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LARGE DEVIATIONS FOR SOLUTIONS OF ONE DIMENSIONAL ITÔ EQUATIONS

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Abstract. The large deviations principle for the sequence of stochastic processes

$$\eta_n(t) = x_0 + \int_0^t b(n\eta_n(s))\,ds + \frac{1}{\varphi(n)}\int_0^t \sigma(n\eta_n(s))\,dw(s)$$

is proved if the limits of integral means exist for the functions $b(x)\sigma^{-2}(x)$ and $\sigma^{-2}(x)$. The rate functional is evaluated.

1. Introduction

This paper is devoted to the study of large deviations for the sequence of solutions of the following stochastic differential equations:

$$\eta_n(t) = x_0 + \int_0^t b(n\eta_n(s))\,ds + \frac{1}{\varphi(n)}\int_0^t \sigma(n\eta_n(s))\,dw(s), \tag{1}$$

$n \in N$, $t \in [0,1]$, defined in a stochastic basis $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$, where $w(t)$ is a Wiener process and where $b(x)$ and $\sigma(x)$ are nonrandom functions such that there exists $\lambda > 1$ for which

$$\frac{1}{\lambda} \le \sigma^2(x) \le \lambda, \qquad |b(x)| \le \lambda. \tag{2}$$

In equation (1), $\varphi(n)$ is a positive increasing function approaching $+\infty$.

Some results on the large deviations are known for the case where the coefficients $b(x)$ and $\sigma(x)$ are twice continuously differentiable periodic functions [1] or where condition (M) of the paper [2] holds. We do not require that the functions $b(x)$ and $\sigma(x)$ be smooth or periodic. Nevertheless, we are able to prove that the large deviations principle holds under assumption (2) if the limits of the corresponding integral means exist. The method of the proof differs from that of the paper [2]. It turns out that the rate of growth of the function $\varphi(n)$ as $n \to \infty$ depends on the asymptotic behavior of integrals of the coefficients of the equation.

Note also that the large deviations principle is proved in [8], [9] for some other types of stochastic processes.

The paper is organized as follows. In Section 2, we prove a result on the large deviations for the sequence (1). Some examples are presented in Section 3.

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For a metric space (X,ρ), we denote by $\mathbf{B}(X,\rho)$ the Borel σ-algebra of its subsets. Recall that [3] a family of probability measures P_n in the space (X,ρ) satisfies the large deviations principle with the rate functional $S(x)$ and normalizing sequence $\psi(n)$ if $\psi(n)\to\infty$ as $n\to\infty$ and

i) the set $\Phi(x)=\{x\colon S(x)\le c\}$ is compact for all $c>0$;
ii) $\overline{\lim}_{n\to\infty}\frac{1}{\psi(n)}\ln \mathsf{P}_n(F)\le -S(F)$ for any closed subset $F\in\mathbf{B}(X,\rho)$;
iii) $\underline{\lim}_{n\to\infty}\frac{1}{\psi(n)}\ln \mathsf{P}_n(G)\ge -S(G)$, for any open subset $G\in\mathbf{B}(X,\rho)$,

where $S(A)=\inf_{x\in A}S(x)$.

For functions defined in the interval $[0,1]$, we introduce the following notation: $C[0,1]$ is the space of continuous functions, $AC_{x_0}[0,1]$ is the set of absolutely continuous functions $x(t)$ such that $x(0)=x_0$.

We also use the standard notation: $I(A)$ denotes the indicator of a set A and $\overline{A}$ is the complement of a set A. By $[a]$ and $\{a\}$ we denote the integer and fractional part of a number a, respectively.

2. Large deviations principle

In the space $C[0,1]$, consider the metric

$$\rho(x,y)=\sup_{0\le t\le 1}|x(t)-y(t)|.$$

Theorem 1. *Assume that*

1)
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{1}{\sigma^2(x)}\,dx=\lim_{T\to\infty}\frac{1}{T}\int_{-T}^0\frac{1}{\sigma^2(x)}\,dx=1/a,$$
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{b(x)}{\sigma^2(x)}\,dx=\lim_{T\to\infty}\frac{1}{T}\int_{-T}^0\frac{b(x)}{\sigma^2(x)}\,dx=B;$$

2) *for all* $c>0$,
$$\lim_{n\to\infty}\sup_{|u|\le c}\frac{\varphi(n)}{\sqrt{n}}\sqrt{\left|\int_0^{un}\theta_1(s)\,ds\right|+\left|\int_0^{un}\theta_2(s)\,ds\right|}=0,$$

where
$$\theta_1(s)=\frac{(b(s)-Ba)}{\sigma^2(s)},\qquad \theta_2(s)=1-\frac{a}{\sigma^2(s)}.$$

Then the family of measures $\mathsf{P}_n(A)=\mathsf{P}\{\eta_n(\cdot)\in A\}$, $A\in\mathbf{B}(C[0,1],\rho)$, *admits the large deviations principle in the space* $(C[0,1],\rho)$ *with normalizing functions*

$$\psi(n)=\varphi^2(n)$$

and rate functional

$$S(x)=\begin{cases}\frac{1}{2a}\int_0^1(\dot x(t)-Ba)^2\,dt, & \text{if } x(\cdot)\in AC_{x_0}[0,1],\\ +\infty, & \text{otherwise.}\end{cases}\tag{3}$$

The proof of Theorem 1 is given below after some auxiliary results.

Lemma 1 ([4])**.** *Consider the following continuous martingale:*

$$x(t)=\int_0^t g(s,\omega)\,dw(s),$$

defined in a stochastic basis $(\Omega,\mathfrak{F},\mathfrak{F}_t,\mathsf{P})$.

Let $g(t,\omega)$ be an $\mathfrak{F}_t$-progressively measurable stochastic process such that

$$\sup_{t\in[0,1]} g^2(t,\omega) \le \lambda$$

almost surely. Then

$$\mathsf{P}\left(\sup_{t\in[0,1]} |x(t)| \ge c\right) \le 2\exp\left\{-\frac{c^2}{2\lambda}\right\}$$

for all $c > 0$.

Lemma 2. *Let a stochastic process $x(t)$ defined in a stochastic basis $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ be given by the equality*

$$x(t) = \int_0^t g(\omega, y(s))\, dw(s),$$

where the stochastic processes $g(\omega, y(t))$ and $y(t)$ are $\mathfrak{F}_t$-progressively measurable and

$$\mathsf{E}\int_0^1 g^2(\omega, y(s))\, ds < \infty.$$

For any positive constant d, put

$$U_d = \left\{\omega\colon \sup_{t\in[0,1]} |y(t)| \le d\right\}.$$

Let $\sup_{|y|\le d} g^2(y,\omega) \le \lambda$ almost surely. Then

$$\mathsf{P}\left(\left\{\sup_{t\in[0,1]} |x(t)| \ge c\right\} \cap U_d\right) \le 2\exp\left\{-\frac{c^2}{2\lambda}\right\}$$

for all $c > 0$.

Proof. We have

$$\mathsf{P}\left(\left\{\sup_{t\in[0,1]} |x(t)| \ge c\right\} \cap U_d\right) = \mathsf{P}\left(I(U_d) \sup_{t\in[0,1]} \left|\int_0^t g(y(s),\omega)\, dw(s)\right| \ge c\right).$$

Since

$$U_d \subseteq \left\{\omega\colon \sup_{t\in[0,s]} |y(t)| \le d\right\}$$

for all $s \in [0,1]$, we establish

$$\begin{aligned}
&I(U_d) \sup_{t\in[0,1]} \left|\int_0^t g(y(s,\omega),\omega)\, dw(s)\right| \\
&\quad \le I(U_d) \sup_{t\in[0,1]} \left|\int_0^t g(y(s),\omega) I\left(\sup_{v\in[0,s]} |y(v)| \le d\right) dw(s)\right| \\
&\qquad + I(U_d) \sup_{t\in[0,1]} \left|\int_0^t g(y(s),\omega) I\left(\sup_{v\in[0,s]} |y(v)| > d\right) dw(s)\right| \\
&\quad = I(U_d) \sup_{t\in[0,1]} \left|\int_0^t g(y(s),\omega) I\left(\sup_{v\in[0,s]} |y(v)| \le d\right) dw(s)\right|.
\end{aligned}$$

Further,

$$I(U_d) \sup_{t\in[0,1]} \left|\int_0^t g(y(s,\omega),\omega)\, dw(s)\right| \le \sup_{t\in[0,1]} \left|\int_0^t g(y(s),\omega) I\left(\sup_{v\in[0,s]} |y(v)| \le d\right) dw(s)\right|$$

almost surely. Applying Lemma 1 we obtain

$$\begin{aligned}
&\mathsf{P}\left(\left\{\sup_{t\in[0,1]}|x(t)|\ge c\right\}\cap U_d\right)\\
&\quad\le \mathsf{P}\left(I(U_d)\sup_{t\in[0,1]}\left|\int_0^t g(y(s),\omega)I\left(\sup_{v\in[0,s]}|y(v)|\le d\right)dw(s)\right|\ge c\right)\\
&\quad\le \mathsf{P}\left(\sup_{t\in[0,1]}\left|\int_0^t g(y(s),\omega)I\left(\sup_{v\in[0,s]}|y(v)|\le d\right)dw(s)\right|\ge c\right)\le 2\exp\left\{-\frac{c^2}{2\lambda}\right\}.\ \square
\end{aligned}$$

In what follows we need a result of the paper [5]. To state this result (Lemma 3 below), consider the sequence of continuous semimartingales $X_\varepsilon(t)$, $t\in[0,1]$, defined in a stochastic basis $(\Omega,\mathfrak{F},\mathfrak{F}_t,\mathsf{P})$,

$$X_\varepsilon(t)=x_0+\int_0^t b_\varepsilon(s)\,ds+\varepsilon^k\int_0^t\alpha_\varepsilon(s)\,dw(s),$$

where $b_\varepsilon(t)$ and $\alpha_\varepsilon(t)$ are $\mathfrak{F}_t$-compatible stochastic processes with $\int_0^t|b_\varepsilon(s)|\,ds<\infty$ and $\int_0^t\alpha_\varepsilon^2(s)\,ds<\infty$ almost surely, ε is a small parameter, and k is a positive number.

Lemma 3 (Theorem A.1 in [5])**.** *Let $0<c_1\le\alpha_\varepsilon^2(s)\le c_2$ and $|b_\varepsilon(s)|\le c_3$. Assume that there are constants $\mathbf{a}>0$ and $\mathbf{b}$ such that*

$$\lim_{\varepsilon\to0}\varepsilon^{2k}\ln\mathsf{P}\left(\sup_{t\in[0,1]}\left|\int_0^t(b_\varepsilon(s)-\mathbf{b})\,ds\right|>\delta\right)=-\infty,$$

$$\lim_{\varepsilon\to0}\varepsilon^{2k}\ln\mathsf{P}\left(\sup_{t\in[0,1]}\left|\int_0^t(\alpha_\varepsilon^2(s)-\mathbf{a})\,ds\right|>\delta\right)=-\infty$$

for all $\delta>0$.

Then the family of measures $\mathsf{P}_\varepsilon(A)=\mathsf{P}\{X_\varepsilon(\cdot)\in A\}$, $A\in\mathbf{B}(C[0,1],\rho)$, admits the large deviations principle in the space $(C[0,1],\rho)$ with the normalizing function $1/\varepsilon^{2k}$ and rate functional

$$S(x)=\begin{cases}\frac{1}{2\mathbf{a}}\int_0^1(\dot x(t)-\mathbf{b})^2\,dt, & \text{if } x(\cdot)\in AC_{x_0}[0,1],\\ +\infty, & \text{otherwise.}\end{cases}$$

Remark. The result of Lemma 3 can also be obtained from Corollary 4.3.8 in [6]. A more general result is proved in [6] in terms of the theory of idempotent measures and maxingales.

Proof of Theorem 1. Denote by φ^{-1} the inverse function for φ. Let $\varepsilon_n=(1/\varphi(n))^{1/k}$ and

$$b_{\varepsilon_n}(s)=b\left(\varphi^{-1}\left(1/\varepsilon_n^k\right)\eta_n(s)\right),\qquad \alpha_{\varepsilon_n}(s)=\sigma\left(\varphi^{-1}\left(1/\varepsilon_n^k\right)\eta_n(s)\right).$$

According to Lemma 3, one only needs to check the following conditions:

$$\lim_{n\to\infty}\frac{1}{\varphi^2(n)}\ln\mathsf{P}\left(\sup_{t\in[0,1]}\left|\int_0^t(b(n\eta_n(s))-aB)\,ds\right|>\varepsilon\right)=-\infty,\tag{4}$$

$$\lim_{n\to\infty}\frac{1}{\varphi^2(n)}\ln\mathsf{P}\left(\sup_{t\in[0,1]}\left|\int_0^t(\sigma^2(n\eta_n(s))-a)\,ds\right|>\varepsilon\right)=-\infty.\tag{5}$$

First we check condition (4). Put

$$H_1(x)=\int_0^x\int_0^r\theta_1(s)\,ds\,dr.$$

Applying Itô's formula to the function $H_1(n\eta_n(t))$, we get

$$\begin{aligned} H_1(n\eta_n(t)) &= H_1(nx_0) + n\int_0^t \left(\int_0^{n\eta_n(s)} \theta_1(r)\,dr\right) b(n\eta_n(s))\,ds \\ &\quad + \frac{n^2}{2\varphi^2(n)}\int_0^t (b(n\eta_n(s)) - aB)\,ds \\ &\quad + \frac{n}{\varphi(n)}\int_0^t \left(\int_0^{n\eta_n(s)} \theta_1(r)\,dr\right) \sigma(n\eta_n(s))\,dw(s). \end{aligned} \tag{6}$$

It follows from (6) that

$$\int_0^t (b(n\eta_n(s)) - aB)\,ds = \frac{2\varphi^2(n)}{n^2} H_1(n\eta_n(t)) - \frac{2\varphi^2(n)}{n^2} H_1(nx_0) - Y_n^1(t) - Y_n^2(t), \tag{7}$$

where

$$Y_n^1(t) = \frac{2\varphi^2(n)}{n}\int_0^t \left(\int_0^{n\eta_n(s)} \theta_1(r)\,dr\right) b(n\eta_n(s))\,ds,$$

$$Y_n^2(t) = \frac{2\varphi(n)}{n}\int_0^t \left(\int_0^{n\eta_n(s)} \theta_1(r)\,dr\right) \sigma(n\eta_n(s))\,dw(s). \tag{8}$$

Now we estimate the terms on the right hand side. First,

$$\begin{aligned} \left|\frac{2\varphi^2(n)}{n^2} H_1(n\eta_n(t))\right| &= \frac{2\varphi^2(n)}{n}\left|\int_0^{\eta_n(t)}\int_0^{nu} \theta_1(s)\,ds\,du\right| \\ &\le \sup_{t\in[0,1]} |\eta_n(t)| \frac{2\varphi^2(n)}{n} \sup_{|u|\le \sup_{t\in[0,1]}|\eta_n(t)|} \left|\int_0^{nu} \theta_1(s)\,ds\right|. \end{aligned} \tag{9}$$

Analogously,

$$\begin{aligned} \left|\frac{2\varphi^2(n)}{n^2} H_1(nx_0)\right| &= \frac{2\varphi^2(n)}{n}\left|\int_0^{x_0}\int_0^{nu} \theta_1(s)\,ds\,du\right| \\ &\le \sup_{|u|\le|x_0|} \frac{2\varphi^2(n)|x_0|}{n}\left|\int_0^{nu} \theta_1(s)\,ds\right|. \end{aligned}$$

Using condition (2) we get

$$\begin{aligned} |Y_n^1(t)| &= \left|\frac{2\varphi^2(n)}{n}\int_0^t \left(\int_0^{n\eta_n(s)} \theta_1(r)\,dr\right) b(n\eta_n(s))\,ds\right| \\ &\le \frac{2\varphi^2(n)\lambda}{n} \sup_{|u|\le \sup_{t\in[0,1]}|\eta_n(t)|} \left|\int_0^{nu} \theta_1(s)\,ds\right|. \end{aligned} \tag{10}$$

For $c > |x_0| + \lambda$, put

$$B_c = \left\{\omega\colon \sup_{t\in[0,1]} |\eta_n(t)| > c\right\}.$$

Applying Lemma 1 we conclude that

$$
\begin{aligned}
\mathsf{P}(B_c) &= \mathsf{P}\left(\sup_{t\in[0,1]}\left|x_0 + \int_0^t b(n\eta_n(s))\,ds + \frac{1}{\varphi(n)}\int_0^t \sigma(n\eta_n(s))\,dw(s)\right| > c\right) \\
&\le \mathsf{P}\left(\sup_{t\in[0,1]}\left|\frac{1}{\varphi(n)}\int_0^t \sigma(n\eta_n(s))\,dw(s)\right| > c - |x_0| - \lambda\right) \\
&\le 2\exp\left\{-\frac{\varphi^2(n)(c-|x_0|-\lambda)^2}{2\lambda}\right\}.
\end{aligned}
\tag{11}
$$

Now, for $\delta > 0$, let

$$
D_\delta = \left\{\omega\colon \sup_{t\in[0,1]} |Y_n^2(t)| > \delta\right\} \cap \overline{B}_c
$$

and

$$
J_{n,c} = \sup_{|u|\le c}\left|\int_0^{nu} \theta(r)\,dr\right|.
$$

Applying Lemma 2 together with (8) we obtain

$$
\mathsf{P}(D_\delta) \le 2\exp\left\{-\frac{n^2\delta^2}{8\varphi^2(n)\lambda J_{n,c}^2}\right\}.
$$

Assumption 2 of Theorem 1 implies that

$$
\lim_{n\to\infty} \frac{\varphi^4(n)}{n^2} J_{n,c}^2 = 0.
$$

Thus we conclude that, given an arbitrary $\delta > 0$, there exists $n(\delta, c, \lambda)$ such that

$$
\begin{aligned}
-\frac{n^2\delta^2}{8\varphi^2(n)\lambda J_{n,c}^2} &= -\frac{\varphi^2(n)\delta^2}{8\lambda\frac{\varphi^4(n)}{n^2}J_{n,c}^2} \\
&< -\frac{\varphi^2(n)(c-|x_0|-\lambda)^2}{2\lambda}
\end{aligned}
\tag{12}
$$

for all $n > n(\delta, c, \lambda)$. Denote

$$
K_1^\delta = \left\{\omega\colon \sup_{t\in[0,1]}\left|\frac{2\varphi^2(n)}{n^2}H_1(n\eta_n(t))\right| > \delta\right\},
$$

$$
K_2^\delta = \left\{\omega\colon \left|\frac{2\varphi^2(n)}{n^2}H_1(nx_0)\right| > \delta\right\},
$$

$$
K_3^\delta = \left\{\omega\colon \sup_{t\in[0,1]}\left|Y_n^1(t)\right| > \delta\right\},
$$

$$
A_\varepsilon = \left\{\omega\colon \sup_{t\in[0,1]}\left|\int_0^t (b(n\eta_n(s)) - Ba)\,ds\right| > \varepsilon\right\}.
$$

Using (7) and bounds (9)–(12) together with condition 2 of Theorem 1 we obtain for sufficiently large n that

$$
\begin{aligned}
\mathsf{P}(A_\varepsilon) &\le \mathsf{P}(A_\varepsilon \cap \overline{B}_c) + \mathsf{P}(B_c) \\
&\le \mathsf{P}\left(\left(K_1^{\varepsilon/4} \cup K_2^{\varepsilon/4} \cup K_3^{\varepsilon/4} \cup D_{\varepsilon/4}\right) \cap \overline{B}_c\right) + \mathsf{P}(B_c) \\
&\le \mathsf{P}\left(\frac{2c\varphi^2(n)}{n} \sup_{|u|\le c} \left|\int_0^{nu} \theta(s)\,ds\right| \ge \frac{\varepsilon}{4}\right) \\
&\quad + \mathsf{P}\left(\frac{2\varphi^2(n)|x_0|}{n} \sup_{|u|\le |x_0|} \left|\int_0^{nu} \theta(s)\,ds\right| \ge \frac{\varepsilon}{4}\right) \\
&\quad + \mathsf{P}\left(\frac{2\varphi^2(n)\lambda}{n} \sup_{|u|\le c} \left|\int_0^{nu} \theta_1(s)\,ds\right| \ge \frac{\varepsilon}{4}\right) + 2\exp\left\{-\frac{n^2\varepsilon^2}{128\varphi^2(n)\lambda J_{n,c}^2}\right\} \\
&\quad + 2\exp\left\{-\frac{\varphi^2(n)(c-|x_0|-\lambda)^2}{2\lambda}\right\} \\
&\le 4\exp\left\{-\frac{\varphi^2(n)(c-|x_0|-\lambda)^2}{2\lambda}\right\}.
\end{aligned}
\tag{13}
$$

Using (13) we get

$$
\lim_{n\to\infty} \frac{1}{\varphi^2(n)} \ln \mathsf{P}(A_\varepsilon) \le -\frac{(c-|x_0|-\lambda)^2}{2\lambda}
$$

for all $c > |x_0| + \lambda$. Passing to the limit as $c \to +\infty$ we prove (4).

Consider the function

$$
H_2(x) = \int_0^x \int_0^r \theta_2(s)\,ds\,dr.
$$

Applying Itô's formula to the stochastic processes $H_2(n\eta_n(t))$ and reasoning as above we prove equality (5). □

Remark. If the functions $b(x)$ and $\sigma(x)$ are bounded and periodic with period T, then

$$
\int_0^{nT} \theta_i(s)\,ds = 0, \qquad i = 1, 2.
$$

This implies that there exists a constant $l > 0$ such that

$$
\left|\int_0^{un} \theta_i(s)\,ds\right| \le l
$$

for all $u \in \mathbf{R}$. Therefore condition 2 of Theorem 1 holds if

$$
\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0.
$$

The latter condition is the same as in [1].

Consider the sequence of stochastic processes $\xi_n(t)$ being the solutions of the following stochastic differential equations:

$$
\xi_n(t) = x_0 + \int_0^t b(|n\xi_n(s)|^\gamma)\,ds + \frac{1}{\varphi(n)} \int_0^t \sigma(|n\xi_n(s)|^\gamma)\,dw(s), \tag{14}
$$

where the functions $b(x)$ and $\sigma(x)$ are periodic with period 1. Assume that condition (2) holds. Put

$$
\int_0^1 \frac{1}{\sigma^2(x)}\,dx = 1/a, \qquad \int_0^1 \frac{b(x)}{\sigma^2(x)}\,dx = B.
$$

Then the following result is valid.

Lemma 4. *Let conditions*

$$\gamma > 0, \qquad \lim_{n\to\infty} \frac{\varphi(n)}{n^{\min(\gamma/2,1/2)}} = 0$$

hold.

Then the family of measures generated by solutions of equation (14) *in the space* $(C[0,1],\rho)$ *admits the large deviations principle with the normalizing function*

$$\psi(n) = \varphi^2(n)$$

and rate functional

$$S(x) = \begin{cases} \frac{1}{2a}\int_0^1 (\dot{x}(t) - Ba)^2\,dt, & \text{if } x(\cdot) \in AC_{x_0}[0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof. We will apply Theorem 1 proved above. We are going to show that condition 1 of Theorem 1 holds. Since $\sigma(|x|^\gamma)$ and $b(|x|^\gamma)$ are even and bounded functions, we only need to show that

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T \frac{1}{\sigma^2(x^\gamma)}\,dx = 1/a, \tag{15}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T \frac{b(x^\gamma)}{\sigma^2(x^\gamma)}\,dx = B. \tag{16}$$

First we prove (16). Changing the variable $y = x^\gamma$ we get

$$\int_1^T \frac{1}{\sigma^2(x^\gamma)}\,dx = \frac{1}{\gamma}\int_1^{T^\gamma} \frac{y^{1/\gamma-1}}{\sigma^2(y)}\,dy.$$

Put

$$F(y) = \int_0^y \frac{1}{\sigma^2(x)}\,dx.$$

Since the function $\frac{1}{\sigma^2(x)}$ is periodic with period 1,

$$F(y) - \frac{1}{a}y = \int_0^y \left(\frac{1}{\sigma^2(x)} - \frac{1}{a}\right)dx = \int_0^{\{y\}} \frac{1}{\sigma^2(x)}\,dx - \frac{1}{a}\{y\}.$$

Using the restrictions imposed on $\sigma(x)$ and inequality $1/a \le \lambda$ we prove the following bound:

$$\frac{1}{a}y - \lambda\{y\} \le F(y) \le \frac{1}{a}y + \lambda\{y\}. \tag{17}$$

Integrating by parts for $T \ge 1$,

$$\frac{1}{\gamma}\int_1^{T^\gamma} \frac{y^{1/\gamma-1}}{\sigma^2(y)}\,dy = \frac{T^{1-\gamma}}{\gamma}F(T^\gamma) - \frac{1}{a\gamma} - \frac{1-\gamma}{\gamma^2}\int_1^{T^\gamma} y^{1/\gamma-2}F(y)\,dy,$$

whence

$$\frac{T}{a} - \frac{2\lambda}{\gamma}\left(T^{1-\gamma}+1\right) \le \frac{1}{\gamma}\int_1^{T^\gamma} \frac{y^{1/\gamma-1}}{\sigma^2(y)}\,dy \le \frac{T}{a} + \frac{2\lambda}{\gamma}\left(T^{1-\gamma}+1\right) \tag{18}$$

for all $T \ge 1$ by (18) and inequality $1/a \le \lambda$.

Using bound (18) we get

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T \frac{1}{\sigma^2(x^\gamma)}\,dx = \lim_{T\to\infty} \frac{1}{T\gamma}\int_1^{T^\gamma} \frac{y^{1/\gamma-1}}{\sigma^2(y)}\,dy = 1/a.$$

Equality (16) is proved. The proof of (17) is the same.

Now we show that condition 2 of Theorem 1 holds. Using bound (18) and restrictions imposed on the function $\sigma^2(x)$ we obtain for $|un| \in [0,1)$ that

$$-1 - a\lambda \le \int_0^{|un|} \left(1 - \frac{a}{\sigma^2(x^\gamma)}\right) dx \le 1 + a\lambda, \tag{19}$$

if $|un| \ge 1$, and

$$\begin{aligned} -1 - a\lambda - \frac{2\lambda a}{\gamma} - \frac{2a|un|^{1-\gamma}\lambda}{\gamma} &\le \int_0^{|un|} \left(1 - \frac{a}{\sigma^2(x^\gamma)}\right) dx \\ &\le 1 + a\lambda + \frac{2\lambda a}{\gamma} + \frac{2a|un|^{1-\gamma}\lambda}{\gamma}. \end{aligned} \tag{20}$$

Since the function $\sigma(|x|^\gamma)$ is even and $a \le \lambda$, relations (19) and (20) imply that

$$\begin{aligned} \left|\int_0^{un} \theta_2(x)\, dx\right| &= \left|\int_0^{|un|} \left(1 - \frac{a}{\sigma^2(|x|^\gamma)}\right) dx\right| \\ &\le 1 + \frac{\lambda^2(2+\gamma)}{\gamma} + \frac{2\lambda^2|un|^{1-\gamma}}{\gamma} I(|un| \ge 1). \end{aligned}$$

Reasoning as above we derive a similar bound for $\int_0^{un} \frac{(b(|x|^\gamma) - Ba)}{\sigma^2(|x|^\gamma)}\, dx$:

$$\left|\int_0^{un} \frac{(b(|x|^\gamma) - Ba)}{\sigma^2(|x|^\gamma)}\, dx\right| \le \frac{\lambda^4(2\gamma+5)}{\gamma} + \frac{6\lambda^4|un|^{1-\gamma}}{\gamma} I(|un| \ge 1). \tag{21}$$

Now (20), (21), and assumptions of Lemma 4 imply that

$$\begin{aligned} &\lim_{n\to\infty} \sup_{|u|\le c} \frac{\varphi(n)}{\sqrt{n}} \sqrt{\left|\int_0^{un} \theta_1(s)\, ds\right| + \left|\int_0^{un} \theta_2(s)\, ds\right|} \\ &\le \lim_{n\to\infty} \sup_{|u|\le c} \frac{\varphi(n)}{\sqrt{n}} \sqrt{\frac{\lambda^4(4\gamma+7)}{\gamma} + \frac{8\lambda^4|un|^{1-\gamma} I(|un| \ge 1)}{\gamma}} \\ &\le \lim_{n\to\infty} \left(\frac{\varphi(n)}{\sqrt{n}} \sqrt{\frac{\lambda^4(4\gamma+15)}{\gamma}} I(\gamma \ge 1) + \frac{\varphi(n)}{n^{\gamma/2}} \sqrt{\frac{9\lambda^4 c^{1-\gamma}}{\gamma}} I(0 < \gamma < 1)\right) \\ &= 0. \end{aligned}$$

□

3. Examples

Below are some examples of stochastic equations whose solutions admit the large deviations principle.

Example 1. Let

$$\eta_n(t) = \frac{1}{\varphi(n)} \int_0^t \sqrt{2 + \frac{1}{1 + (n\eta_n(s))^2}}\, dw(s).$$

In this case, $a = 2$,

$$\theta_2(s) = \frac{1}{2s^2+3}, \qquad \left|\int_0^{un} \theta_2(s)\, ds\right| \le \frac{\pi}{4\sqrt{6}}.$$

Condition 2 of Theorem 1 holds if

$$\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0.$$

The rate functional is given by

$$S(x) = \begin{cases} \frac{1}{4}\int_0^1 \dot{x}^2(t)\,dt, & \text{if } x(\cdot) \in AC_0[0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that since the sequence of functions

$$\sqrt{2 + \frac{1}{1+(nx)^2}}$$

does not converge uniformly on compact sets, Example 1 does not follow from Theorem 3.2.1 in [7].

Example 2. Consider an almost periodic diffusion coefficient

$$\eta_n(t) = \int_0^t \big(1 + \cos(n\eta_n(s))\big)\,ds + \frac{1}{\varphi(n)}\int_0^t \frac{dw(s)}{\sqrt{3 + \sin(n\eta_n(s)) + \sin(\pi n\eta_n(s))}}.$$

In this case, $a = 1/3$, $B = 3$, and

$$\theta_1(s) = 3\cos(s) + \cos(s)\sin(s) + \cos(s)\sin(\pi s), \qquad \theta_2(s) = -\frac{1}{3}(\sin(s) + \sin(\pi s)).$$

Thus

$$\sup_{u\in\mathbf{R}}\left|\int_0^{un} \theta_1(s)\,ds\right| \le 5, \qquad \sup_{u\in\mathbf{R}}\left|\int_0^{un} \theta_2(s)\,ds\right| \le \frac{8}{9}.$$

Condition 2 of Theorem 1 holds if

$$\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0.$$

The rate functional is given by

$$S(x) = \begin{cases} \frac{3}{2}\int_0^1 (\dot{x}(t) - 1)^2\,dt, & \text{if } x(\cdot) \in AC_0[0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

Example 3. Let

$$\xi_n(t) = \frac{1}{\varphi(n)}\int_0^t \frac{dw(s)}{2 + \cos\big(2\pi\sqrt{|n\xi_n(s)|}\,\big)}.$$

In this case $a = 2/9$, $\gamma = 1/2$.

Lemma 4 implies that the rate functional is given by

$$S(x) = \begin{cases} \frac{9}{4}\int_0^1 \dot{x}^2(t)\,dt, & \text{if } x(\cdot) \in AC_0[0,1], \\ +\infty, & \text{otherwise} \end{cases}$$

if

$$\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt[4]{n}} = 0.$$

Example 4. Consider a discontinuous diffusion coefficient,

$$\xi_n(t) = \frac{1}{\varphi(n)}\int_0^t \big(1 + \{n^2\xi_n^2(s)\}\big)\,dw(s).$$

In this case $a = 2$, $\gamma = 2$, and $\min(\gamma/2, 1/2) = 1/2$.

Lemma 4 implies that the rate functional is given by

$$S(x) = \begin{cases} \frac{1}{4}\int_0^1 \dot{x}^2(t)\,dt, & \text{if } x(\cdot) \in AC_0[0,1], \\ +\infty, & \text{otherwise,} \end{cases}$$

if

$$\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0.$$

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