,\text{QL}}^2\tau^2, \hat\beta_{1,\text{QL}}\right) = \hat\beta_{\text{nai-}\alpha}$$

eventually for a random α. Lemma 3.4 implies that $\hat\beta_{\text{QL}} \to \beta$ almost surely. □

Theorem 3.4. *Let assumptions of Theorem* 3.2 *hold. If* $\beta_1 \neq 0$ *or*

$$\exists \varepsilon > 0\colon \quad \mathsf{E}\left[e^{(3\beta_1 - 3\varepsilon)X_0} + e^{(3\beta_1+3\varepsilon)X_0}\right] < \infty, \tag{24}$$

then the system of equations (12) *for the quasi-likelihood estimator eventually has a unique solution in* $\mathbb{R}^2$.

Proof. As in the proof of Theorem 3.3, put $\beta^* = (\beta_0 + \frac{1}{2}\beta_1^2\tau^2, \beta_1)$ and let

$$\alpha_0 = \exp\left(-\beta_1^2\tau^2\right).$$

Using the equality

$$\sum_{i=1}^n s_{\text{nai-}\alpha}\left(X_{0i}, Y_i; \hat\beta_{\text{nai-}\alpha}\right) = 0$$

we evaluate the derivative

$$\begin{aligned} \frac{d\beta_{\text{nai-}\alpha}}{d\alpha} &= -\left(\sum_{i=1}^N \frac{\partial s_{\text{nai-}\alpha}}{\partial b}\right)^{-1}\left(\sum_{i=1}^N \frac{\partial s_{\text{nai-}\alpha}}{\partial \alpha}\right) \\ &= -\left(\sum_{i=1}^N \frac{e^{\hat\beta_0+\hat\beta_1 X_{0i}}(\alpha + (1-\alpha)Y_i)}{\left(\alpha + (1-\alpha)e^{\hat\beta_0+\hat\beta_1 X_{0i}}\right)^2}\begin{pmatrix} 1 & X_{0i} \\ X_{0i} & X_{0i}^2 \end{pmatrix}\right)^{-1} \\ &\quad \times \left(\sum_{i=1}^N \frac{\left(e^{\hat\beta_0+\hat\beta_1 X_{0i}} - 1\right)\left(e^{\hat\beta_0+\hat\beta_1 X_{0i}} - Y_i\right)}{\left(\alpha + (1-\alpha)e^{\hat\beta_0+\hat\beta_1 X_{0i}}\right)^2}\begin{pmatrix} 1 \\ X_{0i} \end{pmatrix}\right). \end{aligned}$$

The derivatives above are evaluated at the point $b = (\hat\beta_0, \hat\beta_1) = \hat\beta_{\text{nai-}\alpha}$.

Then we establish the strong laws of large numbers:

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\partial s_{\text{nai-}\alpha}(X_{0i},Y_i;b)}{\partial b}\to \mathsf{E}\,\frac{e^{b_0+b_1X_0}(\alpha+(1-\alpha)Y)}{\left(\alpha+(1-\alpha)e^{b_0+b_1X_0}\right)^2}\begin{pmatrix}1 & X_0\\ X_0 & X_0^2\end{pmatrix}=:A(b,\alpha), \tag{25}$$

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\partial s_{\text{nai-}\alpha}(X_{0i},Y_i;b)}{\partial b}\to \mathsf{E}\,\frac{\left(e^{b_0+b_1X_0}-1\right)\left(e^{b_0+b_1X_0}-Y\right)}{\left(\alpha+(1-\alpha)e^{b_0+b_1X_0}\right)^2}=:v(b,\alpha). \tag{26}$$

There exists a convex compact set $\Theta\subset\mathbb{R}^2$ containing β^* as an inner point, and there exists an interval $[\alpha_1,\alpha_2]$ with $0<\alpha_1<\alpha_0<\alpha_2<1$ for $\beta_1\neq 0$ or $0<\alpha_1<\alpha_0=\alpha_2=1$ for $\beta_1=0$ such that the derivatives

$$\frac{\partial s_{\text{nai-}\alpha}}{\partial b},\qquad \frac{\partial s_{\text{nai-}\alpha}}{\partial \alpha},\qquad \frac{\partial^2 s_{\text{nai-}\alpha}}{\partial b^2},\qquad \frac{\partial^2 s_{\text{nai-}\alpha}}{\partial \alpha\partial b},\qquad \frac{\partial^2 s_{\text{nai-}\alpha}}{\partial \alpha^2}$$

are bounded in $\Theta\times[\alpha_1,\alpha_2]$ by a random variable with finite expectation. Moreover, the convergence in (25) and in (26) is uniform in $\Theta\times[\alpha_1,\alpha_2]$. The matrix $A(b,\alpha)$ is nondegenerate and continuous with respect to (b,α) in the set $\Theta\times[\alpha_1,\alpha_2]$. The vector $v(b,\alpha)$ is a continuous function of the argument (b,α) in the set $\Theta\times[\alpha_1,\alpha_2]$. In addition, $v(\beta^*,\alpha)=0$ for $\alpha_1\le\alpha\le\alpha_2$.

Thus

$$\lim_{\substack{N\to\infty\\ b\to\beta^*}}\frac{d\hat\beta_{\text{nai-}\alpha}}{d\alpha}=0$$

almost surely; moreover, the convergence is uniform in the interval $[\alpha_1,\alpha_2]$.

Then we separate equations for α from the system (23):

$$\alpha-\exp\left\{-\hat\beta^2_{1,\text{nai-}\alpha}\tau^2\right\}=0. \tag{27}$$

By Lemma 3.4,

$$\frac{d\left(\alpha-\exp\left\{-\hat\beta^2_{1,\text{nai-}\alpha}\tau^2\right\}\right)}{d\alpha}=1+2\tau^2\hat\beta_{1,\text{nai-}\alpha}e^{-\hat\beta^2_{1,\text{nai-}\alpha}\tau^2}\frac{d\hat\beta_{1,\text{nai-}\alpha}}{d\alpha}\to 1$$

almost surely and uniformly in $[\alpha_1,\alpha_2]$. Therefore equation (27) eventually has at most one solution in the interval $[\alpha_1,\alpha_2]$. Now Lemma 3.4 implies that this equation does not eventually have solutions outside the interval $[\alpha_1,\alpha_2]$.

Since equation (27) has at most one solution, we conclude that the system of equations (12) also has at most one solution. The existence of a solution is already proved in Theorem 3.2. □

4. Concluding remarks

In the structural Berkson model of a simple Poisson regression with a homoscedastic normally distributed error in the regressor, sufficient conditions are found that

1. the equation for the simple score estimator eventually has a unique solution in $\mathbb{R}^2$ and that the estimator is strongly consistent;
2. the equation for the quasi-likelihood estimator eventually has a solution (possibly, not unique) in $\mathbb{R}^2$ and that the estimator (defined as an arbitrary measurable solution of this equation) is strongly consistent;
3. the equation for the quasi-likelihood estimator eventually has a unique solution in $\mathbb{R}^2$.

It would also be of interest to obtain similar conditions in the functional Berkson model.

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