Теорія Ймовір. та Матем. Статист.
Вип. 89, 2013

Theor. Probability and Math. Statist.
No. 89, 2014, Pages 127–142
S 0094-9000(2015)00940-6
Article electronically published on January 26, 2015

MINIMAX-ROBUST FILTERING PROBLEM FOR STOCHASTIC SEQUENCES WITH STATIONARY INCREMENTS

UDC 519.21

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Abstract. The problem of optimal estimation is considered for the linear functional

$$A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k),$$

which depends on unknown values of a stochastic sequence $\xi(k)$ with stationary n-th order increments from observations of the sequence $\xi(k)+\eta(k)$ for $k = 0, -1, -2, \dots$. Formulas suitable for calculating the mean-square error and spectral characteristic of the optimal linear estimate of the above functional are derived under the condition of the spectral definiteness, that is in the case where the spectral densities of the sequences $\xi(k)$ and $\eta(k)$ are exactly known. The minimax (robust) method of estimation is applied in the case where spectral densities are not known exactly but sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and the minimax spectral characteristics are proposed for some special sets of admissible spectral densities.

1. Introduction

Classical methods for the solution of extrapolation, interpolation, and filtering problems for stationary stochastic processes and sequences have been developed by A. N. Kolmogorov [11], N. Wiener [26], A. M. Yaglom [28] under the condition of the spectral definiteness, where the spectral densities of the underlying stochastic processes are exactly known. In the case where the spectral densities are not exactly known but a set of admissible spectral densities is given, one can apply the minimax method for solving extrapolation, interpolation, and filtering problems which allows one to determine estimates that minimize the mean-square error for all densities belonging to a given class.

A survey of results on the minimax (robust) methods of data processing is presented by S. A. Kassam and H. V. Poor [10]. U. Grenander [7] is the first to apply the minimax approach to the extrapolation problem for stationary processes. J. Franke [8] and J. Franke and H. V. Poor [9] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. M. P. Moklyachuk [14]–[17] studied the problems of extrapolation, interpolation, and filtering for stationary processes and sequences. Some methods to solve the minimax-robust estimation problems for vector-valued stationary sequences, and processes are developed by M. P. Moklyachuk and O. Yu. Masyutka [19]–[23]. The minimax-robust estimation problems (extrapolation, interpolation, and filtering) for linear functionals that depend on unknown values of periodically correlated stochastic processes are considered

2010 *Mathematics Subject Classification.* Primary 60G10, 60G25, 60G35; Secondary 62M20, 93E10, 93E11.

Key words and phrases. Sequences with stationary increments, robust estimate, mean-square error, least favorable spectral density, minimax spectral characteristic.

by I. I. Dubovets'ka and M. P. Moklyachuk [2]–[6]. M. M. Luz and M. P. Moklyachuk [12]–[13] investigated the minimax interpolation problem for stochastic sequences $\xi(m)$ with stationary n-th order increments; both cases of the estimation from observations of the sequence with an additive noise and from observations without noise are considered in [12]–[13].

In this paper, we investigate the problem of the optimal linear filtering of a functional

$$A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k),$$

which depends on unknown values of a stochastic sequence $\xi(m)$ with stationary n-th order increments based on observations of the sequence $\xi(k)+\eta(k)$ for $k = 0, -1, -2, \dots,$ where $\eta(k)$ is a stochastic sequence with stationary n-th order increments which is uncorrelated with the sequence $\xi(k)$. This filtering problem is solved in the case of the spectral definiteness where the spectral densities of the sequences $\xi(m)$ and $\eta(m)$ are known exactly as well as in the case where the spectral densities of the sequences are not exactly known, but a set of admissible spectral densities is given. Formulas that determine the least favorable spectral densities and minimax (robust) spectral characteristics of the optimal linear estimate of the functional are proposed in this case for concrete classes of admissible spectral densities.

2. Stochastic stationary increment sequence. Spectral representation

Stochastic processes with stationary n-th order increments are introduced and investigated by A. M. Yaglom [27], M. S. Pinsker [25], and A. M. Yaglom and M. S. Pinsker [24].

Definition 2.1. If $\{\xi(m), m \in \mathbb{Z}\}$ is a given stochastic sequence, then

$$\xi^{(n)}(m,\mu) = (1-B_\mu)^n\xi(m) = \sum_{l=0}^{n}(-1)^l C_n^l \xi(m-l\mu) \tag{1}$$

is called an n-th order increment stochastic sequence with step $\mu \in \mathbb{Z}$, where B_μ is a backward shift operator with step $\mu \in \mathbb{Z}$, that is $B_\mu\xi(m) = \xi(m-\mu)$.

For an n-th order increment stochastic sequence $\xi^{(n)}(m,\mu)$, we have

$$\xi^{(n)}(m,-\mu) = (-1)^n\xi^{(n)}(m+n\mu,\mu), \tag{2}$$

$$\xi^{(n)}(m,k\mu) = \sum\nolimits_{l=0}^{(k-1)n} A_l\xi^{(n)}(m-l\mu,\mu), \qquad k \in \mathbb{N}, \tag{3}$$

where the coefficients $\{A_l, l = 0,1,2,\dots,(k-1)n\}$ are determined by the representation

$$(1+x+\dots+x^{k-1})^n = \sum_{l=0}^{(k-1)n} A_l x^l.$$

Definition 2.2. An n-th order increment stochastic sequence $\xi^{(n)}(m,\mu)$ generated by a stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ is widely stationary if the mathematical expectations

$$\mathsf{E}\xi^{(n)}(m_0,\mu) = c^{(n)}(\mu),$$

$$\mathsf{E}\xi^{(n)}(m_0+m,\mu_1)\xi^{(n)}(m_0,\mu_2) = D^{(n)}(m,\mu_1,\mu_2)$$

exist for all m_0, μ, m, μ_1, μ_2 and do not depend on m_0. The function $c^{(n)}(\mu)$ is called the mean value of the n-th order increment sequence, while the function $D^{(n)}(m,\mu_1,\mu_2)$ is called its structural function (or the structural function of an n-th order increment stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$).

A stochastic sequence $\{\xi(m), m \in \mathbb{Z}\}$ that determines an n-th order increment stationary sequence $\xi^{(n)}(m,\mu)$ by formula (1) is called a sequence with stationary n-th order increments.

Theorem 2.1. *The mean value $c^{(n)}(\mu)$ and the structural function $D^{(n)}(m,\mu_1,\mu_2)$ of a stationary n-th order increment stochastic sequence $\xi^{(n)}(m,\mu)$ can be represented in the following forms:*

$$c^{(n)}(\mu) = c\mu^n, \tag{4}$$

$$D^{(n)}(m,\mu_1,\mu_2) = \int_{-\pi}^{\pi} e^{i\lambda m}\left(1-e^{-i\mu_1\lambda}\right)^n\left(1-e^{i\mu_2\lambda}\right)^n \frac{1}{\lambda^{2n}}\, dF(\lambda), \tag{5}$$

where c is a constant and where $F(\lambda)$ is a left-continuous nondecreasing bounded function with $F(-\pi)=0$. The constant c and the function $F(\lambda)$ are determined uniquely by the increment sequence $\xi^{(n)}(m,\mu)$.

On the other hand, a function $c^{(n)}(\mu)$ which has the form (4) with a constant c and a function $D^{(n)}(m,\mu_1,\mu_2)$ which has the form (5) with a function $F(\lambda)$ which satisfies the indicated conditions are the mean value and the structural function of some stationary n-th order increment sequence $\xi^{(n)}(m,\mu)$.

Using representation (5) of the structural function of a stationary n-th order increment sequence $\xi^{(n)}(m,\mu)$ and the Karhunen theorem [1], we obtain the following spectral representation of the stationary n-th order increment sequence $\xi^{(n)}(m,\mu)$:

$$\xi^{(n)}(m,\mu) = \int_{-\pi}^{\pi} e^{im\lambda}\left(1-e^{-i\mu\lambda}\right)^n \frac{1}{(i\lambda)^n}\, dZ(\lambda), \tag{6}$$

where $Z(\lambda)$ is an orthogonal stochastic measure on $[-\pi,\pi)$ connected with the spectral function $F(\lambda)$ by the relation

$$\mathsf{E}Z(A_1)\overline{Z(A_2)} = F(A_1 \cap A_2) < \infty. \tag{7}$$

Example 2.1. Consider an ARIMA(0,1,1) sequence defined by the equation

$$\xi_m = \xi_{m-1} + \varepsilon_m + a\varepsilon_{m-1},$$

where ε_m is a sequence of uncorrelated identically distributed random variables with mean value 0 and variance σ^2. Taking $\eta_m = \xi_m - \xi_{m-1}$, we obtain a moving average sequence $\eta_m = \varepsilon_m + a\varepsilon_{m-1}$. Thus, ξ_m is a stochastic sequence with stationary increments of the first order. The spectral function $F(\lambda)$ of the sequence ξ_m is given by

$$F(\lambda) = \frac{\sigma^2}{4\pi}\int_{-\pi}^{\lambda} \frac{u^2}{1-\cos u}\left(1+2a\cos u + a^2\right) du.$$

Below are some values of the structural function:

$$D^{(1)}(0,1,1) = \sigma^2\left(1+a^2\right), \qquad D^{(1)}(0,1,2) = \sigma^2\left(1+a+a^2\right),$$
$$D^{(1)}(0,2,2) = 2\sigma^2(1+a+a^2),$$
$$D^{(1)}(m,1,1) = \begin{cases} \sigma^2(1+a^2), & m=0,\\ \sigma^2 a, & m=-1,1,\\ 0, & \text{otherwise,}\end{cases}$$

$$D^{(1)}(m,1,2)=\begin{cases}\sigma^2(1+a+a^2), & m=-1,0,\\ \sigma^2a^2, & m=-2,1,\\ 0, & \text{otherwise},\end{cases}$$

$$D^{(1)}(m,2,2)=\begin{cases}2\sigma^2(1+a+a^2), & m=0,\\ \sigma^2(1+2a+a^2), & m=-1,1,\\ \sigma^2a^2, & m=-2,2,\\ 0, & \text{otherwise}.\end{cases}$$

3. Filtering problem for the functional $A\xi$

Let a stochastic sequence $\{\xi(m), m\in\mathbb{Z}\}$ define a stationary n-th order increment sequence $\xi^{(n)}(m,\mu)$ with an absolutely continuous spectral function $F(\lambda)$ which has the spectral density $f(\lambda)$. Let $\{\eta(m), m\in\mathbb{Z}\}$ be a stochastic sequence, uncorrelated with the sequence $\xi(m)$, which determines a stationary n-th order increment sequence $\eta^{(n)}(m,\mu)$ with an absolutely continuous spectral function $G(\lambda)$ which has the spectral density $g(\lambda)$. Without loss of generality we will assume that the mean values of the increment sequences $\xi^{(n)}(m,\mu)$ and $\eta^{(n)}(m,\mu)$ are equal to 0. Suppose that we know the values of the sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2,\dots$. Consider the problem of the mean-square optimal linear estimation of the functional

$$A\xi=\sum_{k=0}^{\infty}a(k)\xi(-k),$$

depending on unknown values of the sequence $\xi(m)$ from observations of the sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2,\dots$. We will consider the case where $\mu>0$.

One can obtain from (1) the following formal equation

$$\xi(-k)=\frac{1}{(1-B_\mu)^n}\xi^{(n)}(-k,\mu)=\sum_{i=k}^{\infty}d_\mu(i-k)\xi^{(n)}(-i,\mu), \tag{8}$$

where $\{d_\mu(i)\colon i\geq 0\}$ are the coefficients in the decomposition

$$\sum_{i=0}^{\infty}d_\mu(i)x^i=\left(\sum_{l=0}^{\infty}x^{\mu l}\right)^n.$$

Equation (8) implies the following relations:

$$\sum_{k=0}^{\infty}a(k)\xi(-k)=\sum_{i=0}^{\infty}\xi^{(n)}(-i,\mu)\sum_{k=0}^{i}a(k)d_\mu(i-k),$$

$$\sum_{k=0}^{\infty}b_\mu(k)\xi^{(n)}(-k,\mu)=\sum_{i=0}^{\infty}\xi(-i)\sum_{l=0}^{\min\{n,[i/\mu]\}}(-1)^lC_n^lb_\mu(i-l\mu),$$

whence we obtain the following representation for the functional $A\xi$:

$$A\xi=\sum_{k=0}^{\infty}a(k)\xi(-k)=\sum_{k=0}^{\infty}b_\mu(k)\xi^{(n)}(-k,\mu)=B\xi,$$

$$b_\mu(k)=\sum_{m=0}^{k}a(m)d_\mu(k-m)=(\mathbf{D}^\mu\mathbf{a})_k,\qquad k\geq 0, \tag{9}$$

where $\mathbf{D}^\mu$ is a linear operator with elements $\mathbf{D}^\mu_{k,j}=d_\mu(k-j)$ if $0\leq j\leq k$ and $\mathbf{D}^\mu_{k,j}=0$ if $j>k$; $\mathbf{a}=(a(0),a(1),a(2),\dots)$. Let $\widehat{A}\xi$ denote the mean-square optimal linear estimate of the functional $A\xi$ from observations of the stochastic sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2,\dots$, and let $\widehat{B}\xi$ denote the mean-square optimal linear estimate of the functional $B\xi$ from observations of the n-th order increment stochastic sequence $\xi^{(n)}(m,\mu)+\eta^{(n)}(m,\mu)$ at points $m=0,-1,-2,\dots$.

Let $\Delta\big(f,g,\widehat{A}\xi\big)=\mathsf{E}\big|A\xi-\widehat{A}\xi\big|^2$ be the mean-square error of the estimate $\widehat{A}\xi$ of the functional $A\xi$, and let $\Delta\big(f,g,\widehat{B}\xi\big)=\mathsf{E}\big|B\xi-\widehat{B}\xi\big|^2$ be the mean-square error of the estimate $\widehat{B}\xi$ of the functional $B\xi$. Since $A\xi=B\xi$, we have

$$\widehat{A}\xi=\widehat{B}\xi. \tag{10}$$

Therefore,

$$\Delta\big(f,g,\widehat{A}\xi\big)=\mathsf{E}\big|A\xi-\widehat{A}\xi\big|^2=\mathsf{E}\big|B\xi-\widehat{B}\xi\big|^2=\Delta\big(f,g,\widehat{B}\xi\big).$$

To find the mean-square optimal estimate of the functional $B\xi$, we use the Hilbert space orthogonal projection method proposed by A. N. Kolmogorov [11]. Suppose that conditions

$$\sum_{k=0}^{\infty}|b_\mu(k)|<\infty,\qquad \sum_{k=0}^{\infty}(k+1)|b_\mu(k)|^2<\infty, \tag{11}$$

$$\sum_{k=0}^{\infty}|(\mathbf{D}^\mu\mathbf{a})_k|<\infty,\qquad \sum_{k=0}^{\infty}(k+1)|(\mathbf{D}^\mu\mathbf{a})_k|^2<\infty \tag{12}$$

are satisfied.

Let $H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$ be the closed linear subspace of the Hilbert space $H=L_2(\Omega,\mathfrak{F},\mathsf{P})$ of the second-order random variables generated by the values

$$\big\{\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu)\colon k\le 0\big\},\qquad \mu>0.$$

Consider a closed linear subspace $L_2^0(f+g)$ of the Hilbert space $L_2(f+g)$ generated by functions

$$\left\{e^{i\lambda k}\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\colon k\le 0\right\}.$$

Then

$$\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu)=\int_{-\pi}^{\pi}e^{i\lambda k}\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda)$$

implies the existence of a one-to-one correspondence between elements

$$e^{i\lambda k}\left(1-e^{-i\lambda\mu}\right)^n/(i\lambda)^n$$

of the space $L_2^0(f+g)$ and elements $\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu)$ of the space $H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$. Every linear estimate $\widehat{B}\xi$ of the functional $B\xi$ admits the representation

$$\widehat{B}\xi=\int_{-\pi}^{\pi}h_\mu(\lambda)\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda), \tag{13}$$

where $h_\mu(\lambda)$ is the spectral characteristic of the estimate $\widehat{B}\xi$. The optimal estimate $\widehat{B}\xi$ is a projection of the element $B\xi$ on the subspace $H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$. This estimate $\widehat{B}\xi$ is determined by the following conditions:

1) $\widehat{B}\xi\in H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$;
2) $\big(B\xi-\widehat{B}\xi\big)\perp H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$.

It follows from condition 2) that, for all $k \le 0$, the function $h_\mu(\lambda)$ satisfies the relation

$$\begin{aligned}
&\mathsf{E}\big(B\xi - \widehat{B}\xi\big)\overline{\big(\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu)\big)}\\
&\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(B_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}-h_\mu(\lambda)\right)e^{-i\lambda k}\left(1-e^{i\lambda\mu}\right)^n\frac{1}{(-i\lambda)^n}f(\lambda)\,d\lambda\\
&\quad\quad-\frac{1}{2\pi}\int_{-\pi}^{\pi}h_\mu(\lambda)e^{-i\lambda k}\left(1-e^{i\lambda\mu}\right)^n\frac{1}{(-i\lambda)^n}g(\lambda)\,d\lambda\\
&\quad=0.
\end{aligned}$$

The latter relation yields

$$\int_{-\pi}^{\pi}\left(B_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{f(\lambda)}{(i\lambda)^n}-h_\mu(\lambda)(f(\lambda)+g(\lambda))\right)\frac{\left(1-e^{i\lambda\mu}\right)^n}{(-i\lambda)^n}e^{-i\lambda k}\,d\lambda=0,$$

where $k \le 0$, whence

$$h_\mu(\lambda)=B_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{(-i\lambda)^nC_\mu\left(e^{i\lambda}\right)}{\left(1-e^{i\lambda\mu}\right)^n(f(\lambda)+g(\lambda))},$$

$$B_\mu(e^{i\lambda})=\sum_{k=0}^{\infty}b_\mu(k)e^{-i\lambda k},\qquad C_\mu\left(e^{i\lambda}\right)=\sum_{k=1}^{\infty}c_\mu(k)e^{i\lambda k}.$$

It follows from condition 1) that the spectral characteristic $h_\mu(\lambda)$ admits the representation

$$h_\mu(\lambda)=h(\lambda)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n},$$

where

$$h(\lambda)=\sum_{k=-\infty}^{0}s(k)e^{i\lambda k}$$

and

$$\int_{-\pi}^{\pi}|h(\lambda)|^2\left|1-e^{i\lambda\mu}\right|^{2n}\frac{f(\lambda)+g(\lambda)}{\lambda^{2n}}\,d\lambda<\infty,$$

$$\frac{(i\lambda)^nh_\mu(\lambda)}{\left(1-e^{-i\lambda\mu}\right)^n}\in L_2^0,$$

(14)

$$\int_{-\pi}^{\pi}\left(B_\mu(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{\lambda^{2n}C_\mu\left(e^{i\lambda}\right)}{\left(1-e^{-i\lambda\mu}\right)^n\left(1-e^{i\lambda\mu}\right)^n(f(\lambda)+g(\lambda))}\right)e^{-i\lambda l}\,d\lambda=0,$$

where $l \ge 1$.

Assume that

$$\int_{-\pi}^{\pi}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}\,d\lambda<\infty,\qquad\int_{-\pi}^{\pi}\frac{\lambda^{2n}}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))}\,d\lambda<\infty. \tag{15}$$

Set

$$R_{k,j}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\lambda(j+k)}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}\,d\lambda,$$

$$P_{k,j}^{\mu}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\lambda(j-k)}\frac{\lambda^{2n}}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))}\,d\lambda,$$

$$Q_{k,j}^{\mu}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\lambda(j-k)}\frac{\left|1-e^{i\lambda\mu}\right|^{2n}f(\lambda)g(\lambda)}{\lambda^{2n}(f(\lambda)+g(\lambda))}\,d\lambda.$$

Then (14) is equivalent to the linear system

$$\sum_{m=0}^{\infty} R_{l,m} b_\mu(m) = \sum_{k=1}^{\infty} P^{\mu}_{l,k} c_\mu(k), \qquad l \geq 1.$$

This system can be rewritten as

$$\mathbf{R}\mathbf{b}_\mu = \mathbf{P}_\mu \mathbf{c}_\mu, \tag{16}$$

where $\mathbf{c}_\mu = (c_\mu(1), c_\mu(2), c_\mu(3), \dots)$, $\mathbf{b}_\mu = (b_\mu(0), b_\mu(1), b_\mu(2), \dots)$, $\mathbf{P}_\mu$, $\mathbf{R}$ are linear operators in the space ℓ_2 defined by $(\mathbf{P}_\mu)_{l,k} = P^{\mu}_{l,k}$, $l, k \geq 1$, $(\mathbf{R})_{l,m} = R_{l,m}$, $l \geq 1$, $m \geq 0$. A solution $\mathbf{c}_\mu$ of equation (16) defines the linear estimate $\widehat{B}\xi$ which is a projection of the element $B\xi$ of the Hilbert space H on the subspace $H^0\big(\xi^{(n)}_\mu + \eta^{(n)}_\mu\big)$. Since the space $H^0\big(\xi^{(n)}_\mu + \eta^{(n)}_\mu\big)$ is closed and convex, the projection $B\xi$ is uniquely determined for an arbitrary sequence $b_\mu(0), b_\mu(1), b_\mu(2), \dots$ satisfying conditions (11). Thus equation (16) has a unique solution for an arbitrary $\mathbf{b}_\mu \neq 0$ and the linear operator $\mathbf{P}_\mu \colon \ell_2 \to X$ has the inverse $(\mathbf{P}_\mu)^{-1}$, where

$$X = \{\mathbf{x}_\mu \in \ell_2 \colon \mathbf{x}_\mu = \mathbf{R}\mathbf{b}_\mu \text{ and } \mathbf{b}_\mu \, \text{satisfies} \, (11)\}.$$

The coefficients are given by

$$c_\mu(k) = \left(\mathbf{P}_\mu^{-1} \mathbf{R}\mathbf{b}_\mu\right)_k,$$

where $(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu)_k$ is the k-th element of the vector $\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu$. Thus, the spectral characteristic $h_\mu(\lambda)$ of the optimal estimate $\widehat{B}\xi$ of the functional $B\xi$ is such that

$$h_\mu(\lambda) = B_\mu\left(e^{i\lambda}\right)\left(1 - e^{-i\lambda\mu}\right)^n \frac{1}{(i\lambda)^n} \frac{f(\lambda)}{f(\lambda) + g(\lambda)} - \frac{(-i\lambda)^n \sum_{k=1}^{\infty} \left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right)_k e^{i\lambda k}}{(1 - e^{i\lambda\mu})^n \left(f(\lambda) + g(\lambda)\right)}. \tag{17}$$

The mean-square error of the estimate is calculated by the formula

$$\begin{aligned} \Delta\big(f, g; \widehat{B}\xi\big) &= \mathsf{E}\big|B\xi - \widehat{B}\xi\big|^2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left|B_\mu\left(e^{i\lambda}\right)\left|1 - e^{i\lambda\mu}\right|^{2n} g(\lambda) + \lambda^{2n} \sum_{k=1}^{\infty} \left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1 - e^{i\lambda\mu}\right|^{2n} (f(\lambda) + g(\lambda))^2} f(\lambda)\, d\lambda \\ &\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left|B_\mu\left(e^{i\lambda}\right)\left|1 - e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n} \sum_{k=1}^{\infty} \left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1 - e^{i\lambda\mu}\right|^{2n} (f(\lambda) + g(\lambda))^2} g(\lambda)\, d\lambda \\ &= \big\langle \mathbf{R}\mathbf{b}_\mu, \mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu \big\rangle + \langle \mathbf{Q}_\mu \mathbf{b}_\mu, \mathbf{b}_\mu \rangle, \end{aligned} \tag{18}$$

where $\mathbf{Q}_\mu$ is a linear operator in the space ℓ_2 with the elements $(\mathbf{Q}_\mu)_{l,k} = Q^{\mu}_{l,k}$, $l, k \geq 0$.

The above reasoning and leads to the following result.

Theorem 3.1. *Let stochastic sequences $\{\xi(m), m \in \mathbb{Z}\}$ and $\{\eta(m), m \in \mathbb{Z}\}$ determine the stationary n-th order increment sequences $\xi^{(n)}(m, \mu)$ and $\eta^{(n)}(m, \mu)$ with absolutely continuous spectral functions $F(\lambda)$ and $G(\lambda)$, respectively. Assume that the spectral densities $f(\lambda)$ and $g(\lambda)$ exist and satisfy conditions* (15). *Further, let the coefficients $\{b_\mu(k) \colon k \geq 0\}$ satisfy conditions* (11). *Then the optimal linear estimate $\widehat{B}\xi$ of the functional $B\xi$ of elements $\xi^{(n)}(m, \mu)$, $m \leq 0$, $\mu > 0$, constructed from observations of the sequence $\xi^{(n)}(m, \mu) + \eta^{(n)}(m, \mu)$ for $m = 0, -1, -2, \dots$ is given by* (13). *The spectral characteristic $h_\mu(\lambda)$ of the optimal estimate $\widehat{B}\xi$ is determined by formula* (17) *and the mean-square error $\Delta\big(f, g; \widehat{B}\xi\big)$ is calculated by* (18).

As a corollary of Theorem 3.1 we obtain the optimal estimate of the unknown value of the increment $\xi^{(n)}(m,\mu)$, $m\le 0$, constructed from observations of the sequence

$$\xi(k)+\eta(k)$$

for $k=0,-1,-2,\dots$. Consider the vector b_μ in (17) whose element at the $(-m)$th position equals 1 and elements at the remaining positions are equal to 0. Then the spectral characteristic $\varphi_m(\lambda,\mu)$ of the estimate

$$\widehat{\xi}^{(n)}(m,\mu)=\int_{-\pi}^{\pi}\varphi_m(\lambda,\mu)\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda) \tag{19}$$

is such that

$$\varphi_m(\lambda,\mu)=e^{i\lambda m}\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{(-i\lambda)^n\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{r}_m\right)_k e^{i\lambda k}}{\left(1-e^{i\lambda\mu}\right)^n\left(f(\lambda)+g(\lambda)\right)}, \tag{20}$$

where $\mathbf{r}_m=(R_{1,-m},R_{2,-m},\dots)$. The mean-square error of the estimate is given by

$$\begin{aligned}\Delta&\left(f,g;\widehat{\xi}^{(n)}(m,\mu)\right)\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|e^{i\lambda m}\left|1-e^{i\lambda\mu}\right|^{2n}g(\lambda)+\lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{r}_m\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2}f(\lambda)\,d\lambda\\ &+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|e^{i\lambda m}\left|1-e^{i\lambda\mu}\right|^{2n}f(\lambda)-\lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{r}_m\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2}g(\lambda)\,d\lambda.\end{aligned} \tag{21}$$

Thus, we proved the following result.

Corollary 3.1. *The optimal linear estimate $\widehat{\xi}^{(n)}(m,\mu)$ of the unknown value of the stochastic increment sequence $\xi^{(n)}(m,\mu)$, $m\le 0$, $\mu>0$, constructed from observations of the sequence $\xi(k)+\eta(k)$ for $k=0,-1,-2,\dots$ is given by* (19). *The spectral characteristic $\varphi_m(\lambda,\mu)$ of the optimal estimate $\widehat{\xi}^{(n)}(m,\mu)$ is determined by formula* (20). *The mean-square error $\Delta(f,g;\widehat{\xi}^{(n)}(m,\mu))$ is calculated by* (21).

Now we consider the smoothing problem for a stationary n-th order increment sequence $\xi^{(n)}(m,\mu)$. The problem is to find the mean-square optimal linear estimate $\widehat{\xi}^{(n)}(0,\mu)$ of the unknown value of the increment $\xi^{(n)}(0,\mu)$, $\mu>0$, constructed from observations of the stochastic sequence $\xi(k)+\eta(k)$ for $k=0,-1,-2,\dots$.

Let $r(k)=R_{k,0}$, $k\in\mathbb{Z}$. Then $\{r(k)\colon k\in\mathbb{Z}\}$ are the Fourier coefficients of the function $\frac{f(\lambda)}{f(\lambda)+g(\lambda)}$. If $\overline{r}(k)$ denotes a conjugate element to $r(k)$, then

$$r(k)=\overline{r}(-k),\qquad k\in\mathbb{Z}.$$

Let $\{V_{k,j}^{\mu}\colon k,j\ge 1\}$ be the coefficients which determine a linear operator $\mathbf{V}_\mu=(\mathbf{P}_\mu)^{-1}$. Then

$$\sum_{l\ge 1}V_{l,j}^{\mu}P_{k,l}=\delta_{k,j},\qquad k,j\ge 1, \tag{22}$$

where $\delta_{k,j}$ is the Kronecker symbol. Using (20) and (22) we obtain the spectral characteristic of the optimal estimate $\widehat{\xi}^{(n)}(0,\mu)$ of the unknown value of the increment $\xi^{(n)}(0,\mu)$:

$$\varphi(\lambda,\mu)=\frac{(1-e^{-i\lambda\mu})^n}{(i\lambda)^n}\sum_{k=0}^{\infty}\overline{r}(k)e^{-i\lambda k}.$$

The optimal estimate of the increment $\xi^{(n)}(0,\mu)$ is calculated by the formula

$$\widehat{\xi}^{(n)}(0,\mu)=\sum_{k=0}^{\infty}\overline{r}(k)\xi^{(n)}(-k,\mu)=\sum_{j=0}^{\infty}(\xi(-j)+\eta(-j))\sum_{l=0}^{\min\{n,[\frac{j}{\mu}]\}}(-1)^l C_n^l\overline{r}(j-l\mu). \tag{23}$$

The mean-square error of the estimate $\widehat{\xi}^{(n)}(0,\mu)$ equals

$$\Delta\left(f,g;\widehat{\xi}^{(n)}(0,\mu)\right)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}^{\mu}_{k,j}\overline{r}(j)r(k)+\sum_{l\in\mathbb{Z}}r(l)g_{\mu}(-l), \tag{24}$$

where $\{g_{\mu}(k)\colon k\in\mathbb{Z}\}$ are the Fourier coefficients of the function $|1-e^{i\lambda\mu}|^{2n}g(\lambda)\lambda^{-2n}$.

Corollary 3.2. *The optimal estimate $\widehat{\xi}^{(n)}(0,\mu)$ of the unknown value $\xi^{(n)}(0,\mu)$ of the stationary n-th order increment sequence $\xi^{(n)}(m,\mu)$, $\mu>0$, constructed from observations of the sequence $\xi(k)+\eta(k)$ for $k=0,-1,-2,\dots$ is given by* (23)*. The value of the mean-square error $\Delta(f,g;\widehat{\xi}^{(n)}(0,\mu))$ of the estimate $\widehat{\xi}^{(n)}(0,\mu)$ equals* (24)*.*

Theorem 3.1 and Corollaries 3.1 and 3.2 determine solutions of the filtering problems for the n-th order increment sequence $\widehat{\xi}^{(n)}(m,\mu)$ and the linear functional $B\xi$ based on the Fourier coefficients of the functions

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))},\qquad \frac{f(\lambda)}{f(\lambda)+g(\lambda)},\qquad \frac{|1-e^{i\lambda\mu}|^{2n}f(\lambda)g(\lambda)}{\lambda^{2n}(f(\lambda)+g(\lambda))}.$$

However, the problem of finding the inverse operator $(\mathbf{P}_{\mu})^{-1}$ for the operator $\mathbf{P}_{\mu}$ determined by the Fourier coefficients of the function

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}$$

is complicated in most cases. Therefore, we propose a method of finding the operator $(\mathbf{P}_{\mu})^{-1}$ under the condition that the functions

$$\frac{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}{\lambda^{2n}},\qquad \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))} \tag{25}$$

admit the canonical factorizations

$$\frac{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}{\lambda^{2n}}=\left|\sum_{k=0}^{\infty}\varphi_{\mu}(k)e^{-i\lambda k}\right|^2, \tag{26}$$

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}=\left|\sum_{k=0}^{\infty}\psi_{\mu}(k)e^{-i\lambda k}\right|^2. \tag{27}$$

Using the coefficients $\varphi_{\mu}(k)$ and $\psi_{\mu}(k)$, $k\geq 0$, of factorizations (26) and (27), we define linear operators Φ_{μ} and Ψ_{μ} in the space ℓ_2. Let $(\Phi_{\mu})_{k,j}=\varphi_{\mu}(k-j)$ and $(\Psi_{\mu})_{k,j}=\psi_{\mu}(k-j)$ if $1\leq j\leq k$, $(\Phi_{\mu})_{k,j}=0$ and $(\Psi_{\mu})_{k,j}=0$ if $j>k$ and $k,j\geq 1$. The operators just defined admit the following relation:

$$\Psi_{\mu}\Phi_{\mu}=\Phi_{\mu}\Psi_{\mu}=I,$$

where I is the identity operator. Moreover, the operator $\mathbf{P}_{\mu}$ admits the factorization $\mathbf{P}_{\mu}=\overline{\Psi}'_{\mu}\Psi_{\mu}$. Thus, $(\mathbf{P}_{\mu})^{-1}=\Phi_{\mu}\overline{\Phi}'_{\mu}$ and the coefficients of the operator $\mathbf{V}_{\mu}=(\mathbf{P}_{\mu})^{-1}$ are given by

$$V^{\mu}_{k,j}=\sum_{p=1}^{\min(k,j)}\varphi_{\mu}(k-p)\overline{\varphi}_{\mu}(j-p),\qquad k,j\geq 1.$$

We summarize all these observations in the following result.

Theorem 3.2. *Let functions* (25) *admit canonical factorizations* (26) *and* (27). *Then the inverse operator* $\mathbf{P}_\mu^{-1}$ *for the operator* $\mathbf{P}_\mu$ *is calculated by the formula* $\mathbf{P}_\mu^{-1} = \Phi_\mu \overline{\Phi}'_\mu$, *where the linear operator* Φ_μ *in* ℓ_2 *is determined by the coefficients* $(\Phi_\mu)_{k,j} = \varphi_\mu(k-j)$ *if* $1 \le j \le k$ *and* $(\Phi_\mu)_{k,j} = 0$ *if* $j < k$, $k, j \ge 1$.

Using Theorem 3.1, we find the optimal estimate

$$\widehat{A}\xi = \int_{-\pi}^{\pi} h_\mu^{(a)}(\lambda)\, dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda) \tag{28}$$

of the functional $A\xi$. The spectral characteristic of the estimate $\widehat{A}\xi$ is calculated by the formula

$$\begin{aligned} h_\mu^{(a)}(\lambda) = A_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n \frac{1}{(i\lambda)^n}\frac{f(\lambda)}{f(\lambda)+g(\lambda)} \\ - \frac{(-i\lambda)^n \sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}}{\left(1-e^{i\lambda\mu}\right)^n (f(\lambda)+g(\lambda))}, \end{aligned} \tag{29}$$

where $A_\mu(e^{i\lambda}) = \sum_{k=0}^{\infty}(\mathbf{D}^\mu\mathbf{a})_k e^{-i\lambda k}$. The mean-square error equals

$$\begin{aligned} &\Delta\left(f,g;\widehat{A}\xi\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2} f(\lambda)\,d\lambda \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2} g(\lambda)\,d\lambda \\ &= \left\langle \mathbf{R}\mathbf{D}^\mu\mathbf{a}, \mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right\rangle + \left\langle \mathbf{Q}_\mu\mathbf{D}^\mu\mathbf{a}, \mathbf{D}^\mu\mathbf{a}\right\rangle. \end{aligned} \tag{30}$$

Theorem 3.3. *Let two uncorrelated stochastic sequences* $\{\xi(m), m \in \mathbb{Z}\}$ *and* $\{\eta(m), m \in \mathbb{Z}\}$ *define stationary n-th order increment sequences* $\xi^{(n)}(m,\mu)$ *and* $\eta^{(n)}(m,\mu)$ *with absolutely continuous spectral functions* $F(\lambda)$ *and* $G(\lambda)$, *respectively. Assume that the spectral densities* $f(\lambda)$ *and* $g(\lambda)$ *exist and satisfy conditions* (15). *Further let conditions* (12) *be satisfied.*

Then the optimal linear estimate $\widehat{A}\xi$ *of the functional* $A\xi$ *of unknown elements* $\xi(m)$, $m \le 0$, *constructed from observations of the sequence* $\xi(m)+\eta(m)$ *for* $m = 0, -1, -2\ldots$ *is given by* (28). *The spectral characteristic* $h_\mu^{(a)}(\lambda)$ *of the optimal estimate* $\widehat{A}\xi$ *is determined by* (29). *The value of the mean-square error* $\Delta(f,g;\widehat{A}\xi)$ *equals* (30).

If the function $|1-e^{i\lambda\mu}|^{2n}\lambda^{-2n}(f(\lambda)+g(\lambda))$ *admits canonical factorization* (26), *then the operator* $\mathbf{P}_\mu^{-1}$ *involved in* (29) *and* (30) *can be represented as* $\mathbf{P}_\mu^{-1} = \Phi_\mu\overline{\Phi}'_\mu$.

Example 3.1. Consider an ARIMA(0,1,2) sequence $\{\xi(m), m \in \mathbb{Z}\}$. The first-order increments of the sequence $\xi(m)$ are stationary and the increments with step $\mu = 1$ form a one-sided moving average sequence of order 2. Let the sequence $\xi(m)$ have the spectral density

$$f(\lambda) = \frac{\lambda^2\left|1-\phi e^{-i\lambda}\right|^2\left|1-\psi e^{-i\lambda}\right|^2}{|1-e^{-i\lambda}|^2}.$$

Consider another stochastic sequence $\{\eta(m), m \in \mathbb{Z}\}$ with stationary increments of order 1 uncorrelated with $\xi(m)$ and such that the increments of the sequence

$$\{\xi(m)+\eta(m), m \in \mathbb{Z}\}$$

with step 1 form a moving average sequence of order 1 and the spectral density has the form

$$f(\lambda)+g(\lambda)=\frac{\lambda^2\left|1-\phi e^{-i\lambda}\right|^2}{|1-e^{-i\lambda}|^2}.$$

Consider the sequence $\{a(k)\colon k\geq 0\}$ of real number defined as follows: $a(0)=1$, $a(k)=-2^{-k}$ for $k\geq 1$. This sequence satisfies conditions (12). The problem is to construct the optimal mean-square linear estimate $\widehat{A}\xi$ of the functional

$$A\xi=\sum_{k=0}^{\infty}a(k)\xi(-k)$$

of unknown values $\xi(k)$, $k\leq 0$, of the sequence $\xi(m)$ from observations $\xi(k)+\eta(k)$, $k=0,-1,-2,\dots$.

To calculate the spectral characteristic of the optimal estimate $\widehat{A}\xi$ of the functional $A\xi$, we use formula (29). The operator $\mathbf{P}_\mu=\mathbf{P}$ is determined by the coefficients

$$(\mathbf{P})_{l,k}=\frac{\psi^p}{1-\psi^2},\qquad |k-l|=p,\ l,k\geq 1.$$

The inverse operator $\mathbf{V}=\mathbf{P}^{-1}$ is defined by the coefficients $(\mathbf{V})_{1,1}=1$, $(\mathbf{V})_{l,l}=1+\phi^2$ if $l\geq 2$, $(\mathbf{V})_{l,k}=-\phi$ if $|l-k|=1$, $l,k\geq 1$, and $(\mathbf{V})_{l,k}=0$ otherwise. The operator $\mathbf{R}$ is defined by the coefficients $(\mathbf{R})_{1,0}=1$ and $(\mathbf{R})_{l,k}=0$ if $l\geq 1$, $k\geq 0$, $(l,k)\neq(1,0)$. The operator $\mathbf{D}^\mu=\mathbf{D}$ is defined by the coefficients $d_\mu(k)=1$, $k\geq 0$.

The spectral characteristic $h_1(\lambda)$ of the estimate $\widehat{A}\xi$ is

$$h_1(\lambda)=\sum_{k=0}^{\infty}s(k)e^{-i\lambda k}\frac{1-e^{-i\lambda}}{i\lambda},$$

where $s(0)=1-\frac{1}{2}\psi+\psi^2+\phi\psi\frac{2-\phi^2}{1-\phi^2}$, $s(k)=2^{-k-1}(2-5\psi+2\psi^2)+\phi^{k+1}\psi$, $k\geq 1$.

Put $A(j)=\min\{n,[j/\mu]\}$, $j\geq 0$. Then the estimate $\widehat{A}\xi$ of the functional $A\xi$ is given by

$$\begin{aligned}\widehat{A}\xi&=\sum_{k=0}^{\infty}s(k)\left(\xi^{(n)}(-k,\mu)+\eta^{(n)}(-k,\mu)\right)\\&=\sum_{j=0}^{\infty}(\xi(-j)+\eta(-j))\sum_{l=0}^{A(j)}(-1)^lC_n^l s(j-l\mu).\end{aligned}$$

4. Minimax-robust method of filtering

The value of the mean-square error $\Delta(h_\mu^{(a)}(f,g);f,g):=\Delta(f,g;\widehat{A}\xi)$ and the spectral characteristic $h_\mu^{(a)}(f,g)$ of the optimal linear estimate $\widehat{A}\xi$ of the functional $A\xi$ of unknown values $\xi(m)$ based on observations of the stochastic sequence $\xi(k)+\eta(k)$ are determined by formulas (29) and (30) under the condition that the spectral densities $f(\lambda)$ and $g(\lambda)$ of stochastic sequences $\xi(m)$ and $\eta(m)$ are known.

In the case where the spectral densities are not exactly known, but a set $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ of admissible spectral densities is given, the minimax (robust) approach can be used to estimate some functionals of unknown values of a stochastic sequence with stationary increments. In other words, we are interested in finding an estimate that minimizes the maximum of the mean-square error for all spectral densities belonging to a given class $\mathcal{D}$ of admissible spectral densities simultaneously.

Definition 4.1. Let $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ be a given class of spectral densities. The spectral densities $f_0(\lambda)\in\mathcal{D}_f$, $g_0(\lambda)\in\mathcal{D}_g$ are called least favorable in the class $\mathcal{D}$ for the optimal linear filtering of the functional $A\xi$ if

$$\Delta(f_0,g_0)=\Delta(h(f_0,g_0);f_0,g_0)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h(f,g);f,g).$$

Definition 4.2. Let $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ be a given class of spectral densities. A spectral characteristic $h^0(e^{i\lambda})$ of the optimal linear estimate of the functional $A\xi$ is called minimax-robust if

$$h^0\left(e^{i\lambda}\right)\in H_{\mathcal{D}}=\bigcap_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}L_2^0(f+g),$$

$$\min_{h\in H_{\mathcal{D}}}\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h;f,g)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta\left(h^0;f,g\right).$$

Using the results obtained above, we prove the following result.

Lemma 4.1. *Let spectral densities $f_\mu^0\in\mathcal{D}_f(\lambda)$, $g_\mu^0\in\mathcal{D}_g(\lambda)$ satisfy conditions* (15). *Then they are least favorable in the class $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ for the optimal linear filtering of the functional $A\xi$ if the operators $\mathbf{P}_\mu^0$, $\mathbf{R}^0$, and $\mathbf{Q}_\mu^0$ constructed with the help of the Fourier coefficients of the functions*

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f_\mu^0(\lambda)+g_\mu^0(\lambda))},\qquad\frac{f_\mu^0(\lambda)}{f_\mu^0(\lambda)+g_\mu^0(\lambda)},\qquad\frac{\left|1-e^{i\lambda\mu}\right|^{2n}f_\mu^0(\lambda)g_\mu^0(\lambda)}{\lambda^{2n}(f_\mu^0(\lambda)+g_\mu^0(\lambda))}$$

determine a solution of the conditional extremum problem

$$\begin{aligned}&\max_{f\in\mathcal{D}}\left(\langle\mathbf{R}\mathbf{D}^\mu\mathbf{a},\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\rangle+\langle\mathbf{Q}_\mu\mathbf{D}^\mu\mathbf{a},\mathbf{D}^\mu\mathbf{a}\rangle\right)\\&\quad=\left\langle\mathbf{R}^0\mathbf{D}^\mu\mathbf{a},\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right\rangle+\left\langle\mathbf{Q}_\mu^0\mathbf{D}^\mu\mathbf{a},\mathbf{D}^\mu\mathbf{a}\right\rangle.\end{aligned}\tag{31}$$

The minimax spectral characteristic is $h^0=h_\mu(f_\mu^0,g_\mu^0)$ if $h_\mu(f_\mu^0,g_\mu^0)\in H_{\mathcal{D}}$.

The function h^0 and the pair (f_μ^0,g_μ^0) form a saddle point of the function $\Delta(h;f,g)$ in the set $H_{\mathcal{D}}\times\mathcal{D}$. Saddle point inequalities

$$\Delta\left(h;f_\mu^0,g_\mu^0\right)\geq\Delta\left(h^0;f_\mu^0,g_\mu^0\right)\geq\Delta\left(h^0;f,g\right)\quad\forall f\in\mathcal{D}_f,\forall g\in\mathcal{D}_g,\forall h\in H_{\mathcal{D}}$$

hold if $h^0=h_\mu(f_\mu^0,g_\mu^0)$ and $h_\mu(f_\mu^0,g_\mu^0)\in H_{\mathcal{D}}$, where (f_μ^0,g_μ^0) is a solution of the following conditional extremum problem

$$\widetilde{\Delta}(f,g)=-\Delta(h_\mu(f_\mu^0,g_\mu^0);f,g)\to\inf,\qquad(f,g)\in\mathcal{D},$$

$$\begin{aligned}&\Delta\left(h_\mu\left(f_\mu^0,g_\mu^0\right);f,g\right)\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n}g_\mu^0(\lambda)+\lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1-e^{i\lambda\mu}\right|^{2n}\left(f_\mu^0(\lambda)+g_\mu^0(\lambda)\right)^2}f(\lambda)\,d\lambda\\&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_\mu(e^{i\lambda})\left|1-e^{i\lambda\mu}\right|^{2n}f_\mu^0(\lambda)-\lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1-e^{i\lambda\mu}\right|^{2n}\left(f_\mu^0(\lambda)+g_\mu^0(\lambda)\right)^2}g(\lambda)\,d\lambda.\end{aligned}$$

This conditional extremum problem is equivalent to the following unconditional extremum problem

$$\Delta_{\mathcal{D}}(f,g)=\widetilde{\Delta}(f,g)+\delta(f,g\mid\mathcal{D}_f\times D_g)\to\inf,$$

where $\delta(f,g|\mathcal{D}_f\times D_g)$ is the indicator function of the set $\mathcal{D}_f\times\mathcal{D}_g$. A solution (f_μ^0,g_μ^0) to this unconditional extremum problem is characterized by the condition $0\in\partial\Delta_{\mathcal{D}}(f_\mu^0,g_\mu^0)$ (see [18]).

5. Least favorable spectral densities in the class $\mathcal{D}_f \times \mathcal{D}_g$

Consider the problem of optimal linear filtering of the functional $A\xi$ for the set of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$, where

$$\mathcal{D}_f^0 = \left\{ f(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda)\, d\lambda \leq P_1 \right\}, \qquad \mathcal{D}_g^0 = \left\{ g(\lambda) \,\middle|\, \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda)\, d\lambda \leq P_2 \right\}.$$

Assume that the densities $f_\mu^0 \in \mathcal{D}_f$, $g_\mu^0 \in \mathcal{D}_g$ and functions

$$h_{\mu,f}\left(f_\mu^0, g_\mu^0\right) = \frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g_\mu^0(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|}{|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f_\mu^0(\lambda)+g_\mu^0(\lambda)\right)}, \tag{32}$$

$$h_{\mu,g}\left(f_\mu^0, g_\mu^0\right) = \frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f_\mu^0(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|}{|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f_\mu^0(\lambda)+g_\mu^0(\lambda)\right)} \tag{33}$$

are bounded. In this case the functional $\Delta(h_\mu(f_\mu^0, g_\mu^0); f, g)$ is continuous and bounded in the space $\mathcal{L}_1 \times \mathcal{L}_1$. The condition $0 \in \partial\Delta_{\mathcal{D}}(f_\mu^0, g_\mu^0)$ implies that the least favorable densities $f_\mu^0(\lambda) \in \mathcal{D}_f$ and $g_\mu^0(\lambda) \in \mathcal{D}_g$ satisfy the equations

$$\begin{aligned}&\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g_\mu^0(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right| \\ &\qquad = \alpha_1|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f_\mu^0(\lambda)+g_\mu^0(\lambda)\right),\end{aligned} \tag{34}$$

$$\begin{aligned}&\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f_\mu^0(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right| \\ &\qquad = \alpha_2|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f_\mu^0(\lambda)+g_\mu^0(\lambda)\right),\end{aligned} \tag{35}$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are some constants such that $\alpha_1 \neq 0$ if $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\mu^0(\lambda)\,d\lambda = P_1$ and $\alpha_2 \neq 0$ if $\frac{1}{2\pi}\int_{-\pi}^{\pi} g_\mu^0(\lambda)\,d\lambda = P_2$.

Therefore the following results hold.

Theorem 5.1. *Let spectral densities $f_\mu^0(\lambda) \in \mathcal{D}_f$ and $g_\mu^0(\lambda) \in \mathcal{D}_g$ satisfy conditions* (15)*, and let functions $h_{\mu,f}(f_\mu^0, g_\mu^0)$ and $h_{\mu,g}(f_\mu^0, g_\mu^0)$ determined by equations* (32) *and* (33) *be bounded.*

Then the spectral densities $f_\mu^0(\lambda)$ and $g_\mu^0(\lambda)$ determined by relations (34) *and* (35) *are least favorable in the class $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ for the optimal linear filtering problem for the functional $A\xi$ if they determine a solution of extremum problem* (31)*. The function $h_\mu(f_\mu^0, g_\mu^0)$ determined by* (29) *is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.*

Theorem 5.2. *Let the spectral density $f(\lambda)$ be known, let $g_\mu^0(\lambda) \in \mathcal{D}_g$, and let conditions* (15) *be satisfied. Assume that the function $h_{\mu,g}(f, g_\mu^0)$ is bounded.*

Then the spectral density $g_\mu^0(\lambda)$ is least favorable in the class $\mathcal{D}_g$ for the optimal linear filtering of the functional $A\xi$ if

$$g_\mu^0(\lambda) = \max\left\{0, \frac{\left|A_\mu(e^{i\lambda})\left|1-e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|}{\alpha_2|\lambda|^n|1-e^{i\lambda\mu}|^n} - f(\lambda)\right\}$$

and the pair (f, g_μ^0) determines a solution of extremum problem (31)*. The function $h_\mu(f, g_\mu^0)$ determined by* (29) *is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.*

6. Least favorable spectral densities in the class $\mathcal{D} = \mathcal{D}_u^v \times \mathcal{D}_\varepsilon$

Consider the problem of the optimal linear filtering of the functional $A\xi$ for the set of spectral densities $\mathcal{D} = \mathcal{D}_u^v \times \mathcal{D}_\varepsilon$, where

$$\mathcal{D}_u^v = \left\{ f(\lambda) \,\middle|\, v(\lambda) \le f(\lambda) \le u(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\,d\lambda \le P_1 \right\},$$

$$\mathcal{D}_\varepsilon = \left\{ g(\lambda) \,\middle|\, g(\lambda) = (1-\varepsilon)g_1(\lambda) + \varepsilon w(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)\,d\lambda \le P_2 \right\}.$$

Here the spectral densities $u(\lambda)$, $v(\lambda)$, and $g_1(\lambda)$ are known and fixed, and the spectral densities $u(\lambda)$ and $v(\lambda)$ are bounded.

Let $f_\mu^0(\lambda) \in \mathcal{D}_u^v$ and $g_\mu^0(\lambda) \in \mathcal{D}_\varepsilon$ be two spectral densities such that the functions $h_{\mu,f}(f_\mu^0, g_\mu^0)$, $h_{\mu,g}(f_\mu^0, g_\mu^0)$ determined by (32) and (33) are bounded.

Then the condition $0 \in \partial\Delta_{\mathcal{D}}(f_\mu^0, g_\mu^0)$ implies the following equations that determine the least favorable densities

$$\begin{aligned}\left| A_\mu\left(e^{i\lambda}\right) \left|1 - e^{i\lambda\mu}\right|^{2n} g_\mu^0(\lambda) + \lambda^{2n} \sum_{k=1}^{\infty} \left(\left(\mathbf{P}_\mu^0\right)^{-1} \mathbf{R}^0 \mathbf{D}^\mu \mathbf{a}\right)_k e^{i\lambda k} \right| \\ = \alpha_1 |\lambda|^n \left|1 - e^{i\lambda\mu}\right|^n \left(f_\mu^0(\lambda) + g_\mu^0(\lambda)\right)\left(\gamma_1(\lambda) + \gamma_2(\lambda) + \alpha_1^{-1}\right),\end{aligned} \tag{36}$$

$$\begin{aligned}\left| A_\mu\left(e^{i\lambda}\right) \left|1 - e^{i\lambda\mu}\right|^{2n} f_\mu^0(\lambda) - \lambda^{2n} \sum_{k=1}^{\infty} \left(\left(\mathbf{P}_\mu^0\right)^{-1} \mathbf{R}^0 \mathbf{D}^\mu \mathbf{a}\right)_k e^{i\lambda k} \right| \\ = \alpha_2 |\lambda|^n \left|1 - e^{i\lambda\mu}\right|^n \left(f_\mu^0(\lambda) + g_\mu^0(\lambda)\right)\left(\varphi(\lambda) + \alpha_2^{-1}\right),\end{aligned} \tag{37}$$

where $\gamma_1 \le 0$ and $\gamma_1 = 0$ if $f_\mu^0(\lambda) \ge v(\lambda)$; $\gamma_2(\lambda) \ge 0$ and $\gamma_2 = 0$ if $f_\mu^0(\lambda) \le u(\lambda)$; $\varphi(\lambda) \le 0$ and $\varphi(\lambda) = 0$ if $g_\mu^0(\lambda) \ge (1-\varepsilon)g_1(\lambda)$.

The following results hold.

Theorem 6.1. *Let spectral densities $f_\mu^0(\lambda) \in \mathcal{D}_u^v$ and $g_\mu^0(\lambda) \in \mathcal{D}_\varepsilon$ satisfy conditions* (15)*. Let functions $h_{\mu,f}(f_\mu^0, g_\mu^0)$ and $h_{\mu,g}(f_\mu^0, g_\mu^0)$ as determined by* (32) *and* (33) *be bounded.*

Then spectral densities $f_\mu^0(\lambda)$ and $g_\mu^0(\lambda)$ determined by equations (36) *and* (37) *are least favorable in the class $\mathcal{D} = \mathcal{D}_u^v \times \mathcal{D}_\varepsilon$ for the optimal linear filtering of the functional $A\xi$ if they determine a solution of extremum problem* (31)*. The minimax spectral characteristic $h_\mu(f_\mu^0, g_\mu^0)$ of the optimal estimate of the functional $A\xi$ is determined by* (29)*.*

Theorem 6.2. *Let the spectral density $f(\lambda)$ be known, $g_\mu^0(\lambda) \in \mathcal{D}_\varepsilon$, and let conditions* (15) *be satisfied. Let the function $h_{\mu,g}(f, g_\mu^0)$ determined by* (29) *be bounded.*

Then the spectral density $g_\mu^0(\lambda)$ is least favorable in the class $\mathcal{D}_\varepsilon$ for the optimal linear filtering of the functional $A\xi$ if

$$g_\mu^0(\lambda) = \max\left\{(1-\varepsilon)g_1(\lambda), f_1(\lambda)\right\},$$

$$f_1(\lambda) = \frac{\alpha_2 \left| A_\mu\left(e^{i\lambda}\right) \left|1 - e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n} \sum_{k=1}^{\infty} \left(\left(\mathbf{P}_\mu^0\right)^{-1} \mathbf{R}^0 \mathbf{D}^\mu \mathbf{a}\right)_k e^{i\lambda k} \right|}{|\lambda|^n |1 - e^{i\lambda\mu}|^n} - f(\lambda),$$

and the pair (f, g_μ^0) determines a solution of extremum problem (31)*. The function $h_\mu(f, g_\mu^0)$ determined by* (29) *is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.*

7. Concluding remarks

In this article we found a solution of the filtering problem for the linear functional

$$A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k),$$

which depends on unknown values of a stochastic sequence $\xi(m)$ with stationary n-th order increments for $m = 0, -1, -2, \dots$. The estimate is constructed from observations of the sequence $\xi(m)+\eta(m)$ for $m = 0, -1, -2, \dots$, where $\eta(m)$ is an uncorrelated with $\xi(m)$ sequence with stationary n-th order increments. We derived formulas for computing the mean-square error and the spectral characteristic of the optimal linear estimate of the functional in the case where the spectral densities of sequences are exactly known. In the case where the spectral densities are not exactly known but a set of admissible spectral densities is specified, the minimax-robust method is applied. Formulas that determine the least favorable spectral densities and minimax (robust) spectral characteristics are derived for some special sets of admissible spectral densities.

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Received 27/NOV/2012
Originally published in English