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COMPARISON THEOREM FOR SOLUTIONS OF PARABOLIC STOCHASTIC EQUATIONS WITH AN ABSORBER

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Abstract. A comparison theorem is proved for solutions of the Cauchy problem for a quasi-linear parabolic stochastic equation. The drift and diffusion coefficients of this equation do not necessarily satisfy the Lipschitz condition. The drift coefficient is assumed to be an absorber.

1. Introduction

The theory of partial stochastic differential equations of parabolic type is one of the central topics of the modern theory of stochastic equations. The papers by Pardoux [1] and Krylov and Rozovskiĭ [2] motivated the research of solution properties of various problems for such equations. Research in this topic still continues today. Prevot [3] and Röckner obtained results concerning equations with perturbations of a white or color noise type. Radchenko [4]–[6] considered stochastic parabolic equations with integrals over random measures.

Comparison theorems are used to study properties of solutions of various problems for partial differential equations, since they allow one to describe properties of classes of solutions with the help of a properly chosen special solution. Comparison theorems are proved for a wide class of problems related to deterministic parabolic equations. The corresponding statements and possible applications can be found in the monograph [7]. Comparison theorems for solutions of partial stochastic differential equations of parabolic type are proved by several authors (see, for example, [8, Theorem 5.1], [9, Theorem 5], [10, Theorem 1], [11, Theorem 1]). One of the main assumptions used in the papers [8], [9], and [10] is the Lipschitz condition imposed on the coefficients of the equation. In contrast, the coefficients of the equations considered in the current paper do not necessarily satisfy the Lipschitz condition.

We show an application of the comparison theorem for equations with power nonlinearities. A comparison theorem is proved in [11] for an equation whose drift and diffusion coefficients are power functions of the phase variable. Moreover, it is assumed in [11] that the exponents for both coefficients, drift and diffusion, are real numbers between zero and one.

Theorem 1 proved below holds for a power diffusion coefficient whose exponent is not less than one.

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2. Definitions, notation, main result

Given a certain stochastic basis $(\Omega, \mathcal{F}, \mathsf{P}, \{\mathcal{F}_t\}_{t\ge 0})$, consider the Cauchy problem

$$(1)\quad \begin{gathered} du^{(j)}(t,x) = a\Delta u^{(j)}(t,x)\,dt + b^{(j)}\big(t,x,u^{(j)}(t,x)\big)\,dt + c\big(t,x,u^{(j)}(t,x)\big)\,dw(t), \\ t\in[0,T], \qquad x\in\mathbf{R}^n, \qquad u^{(j)}(0,x) = u_0^{(j)}(x), \qquad j=1,2. \end{gathered}$$

Here $a>0$, Δ is the Laplace operator, $w(t)$ is a standard Wiener process, and $b^{(j)}(t,x,u)$, $c(t,x,u)$, and $u_0^{(j)}(x)$ are nonrandom functions.

The following notation is used throughout the paper:

$$u_x = \frac{\partial u}{\partial x}, \qquad \nabla = \left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right), \qquad U^{(+)} = \max(0;U), \qquad U^{(-)} = \max(0;-U),$$

$\mathbf{W}_2^1(\mathbf{R}^n)$ is a standard Sobolev space, $\mathbf{L_p}(M)$ is the space of functions f such that $|f|^p$ is integrable over the set M, and $\|u\|_p^p = \int |u(x)|^p\,dx$, $\mathbf{C}(M)$ is the standard space of continuous functions on the set M, and

$$\mathbf{L} = \mathbf{C}\left([0;T];\mathbf{L_2}(\mathbf{H}\times\Omega)\right)\cap\mathbf{L_p}\left([0;T]\times\Omega;\mathbf{V}\right), \qquad p\ge 2.$$

The symbol $\mathcal{K}$ with or without indices denotes positive constants. If necessary, we indicate explicitly the parameters that a constant depends on.

Problem (1) is considered on a triple of spaces $\mathbf{V}\subset\mathbf{H}\equiv\mathbf{H}^*\subset\mathbf{V}^*$, where $\mathbf{V}$ is a Banach space and where $\mathbf{H}$ is a Hilbert space of functions of the argument $x\in\mathbf{R}^n$. We assume that the coefficients of equations and initial conditions of the problems are such that the solutions belong to the space $\mathbf{L}$. Sufficient conditions for this property are given below. A solution of an equation is understood in the sense of [2, Definition 2.1].

Theorem 1. *Assume that, for some* $\lambda>0$,

T1.1. $u_0^{(1)}(x)\le u_0^{(2)}(x)$ *for all* $x\in\mathbf{R}^n$.

T1.2. $b^{(2)}(t,x,u)\ge b^{(1)}(t,x,u)$ *for all* $(t,x,u)\in[0;T]\times\mathbf{R}^n\times\mathbf{R}^1$; *there exists a constant* $\mathcal{K}>0$ *such that* $b^{(1)}(t,x,u)-b^{(1)}(t,x,v)\le\mathcal{K}\cdot(u-v)$ *for all* $u\ge v$ *and all* $(t,x)\in[0;T]\times\mathbf{R}^n$.

T1.3. $\mathsf{E}\big(\int_0^T\big(\int|c(t,x,u^{(j)}(t,x))|e^{-\lambda|x|}\,dx\big)^2\,dt\big)^{1/2}\le\mathcal{K}(T)$, $j=1,2$.

T1.4. *The function* $|c_u(t,x,u)|^2$ *is concave with respect to the argument* u.

T1.5. $\mathsf{E}\int_0^T\int|c_u(t,x,u^{(j)}(t,x))|^2e^{-\lambda|x|}\,dx\,dt\le\mathcal{K}(T)$, $j=1,2$.

T1.6. $\sup_{0\le t\le T}\mathsf{E}\int|u^{(j)}(t,x)|e^{-\lambda|x|}\,dx\le\mathcal{K}(T)$, $j=1,2$.

Then

$$\mathsf{P}\left\{u^{(1)}(t,x)\le u^{(2)}(t,x),\forall(t,x)\in[0;T]\times\mathbf{R}^n\right\}=1.$$

Corollary 1. *Let* $u_0(x)\ge 0$. *If problem* (1) *possesses the zero solution and all the assumptions of Theorem* 1 *hold, then almost all trajectories of a solution* $u\in\mathbf{L}$ *are nonnegative functions.*

Corollary 2. *If the assumptions of Theorem* 1 *hold, then problem* (1) *has at most one solution* $u\in\mathbf{L}$.

3. Proof of the comparison theorem

Prior to the proof of Theorem 1 we consider the following auxiliary result.

Lemma 1. *Let the function* $\phi(u)$ *be defined by*

$$\phi(u)=\begin{cases}-u-\frac{\varepsilon}{2}, & u\le-\varepsilon,\\ -\frac{u^4}{2\varepsilon^3}-\frac{u^3}{\varepsilon^2}, & -\varepsilon<u<0,\\ 0, & u\ge 0,\end{cases}$$

where $\varepsilon > 0$. Then

- **Φ1.** $\phi(u)$ *is a twice continuously differentiable function with respect to u for every $\varepsilon > 0$.*
- **Φ2.** $0 \le \phi(u) \le u^{(-)}$ *and $\phi(u) > 0$ for $u < 0$ and $\phi(u) = 0$ for $u \ge 0$.*
- **Φ3.** $-1 \le \phi_u(u) < 0$ *for $u < 0$, $\phi_u(u) = 0$ for $u \ge 0$, and $0 \le u\phi_u(u) \le 3\phi(u)$.*
- **Φ4.** $0 < \phi_{uu}(u) \le \frac{3}{2\varepsilon}$ *for $u \in (-\varepsilon; 0)$ and $\phi_{uu}(u) = 0$ for $u \notin (-\varepsilon; 0)$.*
- **Φ5.** $\phi(u)$ *converges to $u^{(-)}$ as $\varepsilon \to 0$ uniformly in $u \in \mathbf{R}^1$.*

Proof. To prove property Φ1 we differentiate the function $\phi(u)$ twice with respect to the variable u.

Now we prove Φ2. If $u \le -\varepsilon$, then $\phi(u) = -u - \frac{1}{2}\varepsilon$. Thus $-u - \frac{1}{2}\varepsilon \ge \varepsilon - \frac{1}{2}\varepsilon > 0$ and $-u - \frac{1}{2}\varepsilon < -u = u^{(-)}$.

Further, if $-\varepsilon < u < 0$, then

$$\phi(u) = -\frac{u^4}{2\varepsilon^3} - \frac{u^3}{\varepsilon^2} = -\frac{u^3}{\varepsilon^2}\left(\frac{u}{2\varepsilon} + 1\right) > 0.$$

Now we evaluate $\max_{u\in[-\varepsilon;0]}\left[\phi(u) - u^{(-)}\right]$. First we evaluate the values of the function at the end points: $\phi(-\varepsilon) - \varepsilon = -\frac{1}{2}\varepsilon$, $\phi(0) - 0 = 0$. Then we determine the critical points:

$$(\phi(u) - u^{(-)})_u = -2\left(\frac{u}{\varepsilon}\right)^3 - 3\left(\frac{u}{\varepsilon}\right)^2 + 1 = 0, \qquad u_1 = -\varepsilon, \quad u_2 = \frac{1}{2}\varepsilon \notin (-\varepsilon; 0).$$

Hence $0 < \phi(u) \le u^{(-)}$ for $-\varepsilon < u < 0$. If $u \ge 0$, then $\phi(u) = 0$ and the equality $0 \le \phi(u) \le u^{(-)}$ is obvious.

Next we prove Φ3. If $u \le -\varepsilon$, then the inequality $-1 \le \phi_u(u) < 0$ is obvious. If $u \in (-\varepsilon; 0)$, then

$$\phi_u(u) = -\frac{2u^3}{\varepsilon^3} - \frac{3u^2}{\varepsilon^2}.$$

We evaluate $\min_{u\in[-\varepsilon;0]}\phi_u(u)$ and $\max_{u\in[-\varepsilon;0]}\phi_u(u)$. The values of the function at the end points of the interval are $\phi_u(-\varepsilon) = -1$ and $\phi_u(0) = 0$. Moreover,

$$\phi_{uu}(u) = -\frac{6u}{\varepsilon^2}\left(\frac{u}{\varepsilon} + 1\right) > 0$$

for $u \in (-\varepsilon; 0)$. Thus $-1 < \phi_u(u) < 0$ for $u \in (-\varepsilon; 0)$.

Our current aim is to prove the inequality $u\phi_u(u) \le 3\phi(u)$. If $u \ge 0$, then both sides of the inequality are equal to zero and the inequality holds. Let $u \le -\varepsilon$. Then $u\phi_u(u) = -u$, $3\phi(u) = -3u - 1.5\varepsilon$, and

$$u\phi_u(u) < 3\phi(u) \Leftrightarrow -u < -3u - 1.5\varepsilon \Leftrightarrow u < -0.75\varepsilon.$$

In the case under consideration, $u \le -\varepsilon < -0.75\varepsilon$. Hence the inequality is valid for $u \le -\varepsilon$, too.

Let $-\varepsilon < u < 0$. Then

$$u\phi_u(u) = 3\left(-\frac{u^4}{2\varepsilon^3} - \frac{u^3}{\varepsilon^2}\right) - \frac{u^4}{2\varepsilon^3} \le 3\phi(u).$$

Therefore property Φ3 is proved.

The proof of property Φ4 follows after simple algebra.

Finally we prove property Φ5. Note that

$$\left|\phi(u) - u^{(-)}\right| = \begin{cases} 0.5\varepsilon, & u \le -\varepsilon, \\ \varepsilon\left|-0.5\left(\frac{u}{\varepsilon}\right)^4 - \left(\frac{u}{\varepsilon}\right)^3 + \frac{u}{\varepsilon}\right|, & -\varepsilon < u < 0, \\ 0, & u \ge 0. \end{cases}$$

Put $h(z) = -0.5z^4 - z^3 + z$. Then $-0.5 = h(-1) < h(0) = 0$. The function $h(z)$ increases for $z \in (-1; 0)$. Hence $\left|\phi(u) - u^{(-)}\right| \le 0.5\varepsilon$ for every $u \in \mathbf{R}^1$, that is, $\phi(u)$ converges to $u^{(-)}$ as $\varepsilon \to 0$ uniformly in $u \in \mathbf{R}^1$.

The proof of Lemma 1 is complete. □

Proof of Theorem 1. We apply Itô's formula to the function

$$\phi\left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) e^{-\lambda|x|}$$

and then integrate the equality over $x \in \mathbf{R}^n$ and $s \in (0; t)$:

$$\begin{aligned}
&\int \phi\left(u^{(2)}(t,x) - u^{(1)}(t,x)\right) e^{-\lambda|x|}\,dx \\
&\quad = \int \phi\left(u_0^{(2)}(x) - u_0^{(1)}(x)\right) e^{-\lambda|x|}\,dx \\
&\quad - a \int_0^t\!\!\int \left|\nabla\left(u^{(2)} - u^{(1)}\right)\right|^2 \phi_{uu}\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad + \lambda a \int_0^t\!\!\int \nabla\left(u^{(2)} - u^{(1)}\right) \cdot x|x|^{-1} \phi_u\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad + \int_0^t\!\!\int \left(b^{(2)}\left(s,x,u^{(2)}\right) - b^{(1)}\left(s,x,u^{(1)}\right)\right) \phi_u\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad + 0.5 \int_0^t\!\!\int \left|c\left(s,x,u^{(2)}\right) - c\left(s,x,u^{(1)}\right)\right|^2 \phi_{uu}\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad + \int_0^t\!\!\int \left(c\left(s,x,u^{(2)}\right) - c\left(s,x,u^{(1)}\right)\right) \phi_u\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,dw(s).
\end{aligned} \tag{2}$$

Property Φ3 and assumption T1.3 imply that

$$\mathsf{E}\int_0^t\!\!\int \left(c\left(s,x,u^{(2)}\right) - c\left(s,x,u^{(1)}\right)\right) \phi_u\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,dw(s) = 0.$$

Since $u_0^{(2)}(x) - u_0^{(1)}(x) \ge 0$ in view of T1.1, we get $\phi\big(u_0^{(2)} - u_0^{(1)}\big) = 0$ by Φ2. Then we derive from (2) that

$$\begin{aligned}
&\mathsf{E}\int \phi\left(u^{(2)}(t,x) - u^{(1)}(t,x)\right) e^{-\lambda|x|}\,dx \\
&\quad = -a\,\mathsf{E}\int_0^t\!\!\int \left|\nabla\left(u^{(2)} - u^{(1)}\right)\right|^2 \phi_{uu}\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad + \lambda a\,\mathsf{E}\int_0^t\!\!\int \nabla\left(u^{(2)} - u^{(1)}\right) \cdot x|x|^{-1} \phi_u\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad + \mathsf{E}\int_0^t\!\!\int \left(b^{(2)}\left(s,x,u^{(2)}\right) - b^{(1)}\left(s,x,u^{(1)}\right)\right) \phi_u\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad + 0.5\,\mathsf{E}\int_0^t\!\!\int \left|c\left(s,x,u^{(2)}\right) - c\left(s,x,u^{(1)}\right)\right|^2 \phi_{uu}\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds \\
&\quad = -aA + \lambda a\Lambda + B + 0.5C.
\end{aligned} \tag{3}$$

Now we estimate every term on the right-hand side of (3).

First we deal with A. Property Φ4 implies that $A \ge 0$ and $-aA \le 0$.

Then we estimate Λ. Integrating by parts,

$$\int \nabla \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) \cdot x|x|^{-1} \phi_u \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) e^{-\lambda|x|}\, dx$$

$$= -\int \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) \nabla \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) x|x|^{-1} \times \phi_{uu} \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) e^{-\lambda|x|}\, dx$$

$$- 2\sum_{i=1}^{n} \int \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) \times \phi_u \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right)\Big|_{x_i=0} dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_n$$

$$+ \lambda \int \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) \phi_u \left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) e^{-\lambda|x|}\, dx$$

$$= -\Lambda_1 - \Lambda_2 + \lambda\Lambda_3.$$

Applying property $\Phi 4$ we get

$$|\Lambda_1| \le \int \left|\nabla \left(u^{(2)} - u^{(1)}\right)\right| \left(\phi_{uu} \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\right)^{0.5} \times \left|u^{(2)} - u^{(1)}\right| \left(\phi_{uu} \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\right)^{0.5} dx$$

$$\le \varepsilon \int \left|\nabla \left(u^{(2)} - u^{(1)}\right)\right|^2 \phi_{uu} \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\, dx + \varepsilon^{-1} \int \left|u^{(2)} - u^{(1)}\right|^2 \phi_{uu} \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\, dx$$

$$\le \varepsilon \int \left|\nabla \left(u^{(2)} - u^{(1)}\right)\right|^2 \phi_{uu} \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\, dx + \frac{3\varepsilon}{2\varepsilon} \int e^{-\lambda|x|}\, dx$$

for all $\varepsilon > 0$. Now $\Phi 3$ implies that $-\Lambda_2 \le 0$.

Also, property $\Phi 3$ implies the inequality

$$\Lambda_3 \le 3 \int \phi \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\, dx.$$

As a result,

$$\Lambda \le \varepsilon\, \mathsf{E} \int_0^t \int \left|\nabla \left(u^{(2)} - u^{(1)}\right)\right|^2 \phi_{uu} \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\, dx\, ds + 3\lambda\, \mathsf{E} \int_0^t \int \phi \left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\, dx\, ds + \mathcal{K}(\lambda,\varepsilon,T)\varepsilon.$$

Next we estimate B in equality (3). If $u^{(2)} \ge u^{(1)}$, then $\phi_u \left(u^{(2)} - u^{(1)}\right) = 0$ and the integrand equals zero. If $u^{(2)} \le u^{(1)}$, then assumption T1.2 yields

$$b^{(1)}\left(t,x,u^{(2)}\right) - b^{(1)}\left(t,x,u^{(1)}\right) \ge \mathcal{K}\left(u^{(2)} - u^{(1)}\right)$$

and

$$\left(b^{(1)}\left(t,x,u^{(2)}\right) - b^{(1)}\left(t,x,u^{(1)}\right)\right) \phi_u \left(u^{(2)} - u^{(1)}\right) \le \mathcal{K}\left(u^{(2)} - u^{(1)}\right) \phi_u \left(u^{(2)} - u^{(1)}\right),$$

since $\phi_u \left(u^{(2)} - u^{(1)}\right) < 0$ for $u^{(2)} \le u^{(1)}$. Assumption T1.2 implies that

$$\left(b^{(2)}\left(t,x,u^{(2)}\right) - b^{(1)}\left(t,x,u^{(2)}\right)\right) \phi_u \left(u^{(2)} - u^{(1)}\right) \le 0$$

if $u^{(2)} \le u^{(1)}$. Applying Φ3, we obtain

$$B \le 3\mathcal{K}\,\mathsf{E}\int_0^t\int \phi(u^{(2)}(s,x) - u^{(1)}(s,x))e^{-\lambda|x|}\,dx\,ds.$$

Next we estimate C in equality (3). By the Lagrange mean value theorem,

$$C \le \mathsf{E}\int_0^t\int\int_0^1 \left|c_u\left(s,x,\theta u^{(2)} + (1-\theta)u^{(1)}\right)\right|^2 d\theta \times \left|u^{(2)} - u^{(1)}\right|^2 \phi_{uu}\left(u^{(2)} - u^{(1)}\right) e^{-\lambda|x|}\,dx\,ds.$$

Then property Φ4 implies

$$0 < \phi_{uu}\left(u^{(2)} - u^{(1)}\right) \le \frac{3}{2\varepsilon}$$

if $u^{(2)} - u^{(1)} \in (-\varepsilon;0)$ and $\phi_{uu}(u^{(2)} - u^{(1)}) = 0$ if $u^{(2)} - u^{(1)} \notin (-\varepsilon;0)$. Hence,

$$\left|u^{(2)} - u^{(1)}\right|^2 \phi_{uu}\left(u^{(2)} - u^{(1)}\right) \le 1.5\varepsilon$$

and

$$C \le 1.5\varepsilon\,\mathsf{E}\int_0^t\int\int_0^1 \left|c_u\left(s,x,\theta u^{(2)} + (1-\theta)u^{(1)}\right)\right|^2 d\theta e^{-\lambda|x|}\,dx\,ds.$$

The function $|c_u(t,x,u)|^2$ is concave with respect to the variable u by assumption T1.4, whence

$$\left|c_u\left(s,x,\theta u^{(2)} + (1-\theta)u^{(1)}\right)\right|^2 \le \theta\left|c_u\left(s,x,u^{(2)}\right)\right|^2 + (1-\theta)\left|c_u\left(s,x,u^{(1)}\right)\right|^2$$

and

$$C \le 1.5\varepsilon\,\mathsf{E}\int_0^t\int \left[\left|c_u\left(s,x,u^{(2)}\right)\right|^2 + \left|c_u\left(s,x,u^{(1)}\right)\right|^2\right] e^{-\lambda|x|}\,dx\,ds.$$

Then assumption T1.5 yields the inequality $C \le \mathcal{K}(T)\varepsilon$.

Substituting these bounds into equality (3) and choosing $\varepsilon < \lambda^{-1}$ we get

$$\mathsf{E}\int \phi\left(u^{(2)}(t,x) - u^{(1)}(t,x)\right) e^{-\lambda|x|}\,dx \le \mathcal{K}_1\,\mathsf{E}\int_0^t\int \phi\left(u^{(2)}(s,x) - u^{(1)}(s,x)\right) e^{-\lambda|x|}\,dx\,ds + \mathcal{K}_2\varepsilon.$$

An application of Gronwall's lemma implies that

$$0 \le \mathsf{E}\int \phi\left(u^{(2)}(t,x) - u^{(1)}(t,x)\right) e^{-\lambda|x|}\,dx \le \mathcal{K}_3(\lambda,\varepsilon,T)\varepsilon.$$

We pass to the limit in this inequality as $\varepsilon \to 0$. By property Φ5, the function $\phi(u)$ converges to $u^{(-)}$ as $\varepsilon \to 0$. By property Φ2 we have $0 \le \phi(u) \le u^{(-)}$. The function

$$\left(u^{(2)}(t,x) - u^{(1)}(t,x)\right)^{(-)} e^{-\lambda|x|}$$

is integrable over $x \in \mathbf{R}^n$ for every $t \in [0;T]$ in view of assumption T1.6. By the Lebesgue theorem, one can pass to the limit under the sign of the mathematical expectation in the integral. As a result, we get

$$\mathsf{E}\int \left(u^{(2)}(t,x) - u^{(1)}(t,x)\right)^{(-)} e^{-\lambda|x|}\,dx = 0,$$

whence we conclude that $\left(u^{(2)}(t,x) - u^{(1)}(t,x)\right)^{(-)} = 0$ for all $(t,x) \in [0;T] \times \mathbf{R}^n$ with probability one. This means that

$$\mathsf{P}\left\{u^{(1)}(t,x) \le u^{(2)}(t,x), \forall (t,x) \in [0;T] \times \mathbf{R}^n\right\} = 1.$$

Theorem 1 is proved. □

4. Some applications of the comparison theorem

Below are some examples of how the comparison theorem can be used to study the dynamics of solutions of the Cauchy problem for semi-linear partial stochastic differential equations of parabolic type. It is worth mentioning that the coefficients of the equations in all examples below do not satisfy the Lipschitz condition and thus the results of [8, Theorem 5.1], [9, Theorem 5], [10, Theorem 1] cannot be applied.

Example 1. Consider the following problem:

$$\begin{gathered} du(t,x) = a u_{xx}(t,x)\,dt + b|u(t,x)|^{\beta-1}u(t,x)\,dt + c|u(t,x)|^{\gamma-1}u(t,x)\,dw(t), \\ t \in [0;T], \qquad x \in \mathbf{R}^1, \qquad u(0,x) = u_0(x) \ge 0. \end{gathered} \tag{4}$$

Here $a > 0$, $b < 0$, $c \ne 0$, $\beta > 0$, and $\gamma \ge 1$. Along with (4) consider the problem

$$\begin{gathered} du(t) = b|u(t)|^{\beta-1}u(t)\,dt + c|u(t)|^{\gamma-1}u(t)\,dw(t), \\ t \in [0;T], \qquad u(0) = u_0 > 0. \end{gathered} \tag{5}$$

The process $u(t)$, being a solution of problem (5), is a solution of problem (4) and, in addition, is space homogeneous. This allows us to compare the dynamics of the space inhomogeneous solution of problem (4) with that of the space homogeneous solution of the same problem.

We are going to prove that problem (4) possesses a solution with the required properties.

Theorem 2. *Assume that*

T2.1. $a > 0$, $b < 0$, $c \ne 0$, $\beta > 2\gamma - 1$, $\gamma > 1$.
T2.2. $\|u_0\|_2 < +\infty$.

Then there exists a unique solution $u(t)$ of problem (4) *such that*

$$\begin{aligned} u \in\ & \mathbf{L_1}\left(\Omega; \mathbf{C}\left([0;T]; \mathbf{L_2}\left(\mathbf{R}^1\right)\right)\right) \cap \mathbf{L_2}\left(\Omega \times [0;T]; \mathbf{W_2^1}\left(\mathbf{R}^1\right)\right) \cap \mathbf{L_{2\gamma}}\left(\Omega \times [0;T] \times \mathbf{R}^1\right) \\ & \cap \mathbf{L_{\beta+1}}\left(\Omega \times [0;T] \times \mathbf{R}^1\right). \end{aligned}$$

Proof. The proof of the theorem reduces to checking the assumptions of Theorem 4.1 in [1]. We choose the function spaces as follows:

$$\mathbf{H} = \mathbf{L_2}(\mathbf{R}^1), \qquad \mathbf{V_1} = \mathbf{W_2^1}(\mathbf{R}^1), \qquad \mathbf{V_2} = \mathbf{L_{\beta+1}}(\mathbf{R}^1), \qquad \mathbf{V_3} = \mathbf{L_{2\gamma}}(\mathbf{R}^1),$$
$$\mathbf{V} = \mathbf{V_1} \cap \mathbf{V_2} \cap \mathbf{V_3}.$$

Since $\beta > 1$ and $\gamma > 1$, the Sobolev–Kondrashov–Il'in embedding theorem [12] implies that the space $\mathbf{W_2^1}(\mathbf{R}^1)$ is embedded into the spaces $\mathbf{L_{\beta+1}}(\mathbf{R}^1)$ and $\mathbf{L_{2\gamma}}(\mathbf{R}^1)$. Hence $\mathbf{V} = \mathbf{W_2^1}(\mathbf{R}^1)$. In the case under consideration,

$$\mathbf{L} = \mathbf{L_1}\left(\Omega; \mathbf{C}\left([0;T]; \mathbf{L_2}(\mathbf{R}^1)\right)\right) \cap \mathbf{L_2}\left([0;T] \times \Omega; \mathbf{W_2^1}(\mathbf{R}^1)\right).$$

We define the operators as

$$A_1(u) = -a u_{xx}, \qquad A_2(u) = -b|u|^{\beta-1}u, \qquad A_3(u) = 0,$$
$$B_1(u) = 0, \qquad B_2(u) = 0, \qquad B_3(u) = -c|u|^{\gamma-1}u.$$

The method described in Example 5.2 of [1, p. 133] can be used without any change to check the assumption of Theorem 4.1 [1, p. 126]. We omit this part of the proof. □

Corollary 3. *If all the assumptions of Theorem* 2 *hold and* $\gamma \geq 1.5$*, then all the assumptions of Theorem* 1 *hold for any solution of problem* (4)*, as well.*

Proof. Condition T1.1 is satisfied by an appropriate choice of the initial functions. We choose all of them to be nonnegative. For such a choice of the initial functions, almost all trajectories of the corresponding solutions of problem (4) are nonnegative in view of Corollary 1. This means that the assumptions of Theorem 1 should be checked only for $u \geq 0$.

To check condition T1.2, put

$$b^{(1)}(t,x,u) = b^{(2)}(t,x,u) = bu^{\beta}.$$

Since $b < 0$, we have

$$b(u) - b(v) = b\left(u^{\beta} - v^{\beta}\right) \leq 0 \leq u - v$$

for $0 \leq v \leq u$. Therefore condition T1.2 is valid.

Condition T1.3 follows from $u \in \mathbf{L}_{\mathbf{2}\gamma}\left(\Omega \times [0;T] \times \mathbf{R}^1\right)$ by Theorem 2.

Condition T1.4 also holds, since

$$|c_u(t,x,u)|^2 = |c|^2 \gamma u^{2(\gamma-1)}$$

is concave for $\gamma \geq 1.5$.

Next we check condition T1.5. By the Young inequality,

$$\begin{aligned} &\mathsf{E}\int_0^T \int |c_u(t,x,u(t,x))|^2 e^{-\lambda|x|}\,dx\,ds \\ &\quad= c^2\gamma^2\,\mathsf{E}\int_0^T \int |u(s,x)|^{2(\gamma-1)} e^{-\lambda|x|}\,dx\,ds \\ &\quad\leq c^2\gamma^2\,\mathsf{E}\int_0^T \int |u(s,x)|^{2\gamma}\,dx\,ds + c^2\gamma^2 T\int e^{-\lambda|x|}\,dx \\ &\quad\leq \mathcal{K}(T). \end{aligned}$$

The latter inequality follows from $u \in \mathbf{L}_{\mathbf{2}\gamma}\left(\Omega \times [0;T] \times \mathbf{R}^1\right)$.

Condition T1.6 follows from $u \in \mathbf{L_1}\left(\Omega; \mathbf{C}\left([0;T]; \mathbf{L_2}\left(\mathbf{R}^1\right)\right)\right)$.

Corollary 3 is proved. □

We are going to prove that problem (5) possesses a solution with the required properties.

Theorem 3. *Assume that*

T3.1. $b < 0$, $c \neq 0$, $\beta > 2\gamma - 1$, $\gamma > 1$.

T3.2. $u_0 > 0$.

Then there exists a unique solution $u(t)$ *of problem* (5) *such that almost all its trajectories are continuous and*

$$\mathsf{E}\max_{t\in[0;T]} |u(t)|^2 < +\infty, \qquad \mathsf{E}\int_0^T |u(s)|^{\beta+1}\,ds < +\infty.$$

Proof. We apply Theorem 2.1 [1, p. 93]. The function spaces are chosen as follows: $\mathbf{V} = \mathbf{H} = \mathbf{H}^* = \mathbf{V}^* = \mathbf{R}^1$. In this case, the operators are given by

$$A(t,u) = -b|u|^{\beta-1}u, \qquad B(t,u) = -c|u|^{\gamma-1}u.$$

The method to check the assumptions of Theorem 2.1 [1, p. 93] is literally the same as that used in the proof of the preceding theorem and thus we omit it. □

Corollary 4. *If the assumptions of Theorem* 3 *hold, then the assumptions of Theorem* 1 *hold for any solution of problem* (5)*.*

Corollary 4 is proved similarly to Corollary 3.

Our current aim is to study the dynamics of the process $u(t)$ being a solution of problem (5) and, at the same time, a space homogeneous solution of problem (4). Put

$$\tau_0 = \inf\{t \ge 0\colon u(t) = 0\}, \qquad \tau_\infty = \inf\{t \ge 0\colon u(t) = +\infty\}, \qquad \tau = \tau_0 \wedge \tau_\infty.$$

Theorem 4. *If all the assumptions of Theorem* 3 *hold, then*

$$\mathsf{P}\{\tau = +\infty\} = 1, \qquad \mathsf{P}\left\{\lim_{t\to+\infty} u(t) = 0\right\} = 1.$$

Proof. We apply Theorems 3.1 and 3.2 of [13]. In the case under consideration, the functions $s(u)$ and $k(u)$ are given by

$$s(u) = \exp\left(Qu_0^{\beta-2\gamma+1}\right)\int_{u_0}^{u}\exp\left(-Qy^{\beta-2\gamma+1}\right)dy,$$

$$k(u) = \frac{2}{c^2}\int_{u_0}^{u}\exp\left(-Qy^{\beta-2\gamma+1}\right)\int_{u_0}^{y} z^{-2\gamma}\exp\left(Qz^{\beta-2\gamma+1}\right)dz\,dy,$$

where

$$Q = \frac{2b}{c^2(\beta-2\gamma+1)} < 0.$$

Since $u_0 > 0$ and $Q < 0$, we have $s(0) \ne -\infty$, $s(+\infty) = +\infty$. According to Theorem 3.1(2) of [13]

$$\mathsf{P}\left\{\lim_{t\uparrow\tau} u(t) = 0\right\} = \mathsf{P}\left\{\sup_{0\le t<\tau} u(t) < +\infty\right\} = 1.$$

To prove that τ is infinite, we use the function $k(u)$.

Since $Q < 0$ and $\gamma > 1$,

$$\int_{u_0}^{y} z^{-2\gamma}\exp\left(Qz^{\beta-2\gamma+1}\right)dz = O\left(y^{1-2\gamma}\right)$$

as $y \downarrow 0$. Thus $k(0) = -\infty$.

Since

$$\int_{u_0}^{y} z^{-2\gamma}\exp\left(Qz^{\beta-2\gamma+1}\right)dz = O(1)$$

as $y \to +\infty$ and since

$$\lim_{y\to+\infty}\exp\left(-Qy^{\beta-2\gamma+1}\right) = +\infty,$$

we obtain $k(+\infty) = +\infty$. According to Theorem 3.2(1) of [13], $\mathsf{P}\{\tau = +\infty\} = 1$.

Theorem 4 is proved. □

We have proved that a space homogeneous solution of problem (4) exists and almost all its trajectories tend to zero as $t \to +\infty$ but the value 0 is not attainable. The above results together with Theorem 1 allow us to study the dynamics of space non-homogeneous solutions of problem (4).

If $a > 0$, $b < 0$, $c \ne 0$, $\beta > 2\gamma - 1$, $\gamma \ge 2$, $0 \le u_0(x) \le u_0$, and $\|u_0(\cdot)\|_2 < +\infty$, then Theorem 1 implies

$$\mathsf{P}\left\{0 \le u(t,x) \le u(t), \forall (t,x) \in [0;T]\times\mathbf{R}^1\right\} = 1, \qquad \forall T > 0.$$

Then we deduce from Theorem 4 that almost all trajectories of every solution of problem (4) tend to zero as $t \to +\infty$ for all $x \in \mathbf{R}^1$.

Example 2. Below is another problem which Theorem 1 can be applied to. This example differs from the preceding example, since the drift coefficient contains two nonlinear terms of different orders and one of them is a source, while the other one is an absorber. Theorem 1 helps to study the dynamics of a solution even in this case.

Consider the following Cauchy problem:

$$\begin{aligned} du(t,x) &= au_{xx}(t,x)\,dt + \left(b_1u^2(t,x) - b_2u^3(t,x)\right)\,dt + cu^2(t,x)\,dw(t), \\ &\quad t\in[0;T], \qquad x\in\mathbf{R}^1, \qquad u(0,x)=u_0(x)\ge 0. \end{aligned} \tag{6}$$

Here $a>0$, $b_1>0$, $b_2>0$, and $c\neq 0$.

We are going to show that Theorem 1 can be applied to solutions of problem (6). We prove that problem (6) possesses a solution with the required properties.

Theorem 5. *Assume that*

T5.1. $a>0$, $b_2>b_1>0$, $3(b_2-b_1)>2c^2$, $c\neq 0$.
T5.2. $\|u_0\|_2<+\infty$.

Then there exists a unique solution $u(t)$ *of problem* (6) *such that*

$$\begin{aligned} u &\in \mathbf{L_1}\left(\Omega;\mathbf{C}\left([0;T];\mathbf{L_2}\left(\mathbf{R}^1\right)\right)\right)\cap\mathbf{L_2}\left(\Omega\times[0;T];\mathbf{W_2^1}\left(\mathbf{R}^1\right)\right)\cap\mathbf{L_3}\left(\Omega\times[0;T]\times\mathbf{R}^1\right) \\ &\quad \cap\mathbf{L_4}\left(\Omega\times[0;T]\times\mathbf{R}^1\right). \end{aligned}$$

Proof. We apply Theorem 4.1 of [1, p. 126]. Below we check all the assumptions of this theorem.

The function spaces are chosen as follows:

$$\mathbf{V_1}=\mathbf{W_2^1}(\mathbf{R}^1),\qquad \mathbf{V_2}=\mathbf{L_3}(\mathbf{R}^1),\qquad \mathbf{V_3}=\mathbf{L_4}(\mathbf{R}^1),\qquad \mathbf{V}=\mathbf{V_1}\cap\mathbf{V_2}\cap\mathbf{V_3},$$
$$\mathbf{H}=\mathbf{L_2}(\mathbf{R}^1).$$

The operators are defined by

$$A_1(u)=-au_{xx},\qquad A_2(u)=-b_1u,\qquad A_3(u)=b_2u^3,$$
$$B_1(u)=0,\qquad B_2(u)=0,\qquad B_3(u)=-cu^2.$$

Conditions (4.1)–(4.5) of [1, p. 125] are obvious if $p_1=2$, $p_2=3$, $p_3=4$.

Now we check condition (4.6) of [1, p. 125]. In the case under consideration, this condition is written as

$$\begin{aligned} &2a\|u_x\|_2^2-2b_1\|u\|_3^3+2b_2\|u\|_4^4+\lambda\|u\|_2^2+\nu \\ &\qquad \ge \alpha\left(\|u_x\|_2^2+\|u\|_2^2+\|u\|_3^3+\|u\|_4^4\right)+c^2\|u\|_4^4, \end{aligned} \tag{7}$$

where α, λ, and ν are some positive numbers. Applying the Young inequality for all $\varepsilon>0$, we get

$$\begin{aligned} &2a\|u_x\|_2^2-2b_1\|u\|_3^3+2b_2\|u\|_4^4+\lambda\|u\|_2^2+\nu \\ &\quad \ge \alpha\left(\|u_x\|_2^2+\|u\|_2^2+\|u\|_3^3+\|u\|_4^4\right)+c^2\|u\|_4^4+(2a-\alpha)\|u_x\|_2^2+(\lambda-\alpha)\|u\|_2^2 \\ &\qquad +\left(2b_2-\alpha-c^2-\varepsilon(2b_1+\alpha)\right)\|u\|_4^4+\nu-\varepsilon^{-3}(2b_1+\alpha). \end{aligned}$$

Choosing

$$\lambda\ge\alpha,\qquad \alpha<2a\wedge(2b_2-c^2),\qquad \varepsilon\le(2b_2-c^2-\alpha)/(2b_1+\alpha),\qquad \nu\ge\varepsilon^{-3}(2b_1+\alpha),$$

the latter inequality implies (7).

Therefore condition (4.6) of [1, p. 125] holds.

Finally we check the monotonicity condition (4.7) of [1, p. 125]. In the case under consideration, it is written as

$$
\begin{aligned}
&2a\|u_x - v_x\|_2^2 - 2b_1 \int \left(u^2 - v^2\right)(u-v)\,dx + 2b_2 \int \left(u^3 - v^3\right)(u-v)\,dx \\
&\qquad - c^2 \int \left(u^2 - v^2\right)^2 dx + \lambda \|u-v\|_2^2 \ge 0.
\end{aligned} \tag{8}
$$

Under the assumptions of Theorem 5

$$
2b_1\left(u^2 - v^2\right)(u-v) - 2b_2\left(u^3 - v^3\right)(u-v) + c^2(u^2 - v^2)^2 \le \frac{2}{3} b_1 (u-v)^2 \tag{9}
$$

for all $u, v \in \mathbf{R}^1$. Indeed, the latter inequality is obvious if $u - v = 0$. Let $u \ne v$. Then inequality (9) is equivalent to

$$
2b_1(u+v) - 2b_2\left(u^2 + uv + v^2\right) + c^2\left(u^2 + 2uv + v^2\right) \le \frac{2}{3} b_1. \tag{10}
$$

Given arbitrary $u, v \in \mathbf{R}^1$, we have

$$
u + v \le u^2 + uv + v^2 + \frac{1}{3}. \tag{11}
$$

Estimating the left-hand side of inequality (10) with the help of (11), we get

$$
\begin{aligned}
&2b_1(u+v) - 2b_2\left(u^2 + uv + v^2\right) + c^2\left(u^2 + 2uv + v^2\right) \\
&\quad \le \left[2(b_1 - b_2) + c^2\right] u^2 + 2\left(b_1 - b_2 + c^2\right) uv + \left[2(b_1 - b_2) + c^2\right] v^2 + \frac{2}{3} b_1.
\end{aligned} \tag{12}
$$

This quadratic form is negative definite under the assumptions of Theorem 5. Then (12) and (10) imply inequality (9). Choosing $\lambda = 2b_1/3$ we prove inequality (8).

Hence the monotonicity condition (4.7) of [1, p. 125] holds, as well.

Therefore all the assumptions of Theorem 4.1 of [1, p. 126] hold and Theorem 5 is proved. □

Now we are going to show that Theorem 1 can be applied to study the dynamics of solutions of problem (6). Since equation (6) differs from equation (4) by the drift coefficient, only condition T1.2 should be checked among the assumptions of Theorem 1. Other conditions are already checked in Example 1.

Let $b^{(1)}(t,x,u) = b^{(2)}(t,x,u) = b_1 u^2 - b_2 u^3$. Then condition T1.2 is given by

$$
b_1\left(u^2 - v^2\right) - b_2\left(u^3 - v^3\right) \le \mathcal{K}(u - v), \tag{13}
$$

where $u \ge v$ and $\mathcal{K}$ is a certain positive constant. The latter condition holds for $b_2 > b_1$. Indeed, inequality (11) implies that

$$
b_1\left(u^2 - v^2\right) - b_2\left(u^3 - v^3\right) \le \frac{b_1}{3}(u - v).
$$

Choosing $\mathcal{K} = b_1/3$ we prove inequality (13). As a result, if the assumptions of Theorem 5 hold, then Theorem 1 can be applied to solutions of problem (6).

Along with (6), consider the following problem:

$$
\begin{aligned}
du(t) &= \left(b_1 u^2(t) - b_2 u^3(t)\right) dt + c u^2(t)\, dw(t), \\
& t \in [0; T], \qquad u(0) = u_0 > 0.
\end{aligned} \tag{14}
$$

The process $u(t)$ being a solution of problem (14) is a space homogeneous solution of problem (6). This allows us to compare the dynamics of a space nonhomogeneous solution of problem (6) with the dynamics of the space homogeneous solution of the same problem.

First we prove that problem (14) possesses a solution with the required properties.

Theorem 6. *Assume that*

T6.1. $b_2 > b_1 > 0$, $3(b_2 - b_1) > 2c^2$, $c \neq 0$.

T6.2. $u_0 > 0$.

Then there exists a unique solution $u(t)$ *of problem* (14) *such that almost all its trajectories are continuous and*

$$\mathsf{E} \max_{t\in[0;T]} |u(t)|^2 < +\infty,$$

$$\mathsf{E} \int_0^T |u(s)|^4 \, ds < +\infty.$$

Proof. We apply Theorem 4.1 of [1, p. 126]. First of all we check its assumptions.

The function spaces are chosen as follows:

$$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V} = \mathbf{H} = \mathbf{H}^* = \mathbf{V}^* = \mathbf{R}^1.$$

The operators are given by

$$A_1(t,u) = -b_1 u^2, \qquad A_2 = b_2 u^3, \qquad B_1 = 0, \qquad B_2(t,u) = -cu^2.$$

Checking the rest of the assumptions of Theorem 4.1 in [1, p. 126] is the same as in the proof of Theorem 5. Theorem 6 is proved. □

Theorem 1 can be applied to solutions of problem (14). This can be proved in the same way as in the case of problem (6).

To study the dynamics of a solution of problem (14), we apply Theorem 3.1 of [13, p. 351]. The function $s(u)$ is given by

$$s(u) = \mathcal{K} \int_{u_0}^{u} y^{2b_2 c^{-2}} \exp\left(2b_1 c^{-2} y^{-1}\right) \, dy.$$

Since $b_1 > 0$, $b_2 > 0$, $c \neq 0$, and $u_0 > 0$, we get $s(0) = -\infty$ and $s(+\infty) = +\infty$. Now Theorem 3.1(1) of [13] implies that the process $u(t)$ is recurrent. This means that $\mathsf{P}\{\tau = +\infty\} = 1$ and thus the space homogeneous solution of problem (6) does not have the limit as $t \to +\infty$ with probability 1.

Now we are in position to study the dynamics of a space nonhomogeneous solution of problem (6). Let $0 < \check{u}_0 \leq u_0(x) \leq \hat{u}_0$ and let the processes $\check{u}(t)$ and $\hat{u}(t)$ be the corresponding solutions of problem (14). Applying Theorem 1,

$$\mathsf{P}\left\{0 \leq \check{u}(t) \leq u(t,x) \leq \hat{u}(t), \forall t \geq 0, \forall x \in \mathbf{R}^1\right\} = 1.$$

As shown above, the processes $\check{u}(t)$ and $\hat{u}(t)$ are recurrent and assume only positive values. Therefore the space nonhomogeneous solution $u(t,x)$ is recurrent as well, is positive for all $x \in \mathbf{R}^1$, and has no trend.

Note that we considered the case where the initial function $u_0(x)$ is separated from zero. If one drops this assumption, Theorem 1 still implies that the space nonhomogeneous solution $u(t,x)$ is nonnegative and is totally conservative (that is, there is no explosion and the solution exists for an infinitely long time). The property of recurrence in this case is not valid in general. Also, one cannot say anything about whether or not the process reaches the zero value.

5. Concluding remarks

The examples above show that the most restrictive assumptions of Theorem 1 are conditions T1.2 and T1.4. Condition T1.2 requires, in fact, that the drift coefficient in the main equation contains a strong absorber. This property can be seen from the proof of inequality (13). Condition T1.4, in turn, requires that the diffusion coefficient possess some natural properties. For example, this condition, as we have seen in Example 1, means that the exponent of the diffusion coefficient of the equation should not be less than unity. This is a natural requirement, indeed, since a direct consequence of the comparison theorem is that a solution of the problem is unique. As is well known, typical problems for which a solution of problems (4) and (5) is unique correspond to the cases with exponents that are less than unity.

An advantage of Theorem 1 is that the Lipschitz condition is not required there. This allows one to use this theorem for the investigation of the dynamics of solutions for the Cauchy problems for stochastic parabolic equations with power nonlinearities.

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