S. V. SHKLYAR

ABSTRACT. The Poisson regression Berkson type model with a Gaussian error in the regressor is studied. Simple score and quasi-likelihood estimators of the regression parameters are considered. Sufficient conditions for the strong consistency of the estimators and sufficient conditions for the uniqueness of a solution of estimating equations are found. The proof of the uniqueness does not require that the parameter set be bounded.

1. INTRODUCTION

We consider the log-linear Poisson regression model. Given an observation $X$, the output $Y$ assumes nonnegative integer values with the conditional probabilities

$$P[Y = k \mid X] = e^{-\lambda} \frac{\lambda^k}{k!},$$

where $\lambda = e^{\beta_0 + \beta_1 X}$ and $\beta = (\beta_0, \beta_1)$ is the regression parameter. Here $X$ is the true value of the regressor. We consider the Berkson model for an error in the regressor $X$, namely

$$X = X_0 + U^\text{Berk}.$$  

We assume that the error is Gaussian, $U^\text{Berk} \sim N(0, \tau^2)$, and that it does not depend on $X_0$. We also assume that the error is indifferent in the sense that

$$P[Y = k \mid X_0, U^\text{Berk}] = P[Y = k \mid X].$$

We study the structural model, where the observed values of the regressor $X_0i$ are random and $(X_0i, X_i, Y_i), i=1,2,\ldots$, is a sequence of independent identically distributed realizations. We construct estimators of the parameter $\beta$ under the assumption that the parameter $\tau^2$ is known.

The Poisson regression model with the classical errors in the regressor is considered in the papers [2, 5]. The quasi-likelihood method (also known as the quasi-score method) is considered in the paper [8]. The quasi-likelihood estimator is constructed in [7] for various classical regression models with normally distributed regressor and error of observation; these results are also useful in the so-called Berkson model.

If the regressor and error have a normal distribution in the model with the classical error, then the model reduces to the Berkson model [1, §2.2.3]. The consistency of the quasi-likelihood estimator in generalized linear models with such errors is proved in [3]. We drop the assumption that the set of parameters is bounded; the consistency of the quasi-score estimator and uniqueness of a solution of the estimating equation under the

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assumption of the boundedness of the parametric set is proved in [6] for the classical model.

In the current paper, we consider the structural Berkson model. We change the assumption that the variable $X_0$ has a normal distribution and use the more general conditions (14) and (24). We also find sufficient conditions that solutions of equations determining the simple score and quasi-likelihood estimators are eventually unique. We also find sufficient conditions for the strong consistency of these estimators.

2. The classical model of the Poisson regression

2.1. Statistical model. Values of the regressor are denoted by $X_i$, while those of the output are denoted by $Y_i$. As a rule, we consider the functional model, where the values of the regressor $X_i$ are nonrandom numbers. The distribution of the output $Y_i$ (the conditional distribution of $Y_i$ given $X_i$ if $X_i$ is random) is Poissonian,

$$Y_i \mid X_i \sim \text{Poi}(e^{\beta_0 + \beta_1 X_i}),$$

$$P[Y_i=k \mid X_i] = \frac{1}{k!} \exp \{ -e^{\beta_0 + \beta_1 X_i} + k(\beta_0 + \beta_1 X_i) \},$$

where $\beta = (\beta_0, \beta_1)$ is the regression parameter.

The observations are assumed to be independent. If $X_i$ are nonrandom (in this case, the model is called functional), then $Y_i$ are independent random variables that have Poisson distributions with, in general, different parameters. If $X_i$ are independent and identically distributed random variables (in this case, the model is called structural), then the pairs $(X_i, Y_i)$ are independent and identically distributed random vectors.

2.2. Maximal likelihood estimator. Given the observations

$$(X_i, Y_i), \quad i = 1, 2, \ldots, N,$$

the estimator of the parameter $\beta$ is defined as a point of minimum of the functional

(1) $$Q_{\text{no error}}(b) = \sum_{i=1}^{N} (e^{b_0 + b_1 X_i} - (b_0 + b_1 X_i)Y_i)$$

or as a solution of the system of equations

(2) $$\begin{cases} \sum_{i=1}^{N} e^{b_0 + b_1 X_i} = \sum_{i=1}^{N} Y_i, \\ \sum_{i=1}^{N} X_i e^{b_0 + b_1 X_i} = \sum_{i=1}^{N} X_i Y_i. \end{cases}$$

Since the function $Q_{\text{no error}}(b)$ is convex, the set of points of the global minimum in $\mathbb{R}^2$ of the function defined by (1) coincides with the set of solutions of the system of equations (2).

Lemma 2.1 (Uniqueness of a solution). A solution of equation (2) is unique if and only if

(3) $$\begin{cases} \exists i \leq N \exists j \leq N: \quad X_i < X_j, \quad Y_i > 0, \\ \exists i \leq N \exists j \leq N: \quad X_i < X_j, \quad Y_j > 0. \end{cases}$$

Moreover, if all $X_i$, $i=1, \ldots, N$, are identical and there is a nonzero output $Y_i$, then equation (2) has infinitely many solutions. The equation has no solutions at all in all other cases.
Proof. If \( Y_1 = Y_2 = \cdots = Y_N = 0 \), then the system of equations (2) does not have solutions. Consider the case where there exists a nonzero output, that is, \( Y_i > 0 \) for some \( i \). Put

\[
\bar{X}_Y = \frac{\sum_{i=1}^{N} X_i Y_i}{\sum_{i=1}^{N} Y_i}.
\]

Exclude the variable \( b_0 \) from the system of equations (2):

\[
\frac{\sum_{i=1}^{N} X_i e^{b_1 X_i}}{\sum_{i=1}^{N} e^{b_1 X_i}} = \bar{X}_Y.
\]

The left hand side of the equation is a nondecreasing function (or a constant, or a strictly monotone function) of the argument \( b_1 \) that approaches

\[
\min_{i=1,\ldots,N} X_i \text{ as } b_1 \to -\infty \quad \text{and} \quad \max_{i=1,\ldots,N} X_i \text{ as } b_1 \to +\infty.
\]

The equation possesses exactly one solution if and only if

\[
\min_{i=1,\ldots,N} X_i < \bar{X}_Y < \max_{i=1,\ldots,N} X_i,
\]

which is equivalent to condition (3). The equation possesses infinitely many solutions if and only if

\[
\min_{i=1,\ldots,N} X_i = \bar{X}_Y = \max_{i=1,\ldots,N} X_i.
\]

In all other cases, the equation does not have solutions. \( \square \)

If at least two observations are such that the corresponding \( Y \) are positive and \( X \) are different, then system (2) possesses exactly one solution.

3. The Poisson errors-in-variables model. The Berkson model

3.1. Statistical model. We consider the Berkson model for the scalar log-linear Poisson regression.

3.1.1. A single observation. Let \( X_0 \) be an observed value of the regressor, \( X \) be the true value of the regressor, and \( Y \) be the value of the output (being a nonnegative integer number).

A feature of the Berkson model is that the true value of the regressor equals the observed value of the regressor with the error

\[
X = X_0 + U^{\text{Berk}}.
\]

We also assume that the error is homoscedastic and normally distributed, that is,

\[
X \mid X_0 \sim N \left( X_0, \tau^2 \right).
\]

The parameter \( \tau^2 \) is assumed to be known.

The distribution of \( Y \) depends on the true value of the regressor,

\[
Y \mid (X, X_0) \sim \text{Poi} \left( e^{\beta_0 + \beta_1 X} \right),
\]

\[
P[Y=k \mid X, X_0] = \frac{1}{k!} \exp \left\{ -e^{\beta_0 + \beta_1 X} + k(\beta_0 + \beta_1 X) \right\},
\]

where \( \beta = (\beta_0, \beta_1) \) is the regression parameter.

The assumption that the distribution of \( Y \mid (X, X_0) \) does not depend on \( X_0 \) explicitly, that is,

\[
Y \mid (X, X_0) \sim Y \mid X,
\]

is called the indifferentiability condition.
3.1.2. Functional and structural model. In all cases under consideration we assume that the realizations \((X_{0i}, X_i, Y_i)\) (as random three dimensional vectors) are independent. Observed are the values \((X_{0i}, Y_i), i = 1, 2, \ldots, N\).

**Functional model.** In this model, \(X_{0i}\) are nonrandom numbers and the pairs \((X_i, Y_i)\) are independent but not necessarily identically distributed.

**Structural model.** For this model, \(X_{0i}\) are independent identically distributed random variables. Then the realizations \((X_{0i}, X_i, Y_i)\) also are independent identically distributed random triples.

3.2. Estimators of the regression parameter.

3.2.1. A naive estimator. When constructing a naive estimator, we ignore the error and substitute \(X_0\) instead of \(X\) in the equation for the maximal likelihood estimator in the case of the model without errors \((2)\). Then the estimator is defined as a point of minimum of the function

\[
Q_{\text{naive}}(b) = \sum_{i=1}^{N} \left( e^{b_0 + b_1 X_{0i}} - (b_0 + b_1 X_{0i})Y_i \right)
\]

or as a solution of the equation

\[
\begin{align*}
\sum_{i=1}^{N} e^{b_0 + b_1 X_{0i}} &= \sum_{i=1}^{N} Y_i, \\
\sum_{i=1}^{N} X_{0i} e^{b_0 + b_1 X_{0i}} &= \sum_{i=1}^{N} X_{0i} Y_i.
\end{align*}
\]

3.2.2. The simple score estimator. We consider the regression of \(Y\) with respect to \(X_0\). The conditional expectation equals

\[
E[Y|X_0] = \exp \left\{ \beta_0 + \beta_1 X_0 + \frac{1}{2} \beta_1^2 \tau^2 \right\},
\]

whence \(E[ (\exp \{ \beta_0 + \beta_1 X_0 + \frac{1}{2} \beta_1^2 \tau^2 \} - Y) \cdot a(X_0) ] = 0\) for Borel functions \(a(\cdot)\) such that the latter expectation exists.

The following system of equations for the estimator of the parameter \(\beta\) is unbiased:

\[
\begin{align*}
\sum_{i=1}^{N} e^{b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2} &= \sum_{i=1}^{N} Y_i, \\
\sum_{i=1}^{N} X_{0i} e^{b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2} &= \sum_{i=1}^{N} X_{0i} Y_i.
\end{align*}
\]

The property of a system of equations to be unbiased means that both sides of the equations have the same expectation if the true value of the parameter \(\beta\) is used instead of \(b\). Note that the system of equations \((7)\) in the structural model is unbiased if

\[
E[ X_0 | e^{\beta_1 X_0} ] < \infty.
\]

The estimator is defined as a point of minimum of the function

\[
Q_{SS}(b) = \sum_{i=1}^{N} \left( e^{b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2} - (b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2) Y_i \right),
\]

\[
Q_{SS}(b_0, b_1) = Q_{\text{naive}}(b_0 + \frac{1}{2} b_1^2 \tau^2, b_1).
\]

Relation \((9)\) between target functions has its counterpart describing a relationship between the naive estimator and simple score estimator written as follows:

\[
\hat{\beta}_{1, SS} = \hat{\beta}_{1, \text{naive}},
\]

\[
\hat{\beta}_{0, SS} = \hat{\beta}_{0, \text{naive}} - \frac{1}{2} \hat{\beta}_{1, \text{naive}}^2 \tau^2.
\]
3.2.3. *The quasi-likelihood estimator.* This method of estimation is also called the quasi-score method or weighted least squares method. The system of equations for the estimator is given by

\[
\sum_{i=1}^{N} \frac{E_b[Y_i \mid X_{0i}] - Y_i}{\text{Var}_b[Y_i \mid X_{0i}]} \cdot \frac{\partial E_b[Y_i \mid X_{0i}]}{\partial b} = 0
\]

\((b)\) is the unknown parameter. Since

\[
\text{Var}_b[Y \mid X_0] = e^{b_0 + b_1 X_0 + \frac{1}{2} b_1^2 \tau^2} + e^{2b_0 + 2b_1 X_0 + 2b_1^2 \tau^2} - e^{2b_0 + 2b_1 X_0 + b_1^2 \tau^2}
\]

(also see equality (6), the system of equations for the estimator is written as follows:

\[
\begin{align*}
\sum_{i=1}^{N} \frac{\exp\{b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2\} - Y_i}{1 + \exp\{b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2\}} &= 0, \\
\sum_{i=1}^{N} \frac{\exp\{b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2\} - Y_i}{1 + \exp\{b_0 + b_1 X_{0i} + \frac{1}{2} b_1^2 \tau^2\}} &= 0.
\end{align*}
\]

Note that the factor \((X_{0i} + b_1 \tau^2)\) in the second equation can be changed by \(X_{0i}\), and this results in an equivalent system of equations.

3.2.4. *The maximal likelihood estimator.* The conditional probability equals

\[
P[Y = k \mid X_0] = \frac{1}{k!} f(k, \beta_0 + \beta_1 X_0, \beta_1^2 \tau^2),
\]

where \(f(k, \mu, \sigma^2) = \exp(-e^\zeta + k \zeta, \zeta \sim N(\mu, \sigma^2)).\)

The estimator is defined as a point of maximum of the functional

\[
\sum_{i=1}^{N} \ln f(Y_i, b_0 + b_1 X_{0i}, b_1^2 \tau^2).
\]

An expression for the solution of this problem involves special functions.

3.3. *Uniqueness of a solution and consistency of the simple score estimator in the structural model.* We say that a random event \(A_N\) depending on an index \(N\) *eventually* holds if the probability of

\[
\lim_{N \to \infty} A_N = \{ \exists N_0 \in \mathbb{N}: \text{the random event } A_N \text{ occurs for all } N > N_0 \}
\]

equals one.

**Theorem 3.1.** If the distribution of \(X_0\) is not concentrated at a single point and is such that

\[
\exists \varepsilon > 0: \mathbb{E} \left[ e^{(\beta_1 + \varepsilon) X_0} + e^{(\beta_1 - \varepsilon) X_0} \right] < \infty,
\]

then the system of equations (7) for the simple score estimator eventually has a unique solution in \(\mathbb{R}^2\). This solution is a strongly consistent estimator of the parameter \(\beta\).

**Proof.** The limit value of the normalized target function for the naive estimator is denoted by

\[
q^*_{\text{naive}}(b) = \frac{1}{N} \mathbb{E} Q(b) = \mathbb{E} \left[ e^{b_0 + b_1 X_0} - (b_0 + b_1 X_0) Y \right].
\]

If

\[
|b_1 - \beta_1| \leq \varepsilon,
\]

then the limit value of the normalized target function for the naive estimator is denoted by

\[
q^*_{\text{naive}}(b) = \frac{1}{N} \mathbb{E} Q(b) = \mathbb{E} \left[ e^{b_0 + b_1 X_0} - (b_0 + b_1 X_0) Y \right].
\]
then the expectation exists. If $b$ is such that (13) holds, then
\[
\frac{1}{N} \mathbb{E}[Q(b) \mid X, X_0] = e^{b_0 + b_1 X_0} - (b_0 + b_1 X_0)e^\beta_0 + \beta_1 X,
\]
\[
\frac{1}{N} \mathbb{E}[Q(b) \mid X_0] = e^{b_0 + b_1 X_0} - (b_0 + b_1 X_0)e^\beta_0 + \beta_1 X + \frac{1}{2} \beta_1^2 \tau^2.
\]
Thus the function $q_{\text{naive}}^*(b) = \mathbb{E} \left[ e^{b_0 + b_1 X_0} - (b_0 + b_1 X_0)e^\beta_0 + \beta_1 X + \frac{1}{2} \beta_1^2 \tau^2 \right]$.

The function $q_{\text{naive}}^*(b)$ is convex with respect to the vector variable $b$ (in fact, $q_{\text{naive}}^*(b)$ is strictly convex, since the distribution of $X_0$ is not concentrated at a single point). Since
\[
e^{b_0 + b_1 X_0} - (b_0 + b_1 X_0)e^\beta_0 + \beta_1 X_0 + \frac{1}{2} \beta_1^2 \tau^2
\]
\[
= e^{b_0 + \beta_1 X_0} \left( e^{b_0 - \beta_0 - \frac{1}{2} \beta_1^2 \tau^2 + \left( b_1 - \beta_1 \right) X_0} - b_0 - b_1 X_0 \right)
\]
\[
\geq e^{\beta_0 + \beta_1 X_0} \left( 1 - \beta_0 - \frac{1}{2} \beta_1^2 \tau^2 - \beta_1 X_0 \right),
\]
we get
\[
q_{\text{naive}}^*(b) \geq \mathbb{E} \left[ e^{\beta_0 + \beta_1 X_0} \left( 1 - \beta_0 - \frac{1}{2} \beta_1^2 \tau^2 - \beta_1 X_0 \right) \right].
\]
The equality is attained if and only if
\[
P \left[ b_0 - \beta_0 - \frac{1}{2} \beta_1^2 \tau^2 + \left( b_1 - \beta_1 \right) X_0 = 0 \right] = 1.
\]
Thus the function $q_{\text{naive}}^*(b)$ attains its minimum at the point
\[
(b_0, b_1) = \left( \beta_0 + \frac{1}{2} \beta_1^2 \beta_1^2, \beta_1 \right).
\]
By the strong law of large numbers,
\[
P \left( \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} Q_{\text{naive}}(b) = q_{\text{naive}}^*(b) \right) = 1
\]
for all $b \in [\beta_0 - \varepsilon, \beta_0 + \varepsilon] \times [\beta_1 - \varepsilon, \beta_1 + \varepsilon]$, whence we conclude that
\[
\lim_{N \to \infty} \sup_{|b_0 - \beta_0| \leq \varepsilon, |b_1 - \beta_1| \leq \varepsilon} \left| \frac{1}{N} Q_{\text{naive}}(b) - q_{\text{naive}}^*(b) \right| = 0
\]
almost surely by Theorem 10.8 in [4]. Hence the function $Q_{\text{naive}}(b)$ eventually attains the minimum and, for any point of minimum denoted by $\hat{\beta}_{\text{naive}}$, the almost sure convergence holds:
\[
\lim_{N \to \infty} \hat{\beta}_{0,\text{naive}} = \beta_0 + \frac{1}{2} \beta_1^2 \tau^2,
\]
\[
\lim_{N \to \infty} \hat{\beta}_{1,\text{naive}} = \beta_1.
\]
A solution is unique, since the function $Q_{\text{naive}}(b)$ is analytic and convex. According to relations (10)–(11), this implies that
\[
\lim_{N \to \infty} \hat{\beta}_{SS} = \beta
\]
almost surely. □
3.4. Uniqueness of a solution and consistency of the quasi-likelihood estimator in the structural model.

**Theorem 3.2.** If the distribution of $X_0$ is not concentrated at a single point and is such that
\[
E \left[ e^{-\beta_1 - 2\varepsilon}X_0 + e^{-\beta_1 + 2\varepsilon}X_0 + e^{(2\beta_1 - 2\varepsilon)}X_0 + e^{(2\beta_1 + 2\varepsilon)}X_0 \right] < \infty,
\]
then the system of equations (12) for the quasi-likelihood estimator eventually has a solution in $\mathbb{R}^2$. The estimator defined as a solution of the system of equations (12) and denoted by $\hat{\beta}_{QL}$ is a strongly consistent estimator of the parameter $\beta$, that is,
\[
\lim_{N \to \infty} \hat{\beta}_{QL} = \beta
\]
almost surely.

To prove Theorem 3.2 we consider the family of estimators depending on a parameter $\alpha \in [0, 1]$ and defined as solutions of the following system of equations:
\[
\begin{cases}
\sum_{i=1}^{N} \frac{\exp\{b_0 + b_1 X_{0i}\} - Y_i}{\alpha + (1 - \alpha) \exp\{b_0 + b_1 X_{0i}\}} = 0, \\
\sum_{i=1}^{N} \frac{\exp\{b_0 + b_1 X_{0i}\} - Y_i}{\alpha + (1 - \alpha) \exp\{b_0 + b_1 X_{0i}\}} = 0.
\end{cases}
\]

These estimators can also be defined as the points of minimum of the functional
\[
Q_{nai-\alpha}(b) = \sum_{i=1}^{N} q_{nai-\alpha}(X_{0i}, Y_i; b),
\]
where
\[
q_{nai-\alpha}(x_0, y; b) = \frac{1}{1 - \alpha} \ln (\alpha + (1 - \alpha)e^{b_0 + b_1 x_0}) + \frac{y}{\alpha} \ln (\alpha e^{-b_0 - b_1 x_0} + 1 - \alpha), \quad 0 < \alpha < 1,
\]
\[
q_{nai-\alpha}(x_0, y; b) = b_0 + b_1 x_0 + ye^{-b_0 - b_1 x_0} - y, \quad \alpha = 0,
\]
\[
q_{nai-\alpha}(x_0, y; b) = e^{b_0 + b_1 x_0} - 1 - (b_0 + b_1 x_0)y, \quad \alpha = 1.
\]

Since the functional $Q_{nai-\alpha}(b)$ is convex for all $\alpha \in [0, 1]$, the set of its solutions coincides with the set of solutions of the extremum equation (15).

First we study the asymptotic properties of the estimators for a fixed $\alpha$.

**Lemma 3.1.** Necessary and sufficient conditions for the existence and uniqueness of a solution of equation (15) are given in the following table:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Condition that equation (15) has a unique solution</th>
<th>Condition that equation (15) has infinitely many solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; \alpha \leq 1$</td>
<td>$\exists i \leq N, \exists j \leq N, \exists k \leq N, \exists l \leq N: \ X_{0i} &lt; X_{0j}, X_{0k} &lt; X_{0l}, Y_i &gt; 0, Y_j &gt; 0$</td>
<td>$\forall i \leq N: \ X_{0i} = X_{01}$; $\exists j \leq N: Y_j &gt; 0$</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$\exists i_1 \leq N, \exists i_2 \leq N: Y_{i_1} &gt; 0, Y_{i_2} &gt; 0$, $X_{0i_1} &lt; X_{0} &lt; X_{0i_2}$</td>
<td>$\forall i \leq N: (X_{0i} - \bar{X}_0)Y_i = 0$; $\exists i \leq N: Y_j &gt; 0$</td>
</tr>
</tbody>
</table>

Here
\[
\bar{X}_0 = \frac{1}{N} \sum_{i=1}^{N} X_{0i}
\]
is the average of observed values of the regressor.

If conditions written in the above table do not hold, then equation (15) does not have solutions.
Proof of Lemma 3.1. a) The statement of Lemma 3.1 follows from Lemma 2.1 if \( \alpha = 1 \).

b) If \( x_0, y, \) and \( \alpha, 0 < \alpha \leq 1, \) are fixed, then

\[
\inf_{b \in \mathbb{R}^2} q_{\text{nai-}\alpha}(x_0, y; b) > -\infty;
\]

\[
q_{\text{nai-}\alpha}(x_0, y; b^{(n)}) \to +\infty \iff \max \left( b_0^{(n)} + b_1^{(n)} x_0, -y \left( b_0^{(n)} + b_1^{(n)} x_0 \right) \right) \to +\infty.
\]

The function \( Q_{\text{nai-}\alpha}(b) \) is convex and analytical. Thus the set of points of minimum is convex.

Moreover, the set of points of minimum is a singleton if the function is bounded. This implies that the common for all \( \alpha, 0 < \alpha \leq 1, \) a necessary and sufficient condition for the existence and uniqueness of a solution is written as follows:

\[
\lim_{\|b\| \to \infty} \max_{i=1,\ldots,N} (b_0 + b_1 X_{0i}, -Y_i (b_0 + b_1 X_{0i})) = +\infty.
\]

Equation (15) has a unique solution if and only if the function \( Q_{\text{nai-}\alpha}(b) \) has a single point of minimum. This is equivalent to

\[
\lim_{\|b\| \to \infty} Q_{\text{nai-}\alpha}(b) = +\infty,
\]

or to condition (16).

Therefore, the condition for the existence and uniqueness of a solution of the system of equations (15) for \( 0 < \alpha < 1 \) is the same as that for \( \alpha = 1 \). The proof of the cases where (15) has infinitely many solutions or does not have solutions is omitted here.

c) The case of \( \alpha = 0 \). The system of equations (15) reduces to the following one:

\[
\begin{aligned}
N &= \sum_{i=1}^N e^{-b_0 - b_1 X_{0i}} Y_i, \\
\sum_{i=1}^N X_{0i} &= \sum_{i=1}^N e^{-b_0 - b_1 X_{0i}} X_{0i} Y_i.
\end{aligned}
\]

The latter system does not have solutions if \( Y_1 = Y_2 = \cdots = Y_N = 0 \). Otherwise we exclude the unknown variable \( b_0 \):

\[
X_0 = \sum_{i=1}^N X_{0i} Y_i e^{-b_1 X_{0i}} / \sum_{i=1}^N Y_i e^{-b_1 X_{0i}}.
\]

The right hand side of the equation decreases from \( \max_{Y_i>0} X_{0i} \) to \( \min_{Y_i>0} X_{0i} \) (it is either a constant or strictly monotone function with respect to \( b_1 \)). A solution exists and is unique if and only if \( \min_{Y_i>0} X_{0i} < X_0 < \max_{Y_i>0} X_{0i} \). The equation has infinitely many solutions if \( \min_{Y_i>0} X_{0i} = X_0 = \max_{Y_i>0} X_{0i} \). In the rest of the cases, there are no solutions for this equation.

Let

\[
s_{\text{nai-}\alpha}(x_0, y; b) = \frac{\partial q_{\text{nai-}\alpha}(x_0, y; b)}{\partial b} = \frac{e^{b_0 + b_1 x_0} - y}{\alpha + (1 - \alpha)e^{b_0 + b_1 x_0}} \left( \frac{1}{x_0} \right)
\]

be the elementary estimating function assuming values in \( \mathbb{R}^2 \). Using this notation, the system of equations for the estimator (15) is rewritten as follows:

\[
\sum_{i=1}^N s_{\text{nai-}\alpha}(X_{0i}, Y_i; b) = 0.
\]

Lemma 3.2. Let \( \varepsilon > 0 \) and let the distribution of \( X_0 \) be such that

\[
\mathbb{E} \left[ e^{(\beta_1 + \varepsilon) X_0} + e^{(\beta_1 - \varepsilon) X_0} \right] < \infty.
\]
Let $\Theta$ be a compact set in $\mathbb{R}^2$ belonging to the strip $\{(b_0, b_1): |b_1 - \beta_1| < \varepsilon\}$. Then the random functions $(b, \alpha) \mapsto q_{\text{nai-}\alpha}(X_0, Y; b)$ and $(b, \alpha) \mapsto s_{\text{nai-}\alpha}(X_0, Y; b)$ are bounded in the set $\Theta \times [0, 1]$ by a random variable with finite expectation, that is,

$$
\mathbb{E} \max_{b \in \Theta} \max_{\alpha \in [0, 1]} |q_{\text{nai-}\alpha}(X_0, Y; b)| < \infty,
$$

$$
\mathbb{E} \max_{b \in \Theta} \max_{\alpha \in [0, 1]} \|s_{\text{nai-}\alpha}(X_0, Y; b)\| < \infty.
$$

If condition (17) is changed by a stronger assumption (14), then the derivative $(b, \alpha) \mapsto \frac{\partial}{\partial b} q_{\text{nai-}\alpha}(X_0, Y; b)$ is bounded from above in the set $\Theta \times (0, 1)$ by an integrable random variable, that is,

$$
\mathbb{E} \sup_{b \in \Theta} \sup_{\alpha \in (0, 1)} \left| \frac{\partial q_{\text{nai-}\alpha}(X_0, Y; b)}{\partial \alpha} \right| < \infty.
$$

Here $(X_0, Y)$ is an observation in the Berkson model of the Poisson regression and $\beta = (\beta_0, \beta_1)$ is the true value of the regression parameter.

Proof. We start with a bound proved in [6] (see the proof of Lemma 2.3 therein on pp. 63–65): for all $b \in \mathbb{R}^2$ and for all $\alpha \in [0, 1],$

$$(1 - y)(b_0 + b_1 X_0) \leq q_{\text{nai-}\alpha}(X_0, Y; b) \leq e^{b_0 + b_1 X_0} - 1 + (e^{-b_0 - b_1 X_0} - 1) Y,$$

$$
\left| s_{\text{nai-}\alpha}^{(0)}(X_0, Y; b) \right| \leq \max \left\{ |e^{b_0 + b_1 X_0} - Y|; |1 - Y e^{-b_0 - b_1 X_0}| \right\},
$$

$$
\left| s_{\text{nai-}\alpha}^{(1)}(X_0, Y; b) \right| \leq |X_0| \max \left\{ |e^{b_0 + b_1 X_0} - Y|; |1 - Y e^{-b_0 - b_1 X_0}| \right\}.
$$

Moreover,

$$
\frac{\partial q_{\text{nai-}\alpha}(X_0, Y; b)}{\partial \alpha} \geq y \min \left\{ 1 - e^{b_0 + b_1 X_0} + b_0 + b_1 X_0; -\frac{(e^{-b_0 - b_1 X_0} - 1)^2}{2} \right\},
$$

$$
\frac{\partial q_{\text{nai-}\alpha}(X_0, Y; b)}{\partial \alpha} \leq \max \left\{ b_0 + b_1 X_0 - 1 + e^{-b_0 - b_1 X_0}; \frac{(e^{b_0 + b_1 X_0} - 1)^2}{2} \right\}
$$

for all $b \in \mathbb{R}^2$ and for all $\alpha \in (0, 1)$. Here

$$
s_{\text{nai-}\alpha}^{(j)}(X_0, Y; b) = \frac{\partial}{\partial b^{(j)}} q_{\text{nai-}\alpha}(X_0, Y; b)
$$

is the $j$th coordinate of the vector $s_{\text{nai-}\alpha}(X_0, Y; b), j = 0, 1$.

By condition (17), the bounds for $q_{\text{nai-}\alpha}(X_0, Y; b)$ and $s_{\text{nai-}\alpha}^{(j)}(X_0, Y; b)$ are bounded functions of the argument $b$ in the set $\Theta,$ and the extrema of the functions have finite expectations. If condition (14) holds, then the bounds for $\frac{\partial}{\partial \alpha} q_{\text{nai-}\alpha}(X_0, Y; b)$ are also bounded in the set $\Theta$ by a random variable with finite expectation. \hfill \Box

**Theorem 3.3.** Let $\alpha \in [0, 1]$ be a fixed number. If the distribution of $X_0$ is not concentrated at a single point and if condition (17) holds for some $\varepsilon > 0$, then the system of equations (15) eventually has a unique solution in $\mathbb{R}^2$; moreover, this solution approaches $(\beta_0 + \frac{1}{2} \beta_1^2 r^2, \beta_1)$ almost surely.

Proof. Put $\beta^* = (\beta_0 + \frac{1}{2} \beta_1^2 r^2, \beta_1)$ and

$$
\Theta = \left[ \beta_0 + \frac{1}{2} \beta_1^2 r^2 - \frac{1}{2} \varepsilon, \beta_0 + \frac{1}{2} \beta_1^2 r^2 + \frac{1}{2} \varepsilon \right] \times \left[ \beta_1 - \frac{1}{2} \varepsilon, \beta_1 + \frac{1}{2} \varepsilon \right].
$$

By Lemma 3.2, the expectation $\mathbb{E} |q_{\text{nai-}\alpha}(X_0, Y; b)|$ exists for all $b \in \Theta$. This implies the strong law of large numbers,

$$
\forall b \in \Theta: \quad P \left( \lim_{N \to \infty} \frac{1}{N} Q_{\text{nai-}\alpha}(b) = \mathbb{E} q_{\text{nai-}\alpha}(X_0, Y; b) \right) = 1.
$$
Since the functions $Q_{\text{nai}-\alpha}(b)$ are convex, the convergence is uniform, that is,
\begin{equation}
\mathbb{P} \left( \lim_{N \to \infty} \sup_{b \in \Theta} \left| \frac{1}{N} Q_{\text{nai}-\alpha}(b) - \mathbb{E} q_{\text{nai}-\alpha}(X_0, Y; b) \right| = 0 \right) = 1.
\end{equation}

Our current aim is to show that the limit function attains its minimum in the set $\Theta$ at the point $(\beta_0 + \frac{1}{2} \beta_1^2 \tau^2, \beta_1)$. Indeed,
\begin{equation}
\mathbb{E} q_{\text{nai}-\alpha}(X_0, Y; b) = \min_{\Theta} \mathbb{E} q_{\text{nai}-\alpha}(X_0, Y; \cdot) \iff b = \beta^*
\end{equation}
for all $b \in \Theta$.

Now we prove equivalence \([19]\) in the general case of $0 < \alpha < 1$. It can be checked that the function
\[ g \mapsto \frac{\ln(\alpha + (1 - \alpha)e^g)}{1 - \alpha} + \frac{e^{g_0}}{\alpha} \ln(\alpha e^{-g} + 1 - \alpha) \]
attains its minimum at the point $g = g_0$. Thus all the points of minimum of the function $\mathbb{E}[q_{\text{nai}-\alpha}(X_0, Y; b) \mid X_0]$ with respect to the argument $b$ constitute the line
\[ \{(b_0, b_1) : b_0 + b_1 X_0 = \beta_0 + \beta_1 X_0 + \frac{1}{2} \beta_1^2 \tau^2 \} \]
where
\[ \mathbb{E}[q_{\text{nai}-\alpha}(X_0, Y; b) \mid X_0] = \ln(\alpha + (1 - \alpha)e^{b_0 + b_1 X_0}) + \frac{\exp\left\{ \beta_0 + \beta_1 X_0 + \frac{1}{2} \beta_1^2 \tau^2 \right\}}{\alpha} \ln(e^{-b_0 - b_1 X_0} + 1 - \alpha). \]
Thus the function
\[ \mathbb{E} q_{\text{nai}-\alpha}(X_0, Y; b) \]
attains its minimum at the point $\beta^* = (\beta_0 + \frac{1}{2} \beta_1^2 \tau^2, \beta_1)$. Since the distribution of $X_0$ is not concentrated at a single point, the set of points of minimum
\[ \{(b_0, b_1) : \mathbb{P} (b_0 + b_1 X_0 = \beta_0 + \beta_1 X_0 + \frac{1}{2} \beta_1^2 \tau^2) = 1 \} \]
is not a singleton. Property \([19]\) for $0 < \alpha < 1$ is proved.

The uniqueness of a point of minimum of \([19]\) for the case of $\alpha = 1$ is already shown in the proof of Theorem 3.1. The proof of equality \([19]\) for $\alpha = 0$ is similar to that for $\alpha = 1$.

Since the minimum of the limit function \([19]\) is unique and $Q_{\text{nai}-\alpha}(b)$ is convex in the whole plane $\mathbb{R}^2$, we obtain in view of the convergence in \((18)\) that $\min_{b \in \mathbb{R}^2} Q_{\text{nai}-\alpha}(b) = \min_{b \in \mathbb{R}^2} Q_{\text{nai}-\alpha}(b)$ eventually and that
\begin{equation}
\max_{Q_{\text{nai}-\alpha}(b) = \min_{\Theta} Q_{\text{nai}-\alpha}(\cdot)} \|b - \beta^*\| \to 0
\end{equation}
almost surely. The uniqueness of the point of minimum follows from convergence \((20)\), since the function $Q_{\text{nai}-\alpha}(b)$ is convex and analytical. Thus the function $Q_{\text{nai}-\alpha}(b)$ eventually has a unique point of minimum, and moreover this point approaches $\beta^*$ and is a unique solution of the system of equations \((15)\).

Below is a corollary of Lemma 3.1 and Theorem 3.3.

**Corollary.** If the distribution of $X_0$ is not concentrated at a single point and condition \((17)\) holds for some $\varepsilon > 0$, then the system of equations \((15)\) eventually has a unique solution for all $\alpha \in [0, 1]$.

If the system of equations \((15)\) has a unique solution for some parameter $\alpha \in [0, 1]$, then we denote this solution by $\beta_{\text{nai}-\alpha}$.

A fixed sample is considered in the following result (in particular, the size of the sample is constant).
Lemma 3.3. The estimator $\hat{\beta}_{\text{nai-}\alpha}$ is continuous as a function of $\alpha$ at those points $\alpha \in [0, 1]$ (if they exist) where the system of equations (15) has a unique solution.

Proof. The method of the proof is similar to that of Theorem 3.3. Let $\alpha_0$ be such that the system of equations (15) for $\alpha = \alpha_0$ has a unique solution. Denote this solution by $\hat{\beta}_{\text{nai-}\alpha_0}$. Put
\[
\Theta = \left[ \hat{\beta}_{\text{nai-}\alpha_0} - 1, \hat{\beta}_{\text{nai-}\alpha_0} + 1 \right] \times \left[ \hat{\beta}_{\text{nai-}\alpha_0} - 1, \hat{\beta}_{\text{nai-}\alpha_0} + 1 \right].
\]

The function $Q_{\text{nai-}\alpha}(b)$ is convex and analytical with respect to $b$ and continuous with respect to all arguments $(b, \alpha)$. Thus
\[
\forall b \in \mathbb{R}^2: \lim_{\alpha \to \alpha_0} Q_{\text{nai-}\alpha}(b) = Q_{\text{nai-}\alpha_0}(b).
\]

The limit function $Q_{\text{nai-}\alpha_0}(b)$ attains its minimum at exactly one point, $\hat{\beta}_{\text{nai-}\alpha_0}$. Using a proof similar to that of Theorem 3.3, we establish that the function $Q_{\text{nai-}\alpha}(b)$ has a unique point of minimum (which also is a unique solution of the system of equations (15)) denoted by $\hat{\beta}_{\text{nai-}\alpha}$ for all $\alpha$ belonging to some neighborhood $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \cap [0, 1]$ of $\alpha_0$. Moreover, this solution approaches $\hat{\beta}_{\text{nai-}\alpha_0}$:
\[
\lim_{\alpha \to \alpha_0} \hat{\beta}_{\text{nai-}\alpha} = \hat{\beta}_{\text{nai-}\alpha_0}.
\]

Corollary. If the distribution of $X_0$ is not concentrated at a single point and condition (17) holds for some $\varepsilon > 0$, then the estimator $\hat{\beta}_{\text{nai-}\alpha}$ eventually is continuous in $[0, 1]$ as a function of the argument $\alpha$.

Lemma 3.4. If the distribution of $X_0$ is not concentrated at a single point and condition (14) holds for some $\varepsilon > 0$, then the convergence is uniform as the size of the sample increases, that is,
\[
P \left( \sup_{\alpha \in [0, 1]} \| \beta_{\text{nai-}\alpha} - \beta^* \| \to 0 \right) = 1,
\]
where $\beta^* = (\beta_0 + \frac{1}{2}\beta_1^2 \varepsilon^2, \beta_1)$.

Proof. Put
\[
\Theta = \left[ \beta_0 + \frac{1}{2} \beta_1^2 \varepsilon^2 - \frac{1}{2} \varepsilon, \beta_0 + \frac{1}{2} \beta_1^2 \varepsilon^2 + \frac{1}{2} \varepsilon \right] \times \left[ \beta_1 - \frac{1}{2} \varepsilon, \beta_1 + \frac{1}{2} \varepsilon \right].
\]

By Lemma 3.2, the function $q_{\text{nai-}\alpha}(X_0, Y; b)$ and its derivatives with respect to $b$ and $\alpha$ are bounded in $\Theta \times [0, 1]$ and in $\Theta \times (0, 1)$, respectively. This implies that the function $E_{q_{\text{nai-}\alpha}}(X_0, Y; b)$ is continuous in $\Theta \times [0, 1]$ and that the convergence is uniform, that is,
\[
P \left( \lim_{N \to \infty} \max_{b \in \Theta} \max_{\alpha \in [0, 1]} \left| \frac{1}{N} Q_{\text{nai-}\alpha}(b) - E_{q_{\text{nai-}\alpha}}(X_0, Y; b) \right| = 0 \right) = 1.
\]

It remains to show that
\[
\max_{\alpha \in [0, 1]} \| \hat{\beta}_{\text{nai-}\alpha} - \beta^* \| < \varepsilon_1
\]
eventually for all $\varepsilon_1 > 0$. Choose $0 < \varepsilon_1 \leq \frac{1}{2} \varepsilon$.
Since the function $E_{q_{\text{nai-}\alpha}}(X_0, Y; b)$ for a fixed $\alpha \in [0, 1]$ attains its minimum at $b = \beta^*$, we conclude that
\[
\min_{\alpha \in [0, 1]} \min_{b - \beta^* = \varepsilon_1} (E_{q_{\text{nai-}\alpha}}(X_0, Y; b) - E_{q_{\text{nai-}\alpha}}(X_0, Y; \beta^*)) > 0.
\]
In view of (21), we eventually obtain
\[
\min_{\alpha \in [0, 1]} \min_{b - \beta^* = \varepsilon_1} \frac{1}{N} (Q_{\text{nai-}\alpha}(b) - Q_{\text{nai-}\alpha}(\beta^*)) > 0.
\]
By the convexity, this implies that all points of minimum of the functions $Q_{nai-\alpha}(b)$ eventually belong to the circle $\{b: ||b - \beta|| < \varepsilon_1\}$ for all $\alpha$. The function $Q_{nai-\alpha}(b)$ is analytical, whence we conclude that the point of minimum (for all $\alpha$) is unique. If the point of minimum is unique we denote it by $\hat{\beta}_{nai-\alpha}$. This means that (22) holds eventually. \hfill \Box

**Proof of Theorem 3.2.** Existence of a solution. Introducing an auxiliary variable $\alpha \in [0, 1]$, we rewrite the system of equations (12) as follows:

$$
\sum_{i=1}^{N} s_{nai-\alpha}(b_0 + \frac{1}{2}b_1^2\tau^2, b_1) = 0,
\alpha = \exp\{-b_1^2\tau^2\}.
$$

If $\alpha$ is fixed, then the first equation of the latter system has a unique solution $b$, and the system is equivalent to the following one:

$$
\begin{cases}
(b_0 + \frac{1}{2}b_1^2\tau^2, b_1) = \hat{\beta}_{nai-\alpha}, \\
\alpha = \exp\{-b_1^2\tau^2\} = 0.
\end{cases}
$$

The equation $\alpha - \exp\{-\hat{\beta}_{nai-\alpha}^2\tau^2\} = 0$ has a solution in $[0, 1]$, since the left hand side is continuous and assumes values of different signs at the points $\alpha = 0$ and $\alpha = 1$. Thus system (12) has solutions.

**Consistency.** The first equation of system (23) implies that

$$
\left(\hat{\beta}_{0,QL} + \frac{1}{2}\hat{\beta}_{1,QL}\tau^2, \hat{\beta}_{1,QL}\right) = \hat{\beta}_{nai-\alpha}
$$

eventually for a random $\alpha$. Lemma 3.4 implies that $\hat{\beta}_{QL} \rightarrow \beta$ almost surely. \hfill \Box

**Theorem 3.4.** Let assumptions of Theorem 3.2 hold. If $\beta_1 \neq 0$ or

$$
\exists \varepsilon > 0: \quad \mathbb{E}\left[e^{(3\beta_1-3\varepsilon)X_0} + e^{(3\beta_1+3\varepsilon)X_0}\right] < \infty,
$$

then the system of equations (12) for the quasi-likelihood estimator eventually has a unique solution in $\mathbb{R}^2$.

**Proof.** As in the proof of Theorem 3.3 put $\beta^* = (\beta_0 + \frac{1}{2}\beta_1^2\tau^2, \beta_1)$ and let $\alpha_0 = \exp\{-\beta_1^2\tau^2\}$.

Using the equality

$$
\sum_{i=1}^{n} s_{nai-\alpha}(X_{0i}, Y_i; \hat{\beta}_{nai-\alpha}) = 0
$$

we evaluate the derivative

$$
\frac{d\hat{\beta}_{nai-\alpha}}{d\alpha} = -\left(\sum_{i=1}^{N} \frac{\partial s_{nai-\alpha}}{\partial b}\right)^{-1} \left(\sum_{i=1}^{N} \frac{\partial s_{nai-\alpha}}{\partial \alpha}\right)
$$

$$
= -\left(\sum_{i=1}^{N} e^{\hat{\beta}_0 + \hat{\beta}_1X_{0i}}(\alpha + (1 - \alpha)Y_i) \left(1, X_{0i}, X_{0i}^2, X_{0i}Y_i, X_{0i}^2Y_i\right)\right)^{-1}
$$

$$
\times \left(\sum_{i=1}^{N} \frac{e^{\hat{\beta}_0 + \hat{\beta}_1X_{0i}} - 1}{\alpha + (1 - \alpha)\hat{\beta}_0 + \hat{\beta}_1X_{0i}} \left(1, X_{0i}, X_{0i}^2, X_{0i}Y_i, X_{0i}^2Y_i\right)\right).
$$

The derivatives above are evaluated at the point $b = (\hat{\beta}_0, \hat{\beta}_1) = \hat{\beta}_{nai-\alpha}$. 
Then we establish the strong laws of large numbers:

\[ \frac{1}{N} \sum_{i=1}^{N} \frac{\partial s_{\text{nai-}A}(X_{0i}, Y_i; b)}{\partial b} \to E e^{b_0 + b_1 X_0 (\alpha + (1 - \alpha) Y)} \left( \frac{1}{\alpha + (1 - \alpha)e^{b_0 + b_1 X_0}} \right)^2 \left( \begin{array}{c} X_0 \\ X_0^\tau \end{array} \right) =: A(b, \alpha), \]

\[ \frac{1}{N} \sum_{i=1}^{N} \frac{\partial s_{\text{nai-}A}(X_{0i}, Y_i; b)}{\partial b} \to E \left( e^{b_0 + b_1 X_0 - 1} \left( e^{b_0 + b_1 X_0} - Y \right) \right) \left( \alpha + (1 - \alpha)e^{b_0 + b_1 X_0} \right)^2 =: v(b, \alpha). \]

There exists a convex compact set \( \Theta \subset \mathbb{R}^2 \) containing \( \beta^* \) as an inner point, and there exists an interval \( [\alpha_1, \alpha_2] \) with \( 0 < \alpha_1 < \alpha_0 < \alpha_2 < 1 \) for \( \beta_1 \neq 0 \) or \( 0 < \alpha_1 < \alpha_0 = \alpha_2 = 1 \) for \( \beta_1 = 0 \) such that the derivatives

\[ \frac{\partial s_{\text{nai-}A}}{\partial b}, \frac{\partial s_{\text{nai-}A}}{\partial \alpha}, \frac{\partial^2 s_{\text{nai-}A}}{\partial b^2}, \frac{\partial^2 s_{\text{nai-}A}}{\partial \alpha \partial b}, \frac{\partial^2 s_{\text{nai-}A}}{\partial \alpha^2} \]

are bounded in \( \Theta \times [\alpha_1, \alpha_2] \) by a random variable with finite expectation. Moreover, the convergence in (25) and in (26) is uniform in \( \Theta \times [\alpha_1, \alpha_2] \). The matrix \( A(b, \alpha) \) is nondegenerate and continuous with respect to \( (b, \alpha) \) in the set \( \Theta \times [\alpha_1, \alpha_2] \). The vector \( v(b, \alpha) \) is a continuous function of the argument \( (b, \alpha) \) in the set \( \Theta \times [\alpha_1, \alpha_2] \). In addition, \( v(\beta^*, \alpha) = 0 \) for \( \alpha_1 \leq \alpha \leq \alpha_2 \).

Thus

\[ \lim_{\substack{N \to \infty \cr \beta \to \beta^*}} \frac{d^2 \hat{\beta}_{\text{nai-}A}}{d \alpha} = 0 \]

almost surely; moreover, the convergence is uniform in the interval \( [\alpha_1, \alpha_2] \).

Then we separate equations for \( \alpha \) from the system (23):

\[ \alpha - \exp \left\{ -\hat{\beta}_{1, \text{nai-}A}^2 \tau^2 \right\} = 0. \]

By Lemma 3.4

\[ \frac{d}{d \alpha} \left( \alpha - \exp \left\{ -\hat{\beta}_{1, \text{nai-}A}^2 \tau^2 \right\} \right) = 1 + 2\tau^2 \hat{\beta}_{1, \text{nai-}A}^2 e^{-\hat{\beta}_{1, \text{nai-}A}^2 \tau^2} \frac{d \hat{\beta}_{1, \text{nai-}A}}{d \alpha} \to 1 \]

almost surely and uniformly in \( [\alpha_1, \alpha_2] \). Therefore equation (27) eventually has at most one solution in the interval \( [\alpha_1, \alpha_2] \). Now Lemma 3.4 implies that this equation does not eventually have solutions outside the interval \( [\alpha_1, \alpha_2] \).

Since equation (27) has at most one solution, we conclude that the system of equations (12) also has at most one solution. The existence of a solution is already proved in Theorem 3.2.

\[ \Box \]

4. Concluding remarks

In the structural Berkson model of a simple Poisson regression with a homoscedastic normally distributed error in the regressor, sufficient conditions are found that

1. the equation for the simple score estimator eventually has a unique solution in \( \mathbb{R}^2 \) and that the estimator is strongly consistent;
2. the equation for the quasi-likelihood estimator eventually has a solution (possibly, not unique) in \( \mathbb{R}^2 \) and that the estimator (defined as an arbitrary measurable solution of this equation) is strongly consistent;
3. the equation for the quasi-likelihood estimator eventually has a unique solution in \( \mathbb{R}^2 \).

It would also be of interest to obtain similar conditions in the functional Berkson model.
Bibliography


Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Volodymyrs’ka Street, 64, Kyiv 01601, Ukraine

E-mail address: shklyar@mail.univ.kiev.ua

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