CONVERGENCE OF STOCHASTIC INTEGRALS TO A CONTINUOUS LOCAL MARTINGALE WITH CONDITIONALLY INDEPENDENT INCREMENTS

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Abstract. For each $T > 0$, let a tensor-valued stochastic process $Y_T$ be defined by

$$Y_T(t) = \int_0^t DZ_T(s) \otimes \vartheta_T(s),$$

where $Z_T$ is an $\mathbb{R}^d$-valued locally square integrable martingale with respect to some filtration $\mathcal{F}_T$ and where $\vartheta_T$ is an $\mathbb{R}^d$-valued $\mathcal{F}_T$-predictable stochastic process such that $\int_0^t |\vartheta_T(s)|^2 D\text{tr}(Z_T)(s) < \infty$ for all $t$. In this paper, conditions are found for the convergence $\mathbb{L}^2(Y_T, (Y_T)) \xrightarrow{law} (Y, (Y))$, where $Y$ is a continuous local martingale with conditionally independent increments given $(Y)$.

INTRODUCTION

The results on the convergence in distribution of a sequence of semimartingales to a stochastic process with independent increments is the classics of stochastic analysis [1, 2]. In the monographs [1, 2] as well as in a number of papers on this topic (see, for example, [3]), one of the main assumptions is that the predictable characteristics converge in probability. This assumption is restrictive and, moreover, simply unnatural if prelimit processes are defined on different probability spaces and predictable characteristics of the limit process are random. A sufficient condition is found in [4] for the asymptotic independence of increments of stochastic processes (possibly defined on different probability spaces) if their jumps are asymptotically small. However, that condition is not easy to check in general. Example 4.8 in [4] exhibits, nevertheless, the case where the condition is easy to check.

The aim of this paper is to find sufficient conditions for the asymptotic independence of increments for a subclass of locally square integrable martingales. The conditions below are more general than in Example 4.8 of [4] and, at the same time, can effectively be checked for the class under consideration. By using the results of [4], such a property of prelimit processes allows us to change the assumption on the convergence of quadratic characteristics in probability by that for the convergence in distribution.

Throughout below the symbol $\mathbb{T}$ stands for both sets of positive real numbers and positive integer numbers. Recall that an arbitrary function on a directed set is called a net ($\mathbb{T}$ plays the role of a directed set in this paper). A special case of a net is a sequence (or a function) on $\mathbb{N}$. Considering a net $(X_T, T \in \mathbb{T})$ of stochastic processes, we admit the possibility that the probability spaces are, generally speaking, different for
different $T$. In view of this remark, any appearance of symbols like $X_T$ or $Y_T$ indicates that $P$ and $E$ should be understood as $P_T$ and $E_T$, respectively.

All vectors are thought of, unless otherwise stated, as columns. All matrices are tacitly assumed square ones. We use the Euclidean norm $|\cdot|$ of vectors and the operator norm $\|\cdot\|$ of matrices. By $\mathbb{R}^d$ and $\mathbb{R}^{d*}$ we denote the spaces of $d$-dimensional column vectors and row vectors, respectively. $\mathcal{S}$ and $\mathcal{S}_+$ signify, respectively, the class of all symmetric square matrices of a fixed size (defined by context) with real entries and its subclass of nonnegative (in the spectral sense) matrices. The coordinates of row vectors are written with subscript, and those of column vectors with superscript. Since matrices of a fixed size are elements of a finite-dimensional vector space, we treat them, if necessary, as vectors, in which case we write $|\cdot|$ instead of $\|\cdot\|$. The same concerns the tensors considered below.

Some notions of the tensor algebra used throughout the paper can be found in §24 of [5] or in §23 and §24 of [6]. The tensor multiplication is denoted by $\otimes$. In particular, $a \otimes b = ab^\top$ for $(0, 1)$-tensors or, which is the same, for vector columns.

If a reader wants to concentrate on purely probabilistic aspects of the below results, then one can omit tensors by assuming that the dimensions $d$ and $p$ are equal to unity (then the tensor product is the product of real numbers). It is worth mentioning, however, that an estimator of a matrix parameter of a stochastic process is a random tensor and, as a function of the duration of observation, is a tensor valued stochastic process. Therefore if one wants function theorems to be applicable for an investigation of asymptotic properties of estimators of parameters (including matrix parameters) of stochastic processes, then the restriction to the scalar case is not appropriate.

Let $X_T, T \in \mathbb{T}$, and $X$ be $\mathbb{R}^p$-valued or $\mathcal{S}$-valued stochastic processes whose trajectories belong to the Skorokhod space $D$ (that is, the trajectories are right continuous processes on $\mathbb{R}^+$ that have no discontinuities of the second kind). We write

$$X_T \xrightarrow{D} X$$

if the measures generated by the processes $X_T$ in the Borel $\sigma$-algebra of $D$ weakly converge as $T \to \infty$ to the measure generated by the process $X$. If, in addition, the process $X$ is continuous, then we write $X_T \xrightarrow{C} X$.

A net $(X_T)$ is called relatively compact in $D$ (in $C$) if every its cofinal subnet contains a further cofinal net that converges in the appropriate sense.

The convergence in distribution in $\mathbb{R}^p$ is denoted by $\xrightarrow{d}$. For $f \in D$, let

$$\Delta f(t) = f(t) - f(t-)$$

the integral $\int_{t_1}^{t_2}$ is understood as $\int_{[t_1, t_2]}$. The parameter $T$ is used to distinguish between the stochastic processes and does not correspond to the time variable. The time variable is written in parentheses as, for example, in equality (1) below. Each appearance of the symbol $\to$ corresponds to the passing to the limit as $T \to \infty$.

All locally square integrable martingales are assumed to be right continuous and such that the left limits exist. This assumption does not restrict the generality in view of Theorem 1.3.2 in [7].

The net $(Y_T)$ of tensor valued stochastic processes for which we find conditions for the convergence to a process $Y$ with conditionally independent increments is given by

$$Y_T(t) = \int_0^t dZ_T(s) \otimes \partial_T(s),$$

where $\partial_T(s)$ represents the infinitesimal time change of the process $Z_T$ at time $s$. This expression is consistent with the definition of the tensor $\otimes$ and provides a way to construct such processes.

In the context of this paper, the main focus is on the analysis of stochastic processes and their asymptotic behaviors. The use of tensors allows for a more comprehensive analysis of matrix-valued parameters, which is crucial for understanding the behavior of estimators in stochastic processes.
where, for each $T$, $Z_T$ is an $\mathbb{R}^d$-valued locally square integrable martingale with respect to some filtration $\mathbb{F}_T$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while $\vartheta_T$ is an $\mathbb{R}^d$-valued $\mathbb{F}_T$-predictable stochastic process such that

$$
(2) \quad \int_0^t |\vartheta_T(s)|^2 \, d\text{tr}(Z_T)(s) < \infty, \quad t \in \mathbb{R}_+.
$$

The main results of this paper, Theorems 4–6, provide more information than simply the convergence $Y_T \xrightarrow{C} Y$. To investigate the asymptotic properties of estimators of parameters of stochastic processes, one needs to use all the information provided by Theorems 4–6 that does not use the unnatural assumption mentioned above, while the $\sigma$-algebra for which the increments of the limit process are conditionally independent arises as a result of a limit procedure and is a generalization of the $\sigma$-algebra of invariant sets well known in ergodic theory.

1. Some preliminary results

Lemma 1. For each $d \times d$ matrix $B \in \mathcal{S}_+$,

$$
\|B\| \leq \text{tr} B \leq d \cdot \|B\|.
$$

Proof. The norm of a real symmetric matrix, as is well known, equals the maximal absolute value of its eigen numbers. \qed

Corollary 1. For each $d \times d$ matrix $A$,

$$
\text{tr} AA^\top \leq d \cdot \|A\|^2.
$$

In the space of tensors of order four, consider the norm such that $\|A \otimes B\| = \|A\| \|B\|$ for all square matrices $A$ and $B$. Lemma 1 implies the following result.

Corollary 2. Let $A$ and $B$ be two matrices of the same sizes, and let $B \in \mathcal{S}_+$. Then $\|A \otimes B\| \leq \|A\| \text{tr} B$.

We write $A \leq B$ for two matrices $A$ and $B$ if $B - A \in \mathcal{S}_+$.

Lemma 2. Let $D_1(\cdot)$ and $D_2(\cdot)$ be measurable matrix valued functions on $[t_1, t_2]$. Assume that $D_2(\cdot)$ is increasing and right continuous. Then

$$
\left\| \int_{t_1}^{t_2} dD_2(s) \otimes D_1(s) \right\| \leq \int_{t_1}^{t_2} \|D_1(s)\| \, d\text{tr} D_2(s)
$$

if the right-hand side of the inequality is finite.

Proof. The result follows easily from Corollary 1 for piecewise constant functions $D_2(\cdot)$ (in this case, the integrals become the sums). The general case is derived from this particular one by using a standard procedure. \qed

Given an arbitrary matrix $A$, we denote by $A$ the vector obtained by writing its columns one after another.

Lemma 3. For each $d \times d$ matrix $A$, $|A| \leq \sqrt{d} \|A\|$. \qed

Proof. Denoting by $a_i$ the column $i$ of the matrix, one can write $\sum |a_i|^2 = \text{tr} AA^\top$. It remains to apply Corollary 1.

In what follows we write $Y_T$ rather than $Y_T$.

Lemma 4. For every $T > 0$, let $Y_T$ be a $(0,2)$-tensor valued locally square integrable martingale. Assume that there exists a vector valued locally square integrable martingale $X$ such that $(Y_T, (Y_T)) \xrightarrow{D} (X, \langle X \rangle)$. Then $(Y_T, (Y_T)) \xrightarrow{D} (Y, (Y'))$, where $Y$ is a $(0,2)$-tensor valued locally square integrable martingale such that $Y = X$. \qed
Denote by \( \sharp \) the linear operation in the space of tensors of order four that acts as follows:
\[
(a_1 \otimes a_2 \otimes a_3 \otimes a_4)^\sharp = a_1 \otimes a_3 \otimes a_2 \otimes a_4.
\]
If \( Z_1 \) and \( Z_2 \) are tensor valued locally square integrable martingales, then we denote the compensator of the process \( Z_1 \otimes Z_2 \) by \( \langle Z_1, Z_2 \rangle \). We write \( \langle Z \rangle \) instead of \( \langle Z, Z \rangle \).

**Lemma 5.** Let \( Z \) be an \( \mathbb{R}^d \)-valued locally square integrable martingale with respect to the flow of the \( \sigma \)-algebras \( \mathcal{F} = (\mathcal{F}(t), t \in \mathbb{R}_+) \), and let \( \vartheta \) be an \( \mathbb{R}^d \)-valued \( \mathbb{F} \)-predictable stochastic process such that \( \int_0^t |\vartheta(s)|^2 \, d\langle Z \rangle(s) < \infty \) for all \( t \). Then

1. for every \( t \), the stochastic integral \( Y(t) \equiv \int_0^t dZ(s) \otimes \vartheta(s) \) is well defined;
2. \( Y \) is a locally square integrable martingale with quadratic characteristic

\[
\langle Y \rangle(t) = \left( \int_0^t d\langle Z \rangle(s) \otimes \vartheta(s)^\otimes 2 \right)^\sharp;
\]
3. \( \text{tr} \langle Y \rangle(t) = \int_0^t |\vartheta(s)|^2 \, d\langle Z \rangle(s) \).

**Proof.** The first two statements for the case of \( d = 1 \) are contained in Theorem I.4.40 of [2]. It is obvious that the first statement remains valid for the general case as well. It is also obvious that the process \( Y \) is a locally square integrable martingale for \( d > 1 \), since this property holds for \( d = 1 \).

The second statement holds in the case of a general \( d \) for integrands

\[
v(t) = \alpha_0 I_{\{0\}}(t) + \sum_{i=1}^m \alpha_i I_{[s_{i-1}, s_i)}(t),
\]
where \( 0 = s_0 < \cdots < s_m \) and \( \alpha_i \) is an \( \mathbb{R}^d \)-valued \( \mathcal{F}(s_i+) \)-measurable random variable for each \( i \in \{0, \ldots, m\} \). Indeed, in this case

\[
Y(t) = \sum_{i=1}^m (Z(s_i \wedge t) - Z(s_{i-1} \wedge t)) \otimes \alpha_i,
\]

\[
Y^\otimes 2(t) = \left( \sum_{i,j=1}^m \left( (Z(s_i \wedge t) - Z(s_{i-1} \wedge t)) \otimes (Z(s_j \wedge t) - Z(s_{j-1} \wedge t)) \right) \otimes \alpha_i \otimes \alpha_j \right)^\sharp,
\]
whence

\[
\langle Y \rangle(t) = \left( \sum_{i=1}^m (\langle Z \rangle(s_i \wedge t) - \langle Z \rangle(s_{i-1} \wedge t)) \otimes \alpha_i^\otimes 2 \right)^\sharp.
\]

The latter equality is nothing else as [3]. The general case follows from this particular one in a standard way.

By construction, the vector \( \overline{Y}(t) \) consists of columns

\[
\int_0^t \vartheta(s) \, dZ_1(s), \ldots, \int_0^t \vartheta(s) \, dZ_d(s)
\]
written one after another, where \( Z_i \) is the component \( i \) of the vector process \( Z \). Thus the \( d^2 \times d^2 \) matrix \( \langle Y \rangle(t) \) consists of \( d^2 \) blocks \( \int_0^t \vartheta(s)^\otimes 2 \, d\langle Z_i, Z_j \rangle(s) \) of size \( d \times d \). Hence,

\[
\text{tr} \langle Y \rangle(t) = \sum_{i=1}^d \int_0^t \text{tr} \vartheta(s)^\otimes 2 \, d\langle Z_i \rangle(s).
\]

It remains to note that \( \text{tr} x^\otimes 2 = |x|^2 \) for an arbitrary \( x \in \mathbb{R}^d \) to establish the third statement of the lemma. \( \square \)
For all \(a, b \in \mathbb{R}^d\) and \(u, v \in \mathbb{R}^{d*}\), we have
\[
(ua)(vb) = uab^\top v^\top = u(a \otimes b)v^\top.
\]

Repeating the proof of Lemma 5 with obvious changes, we obtain the following result.

**Lemma 6.** Let \(\zeta\) be an \(\mathbb{R}^d\)-valued locally square integrable martingale with respect to the flow of \(\sigma\)-algebras \(\mathcal{F} = (\mathcal{F}(t), t \in \mathbb{R}_+)\), and let \(\psi\) be an \(\mathbb{R}^{d*}\)-valued \(\mathcal{F}\)-predictable stochastic process such that \(\int_0^t |\psi(s)|^2 \, d\text{tr}(\zeta)(s) < \infty\) for all \(t\). Then

1) for all \(t\), the stochastic integral \(\Psi(t) = \int_0^t \psi(s) \, d\zeta(s)\) is well defined;
2) \(\Psi\) is a locally square integrable martingale with characteristic
\[
\langle \Psi \rangle(t) = \int_0^t \psi(s) \, d\langle \zeta \rangle(s)(\psi(s))^\top.
\]

**Corollary 3.** Let \(\zeta, \psi,\) and \(\Psi\) be the same as in Lemma 6. Also let
\[
\eta(t) = \int_0^t \chi(s)\psi(s) \, d\zeta(s),
\]
where \(\chi\) is an \(\mathcal{F}\)-predictable \([0, 1]\)-valued stochastic process. Then
\[
\langle \eta \rangle(t_2) - \langle \eta \rangle(t_1) \leq \langle \Psi \rangle(t_2) - \langle \Psi \rangle(t_1)
\]
for all \(t_2 > t_1 \geq 0\).

Put \(\Pi(t, c) = \{(t_1, t_2) : (t_2 - c)_+ \leq t_1 < t_2 \leq t\}\), and recall the well known criterion of the relative compactness in the space \(C\) (see, for example Proposition VI.3.26 in [2] and Remarks VI.3.25 and VI.3.9 therein).

**Theorem 1.** A net \((\xi_T, T \in \mathbb{T})\) of stochastic processes whose trajectories belong to the space \(D\) is relatively compact in the space \(C\) if and only if
\[
\lim_{L \to \infty} \lim_{T \to \infty} \mathbb{P}\left\{ \sup_{s \leq t} |\xi_T(s)| > L \right\} = 0,
\]
\[
\lim_{c \to 0} \lim_{T \to \infty} \mathbb{P}\left\{ \sup_{(t_1, t_2) \in \Pi(t,c)} |\xi_T(t_2) - \xi_T(t_1)| > \varepsilon \right\} = 0
\]
for all positive \(t\) and \(\varepsilon\).

Theorem 1 is usually stated in the literature for the case of \(\mathbb{T} = \mathbb{N}\) which obviously implies the case of \(\mathbb{T} = \{T \in \mathbb{R} : T > 0\}\).

**Corollary 4** (of Theorem 1 and Lemma 2). The relative compactness in the space \(C\) of the net \((V_T)\) of right continuous increasing \(\mathcal{G}\)-valued stochastic processes is equivalent to the relative compactness of the net \((\text{tr} V_T)\).

Theorem 1, Lemma 2, and an obvious inequality
\[
|\varphi_T(s)|^2 \leq N^2 + |\varphi_T(s)|^2 I\{|\varphi_T(s)| > N\}
\]
allow one to state the following result.

**Corollary 5.** For every \(T \in \mathbb{T}\), let a stochastic process \(U_T\) be defined by
\[
U_T(t) = \int_0^t dH_T(s) \otimes \varphi_T(s)^\otimes 2,
\]
where $H_T$ is a right-continuous-increasing and $\mathcal{G}$ valued stochastic process and where $\varphi_T$ is an $\mathbb{R}^d$-valued measurable stochastic process. Assume that a net $(\text{tr } H_T)$ is relatively compact in the space $C$ and

$$
\lim_{N \to \infty} \lim_{T \to \infty} \mathbb{P} \left\{ \int_0^t |\varphi_T(s)|^2 I\{|\varphi_T(s)| > N\} \, d\text{tr } H_T(s) > \varepsilon \right\} = 0
$$

for all positive $\varepsilon$ and $t$. Then the net $(U_T)$ also is relatively compact in the space $C$.

Put $S = \{(s, t) \in \mathbb{R}^2 : 0 < s < t\}$. Let $(\Theta, \mathcal{E})$ be a measurable space, and let

$$
\mathbb{H} \equiv (\mathcal{H}(t), t \in \mathbb{R}_+)
$$

be a filtration in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (the filtration may depend on $T$). We say that a $\Theta$-valued random function $\boldsymbol{x}$ defined on $S \times \Theta$ is uniformly $(\mathbb{H}, \mathcal{E})$-adapted if, for all $t > s > 0$, the random function $\boldsymbol{x}(s, t, \cdot)$ of the variable $\theta \in \Theta$ is $\mathcal{H}(s) \otimes \mathcal{E}$-measurable with respect to $(\omega, \theta) \in \Omega \times \Theta$.

The Borel $\sigma$-algebra in the metric space $\mathfrak{X}$ is denoted by $\mathcal{B}(\mathfrak{X})$.

**Lemma 7.** For every $T \in \mathbb{T}$, let $Q_T$ be an $\mathbb{F}_T$-adapted stochastic process assuming values in a metric space $(\mathfrak{X}, \rho)$. Assume that there exists a family $\{\varphi_T^r, r > 0, T \in \mathbb{T}\}$ of $\mathfrak{X}$-valued random functions defined on $S \times \mathfrak{X}$ such that each $\varphi_T^r$ is uniformly $(\mathbb{F}_T, \mathcal{B}(\mathfrak{X}))$-adapted and

$$
\lim_{r \to 0} \lim_{T \to \infty} \mathbb{P} \left\{ \rho(Q_T(t), \varphi_T^r(s, t, Q_T(s))) > \varepsilon \right\} = 0
$$

for all $t > s > 0$ and $\varepsilon > 0$. Then

$$
\mathbb{E}(f(Q_T(t)) \mid \mathcal{F}_T(s)) - f(Q_T(t)) \xrightarrow{\mathbb{P}} 0
$$

for all $t > s > 0$ and all bounded uniformly continuous functions $f : \mathfrak{X} \to \mathbb{R}$.

**Proof.** Fix $t > s > 0$ and a bounded uniformly continuous function defined on $\mathfrak{X}$. Put $q_T^r = \varphi_T^r(s, t, Q_T(s))$ and $\gamma_T^r = f(Q_T(t)) - f(q_T^r)$. Equality (6) together with the obvious inequality

$$
\mathbb{E}\gamma_T^r \leq 2\|f\|_{\infty} \mathbb{P}\{\rho(Q_T(t), q_T^r) > \varepsilon\} + \sup_{\rho(x, y) \leq \varepsilon} |f(x) - f(y)|
$$

implies that

$$
\lim_{r \to 0} \lim_{T \to \infty} \mathbb{E} |\gamma_T^r| = 0.
$$

Moreover

$$
\lim_{r \to 0} \lim_{T \to \infty} \mathbb{E} |\mathbb{E}(\gamma_T^r \mid \mathcal{G}_T)| = 0
$$

for all $\sigma$-algebras $\mathcal{G}_T \subset \mathcal{F}_T$. Since $Q_T$ is $\mathbb{F}_T$-adapted and $\varphi_T^r$ is uniformly $(\mathbb{F}_T, \mathcal{B}(\mathfrak{X}))$-adapted, the random variable $q_T^r$ is $\mathcal{F}_T(s)$-measurable. The latter relation with $\mathcal{G}_T = \mathcal{F}_T(s)$ is rewritten as follows:

$$
\lim_{r \to 0} \lim_{T \to \infty} \mathbb{E} |\mathbb{E}(f(Q_T(t)) \mid \mathcal{F}_T(s)) - f(q_T^r)| = 0.
$$

Comparing with (8) we conclude that

$$
\lim_{T \to \infty} \mathbb{E} |\mathbb{E}(f(Q_T(t)) \mid \mathcal{F}_T(s)) - f(Q_T(t))| = 0,
$$

which is equivalent to (7). \hfill \square

Put $g_r(x) = (|x| \vee r) \text{sgn } x$ and $h_r(x) = (|x| \vee r)^{-1} \text{sgn } x$, $x \in \mathbb{R}$, $r > 0$. 

Lemma 8. 1) Let \((R_T)\) and \((U_T)\) be two nets of \(\mathbb{R}\)-valued and \(\mathbb{R}^p\)-valued stochastic processes, respectively; and

2) let \((\Xi_T)\) be a net of \(\mathbb{R}^p\)-valued random variables such that

\[
\lim_{n \to \infty} \lim_{T \to \infty} P\{ |R_T(t)| > L \} = 0, \tag{9}
\]

\[
\lim_{n \to \infty} \lim_{T \to \infty} P\{ |R_T(t)| \leq r \} = 0, \tag{10}
\]

\[
U_T(t) - \Xi_T R_T(t) \overset{P}{\to} 0 \tag{11}
\]

for all \(t > 0\).

Then

\[
\lim_{r \to 0} \lim_{T \to \infty} P\{ |U_T(t) - g_r(R_T(t))h_r(R_T(s))U_T(s)| > \varepsilon \} = 0 \tag{12}
\]

for all \(t > s > 0\) and \(\varepsilon > 0\).

Proof. Put \(J_T(t) = U_T(t)/R_T(t)\) (if \(R_T(t) \neq 0\)). It is clear that

\[U_T(t) - g_r(R_T(t))h_r(R_T(s))U_T(s) = R_T(t)(J_T(t) - J_T(s))\quad \text{for} \quad |R_T(t)| \wedge |R_T(s)| \geq r.\]

Hence

\[
\{ |U_T(t) - g_r(R_T(t))h_r(R_T(s))U_T(s)| > \varepsilon \} \subset \{ |R_T(t)| \leq r \} \cup \{ |R_T(s)| \leq r \} \cup \{ |R_T(t)| \wedge |J_T(t) - J_T(s)| > \varepsilon \} \cap \{ |R_T(s)| > r \}
\]

It is also clear that

\[
\{ |R_T(t)| \cdot |J_T(t) - J_T(s)| > \varepsilon \} \subset \{ |R_T(t)| > L \} \cup \{ |J_T(t) - J_T(s)| > \varepsilon / \lambda \}
\]

for all \(L > 0\). Thus

\[
P\{ |U_T(t) - g_r(R_T(t))h_r(R_T(s))U_T(s)| > \varepsilon \}
\]

\[
\leq P\{ |R_T(t)| \leq r \} + \{ |R_T(s)| \leq r \} + P\{ |R_T(t)| > L \}
\]

\[
+ P\{ |J_T(t) - \Xi_T| > \varepsilon /2L, |R_T(t)| > r \}
\]

\[
+ P\{ |J_T(s) - \Xi_T| > \varepsilon /2L, |R_T(s)| > r \}
\]

for all \(t > s > 0\) and positive \(\varepsilon, r,\) and \(L\). In view of (9) and (10), this means that the result desired follows from

\[
\lim_{T \to \infty} P\{ |J_T(t) - \Xi_T| > \delta, |R_T(t)| > r \} = 0
\]

for all positive \(t, \delta,\) and \(r\). In turn, this property follows from (11) and an obvious inclusion \(\{ |R_T(t)| > r \} \cap \{ |J_T(t) - \Xi_T| > \delta \} \subset \{ |U_T(t) - \Xi_T R_T(t)| > r \delta \}. \quad \square
\]

The following result is proved similarly.

Lemma 9. 1) Let \((R_T)\) and \((U_T)\) be two nets of \(\mathbb{R}\)-valued and \(\mathbb{R}^p\)-valued stochastic processes, respectively;

2) let \((\Xi_T)\) be a net of \(\mathbb{R}^p\)-valued random variables such that relations (9) and (11) hold for all \(t > 0\); and

3) let \(R_T(t) \neq 0\) for all \(t > 0\) and sufficiently large \(T \in \mathbb{T}\).

Then

\[
U_T(t) - R_T(t)R_T(s)^{-1}U_T(s) \overset{P}{\to} 0
\]

for all \(t > s > 0\) and \(\varepsilon > 0\).
2. General results on the asymptotic conditional independence of increments of locally square integrable martingales

Two theorems of the paper [4] are presented below in a more convenient form. The proof of the main results uses only Theorem 3 while Theorem 2 helps to better understand the key assumption A given below:

A. there exists a family \( \{ \phi_T^r, r > 0, T \in \mathbb{T} \} \) of \( \mathcal{S}_+ \)-valued random functions on \( S \times \mathcal{S}_+ \) such that every \( \phi_T^r \) is uniformly \( (\mathbb{F}_T, \mathcal{B}(\mathcal{S}_+)) \)-adapted and

\[
\lim_{r \to 0} \lim_{T \to \infty} \mathbb{P}\{ \| \langle X_T \rangle(t) - \phi_T^r(s, t, \langle X_T \rangle(s)) \| > \varepsilon \} = 0
\]

for all \( t > s > 0 \) and \( \varepsilon > 0 \).

**Theorem 2.** For every \( T \in \mathbb{T} \), let \( X_T, X_T^1, X_T^2, \ldots \) be \( \mathbb{R}^p \)-valued locally square integrable martingales with respect to a flow \( \mathbb{F}_T = (\mathcal{F}_T(t), t \in \mathbb{R}_+) \). Assume that

1) condition A holds and that a net \( \langle \text{tr}(X_T), T \in \mathbb{T} \rangle \) is relatively compact in the space \( C \);

2) \( \langle zX_T^m \rangle(t_2) - \langle zX_T^m \rangle(t_1) \leq \langle zX_T \rangle(t_2) - \langle zX_T \rangle(t_1) \)

for all \( t_2 > t_1 \geq 0, m \in \mathbb{N}, \) and \( z \in \mathbb{R}^p \); e

3) \( \lim_{T \to \infty} \mathbb{E} \max_{s \leq t} |\Delta X_T^m(s)|^2 = 0 \)

and

\( \lim_{l \to \infty} \lim_{T \to \infty} \mathbb{P}\{ \text{tr}(X_T^l - X_T, T) > \varepsilon \} = 0 \)

for all \( t > 0, \varepsilon > 0 \) and \( m \in \mathbb{N} \).

Then

\[
\mathbb{E} \left( \exp\left\{ i \sum_{j=1}^n z_j (X_T(t_j) - X_T(t_{j-1})) \right\} \bigg| \mathcal{F}_T(s) \right) - \exp\left\{ -\frac{1}{2} \sum_{j=1}^n z_j (\langle X_T \rangle(t_j) - \langle X_T \rangle(t_{j-1})) z_j^\top \right\} \overset{\mathbb{P}}{\to} 0
\]

for all \( n \in \mathbb{N}, z_1, \ldots, z_n \in \mathbb{R}^p, \) and \( t_n > \cdots > t_1 > t_0 \geq s > 0 \).

**Proof.** According to Corollary 4 the net \( \langle \langle X_T \rangle, T \in \mathbb{T} \rangle \) is relatively compact in \( C \), since \( \langle \text{tr}(X_T), T \in \mathbb{T} \rangle \) possesses this property. In view of Lemma 7 condition A implies that

\( \mathbb{E}(F(\langle zX_T \rangle(t))|\mathcal{F}_T(s)) - F(\langle zX_T \rangle(t)) \overset{\mathbb{P}}{\to} 0 \)

for all \( t > s > 0 \) and \( z \in \mathbb{R}^{d^*} \) and for a bounded uniformly continuous function \( F \). Thus for the case of \( T = \mathbb{N} \) Theorem 2 follows from Theorem 4.6 of [4].

From the part proved above, we conclude that relation (16) holds for every increasing to infinity sequence \( \langle T_k, k \in \mathbb{N} \rangle \) of positive real functions and for all \( n \in \mathbb{N} \) if \( T \) approaches infinity in the set \( \{ T_k \} \) rather than in \( \mathbb{N} \), where \( z_1, \ldots, z_n \in \mathbb{R}^p \) and \( t_n > \cdots > t_1 > t_0 \geq s > 0 \). This implies Theorem 2 for the case of \( T = \{ T \in \mathbb{R}: T > 0 \} \).

**Theorem 3.** For every \( T \in \mathbb{T} \), let \( X_T, X_T^1, X_T^2, \ldots \) be \( \mathbb{R}^p \)-valued locally square integrable martingales with respect to the flow \( \mathbb{F}_T \). Assume that

1) condition A holds and relations (13)–(15) are satisfied;
2) \[
\lim_{L \to \infty} \sup_{l} \lim_{T \to \infty} P \left\{ \left| X^{i}_{T}(0) \right| > L \right\} = 0;
\]

3) for all \( \varepsilon > 0 \) and \( t > 0 \),
\[
\lim_{l \to \infty} \lim_{T \to \infty} P \left\{ \left| X^{i}_{T}(0) - X_{T}(0) \right| > \varepsilon \right\} = 0;
\]

4) there exist an \( \mathbb{R}^{p} \)-random variable \( \check{X} \) and an \( \mathcal{G}_{t} \)-valued stochastic process \( H \) defined on a common probability space and such that
\[
(X_{T}(0), \langle X_{T} \rangle) \xrightarrow{C} (\check{X}, H).
\]

Then \( (X_{T}, \langle X_{T} \rangle) \xrightarrow{C} (X, H) \), where \( X \) is a continuous local martingale with the initial value \( \check{X} \) and quadratic characteristic \( H \) such that
\[
E \left( \exp \left\{ i \sum_{j=1}^{n} z_{j} (X(t_{j}) - X(t_{j-1})) \right\} \mid \check{X}, H(\cdot) \right) = \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} z_{j} (H(t_{j}) - H(t_{j-1})) z_{j}^{T} \right\}
\]
for all \( n \in \mathbb{N} \), where \( t_{n} > \cdots > t_{0} \geq 0 \) and \( z_{1}, \ldots, z_{n} \in \mathbb{R}^{p} \).

**Proof.** Rewriting inequality (13) as
\[
z (\langle X^{n}_{T} \rangle (t_{2}) - \langle X^{m}_{T} \rangle (t_{1})) z^{T} \leq z (\langle X_{T} \rangle (t_{2}) - \langle X_{T} \rangle (t_{1})) z^{T},
\]
fixing an orthonormal basis \( e^{1}, \ldots, e^{p} \) in \( \mathbb{R}^{p} \), letting sequentially \( z = e^{1}, \ldots, z = e^{p} \), and adding together these \( p \) inequalities, we get
\[
\text{tr} (\langle X^{n}_{T} \rangle (t_{2}) - \langle X^{m}_{T} \rangle (t_{1})) \leq \text{tr} (\langle X_{T} \rangle (t_{2}) - \langle X_{T} \rangle (t_{1})).
\]

We further deal with the case of \( T = \mathbb{N} \) (the case of \( T = \{ T \in \mathbb{R} : T > 0 \} \) is reduced to \( T = \mathbb{N} \) in the same way as in the proof of Theorem [2].


The sequence \( \langle X_{T} \rangle, T \in \mathbb{N} \) is relatively compact in the space \( C \) by assumption [18], whence
\[
\lim_{L \to \infty} \sup_{l} \lim_{T \to \infty} P \left\{ \text{tr} \langle X^{i}_{T} \rangle (t) > L \right\} = 0,
\]

\[
\lim_{c \to 0} \sup_{l} \lim_{T \to \infty} P \left\{ \sup_{(t_{1}, t_{2}) \in \Pi(t, c)} \text{tr} (\langle X^{i}_{T} \rangle (t_{2}) - \langle X^{i}_{T} \rangle (t_{1})) > \varepsilon \right\} = 0
\]
for all positive \( t \) and \( \varepsilon \) by inequality [20] and Theorem [1].

Therefore all the assumptions of Theorem 5.6 in [4] hold (three of them are just checked, while the rest are assumed to hold in Theorem [3]). This completes the proof of Theorem [3].

**Remark 1.** It is clear that property [19] for an \( \mathbb{R}^{p} \)-valued process \( X \) is equivalent to the following pair of properties:

(i) \( X \) has conditionally independent \( H(\cdot) \) increments with respect to \( \check{X} \);

(ii) for all \( t > s \geq 0 \) and \( z \in \mathbb{R}^{p} \),
\[
E \left( e^{iz(X(t) - X(s))} \mid \check{X}, H(\cdot) \right) = e^{-z(\langle X(\cdot) \rangle - \langle X(s) \rangle) z^{T} / 2}.
\]
In Theorem 4 below, we use the following notation

\[ U_T(t) = \int_0^t d(Z_T)(s) \otimes \vartheta_T(s) \otimes 2. \]

Also, condition A changes as follows:

B. There exists a family \( \{ z_T, r > 0, T \in \mathbb{T} \} \) of \( \mathbb{R}^d \)-valued random functions such that every \( z_T \) is uniformly \( (\mathbb{F}_T, \mathcal{B}(\mathbb{R}^d)) \)-adapted and

\[ \lim_{r \to 0} \lim_{T \to \infty} \mathbb{P} \left\{ \left| U_T(t) - z_T^r(s, t, U_T(s)) \right| > \varepsilon \right\} = 0 \]

for all \( t > s > 0 \) and \( \varepsilon > 0 \).

**Theorem 4.** For every \( T > 0 \), let a stochastic process \( Y_T \) be defined by equality 11, where \( Z_T \) is an \( \mathbb{R}^d \)-valued locally square integrable martingale with respect to the flow \( \mathbb{F}_T \) and where \( \vartheta_T \) is an \( \mathbb{R}^d \)-valued \( \mathbb{F}_T \)-predictable stochastic process satisfying condition 2. Assume also that

\[ \lim_{T \to \infty} \mathbb{E} \max_{s \leq t} |\Delta Z_T(s)|^2 = 0, \]

\[ \lim_{N \to \infty} \lim_{T \to \infty} \mathbb{P} \left\{ \int_0^t |\vartheta_T(s)|^2 \mathbb{I}\{ |\vartheta_T(s)| > N \} d|Z_T(s)| > \varepsilon \right\} = 0 \]

for all positive \( t \) and \( \varepsilon \). Finally, assume that the stochastic processes \( U_T \) defined by equality 21 satisfy condition B and there exists an \( \mathbb{R}^{d'} \)-valued stochastic process \( G \) such that

\[ \langle Y_T \rangle \overset{C}{\to} G. \]

Then

1) \( (Y_T, \langle Y_T \rangle) \overset{C}{\to} (Y, \langle Y \rangle) \), where \( Y \) is a continuous local martingale with initial value \( O \) (\( O \) denotes the zero tensor) and with quadratic characteristic \( \langle Y \rangle = G \); 2) the process \( Y \) has conditionally independent increments with respect to \( G \); 3) for all \( t > s \geq 0 \) and \( z \in \mathbb{R}^{d^2} \),

\[ \mathbb{E} \left( e^{iz(\langle Y \rangle(t) - \langle Y \rangle(s))} \bigg| \langle Y \rangle(s) \right) = e^{-z(\langle Y \rangle(t) - \langle Y \rangle(s))z^T/2}. \]

**Proof.** Put

\[ Y_T^m(t) = \int_0^t dZ_T(s) \otimes \vartheta_T(s) I\{|\vartheta_T(s)| \leq m\}, \]

\( X_T = Y_T \), and \( X_T^m = Y_T^m \). By Lemma 5 \( \langle Y_T \rangle = U_T^2 \) and thus B implies A. We are going to show that the rest of the assumptions of Theorem 3 hold for \( X_T, X_T^1, X_T^2, \ldots \).

Two assumptions concerning the initial values are trivial in the case under consideration, since all processes start from zero:

By construction,

\[ \Delta Y_T^m(s) = \Delta Z_T(s) \otimes \vartheta_T(s) I\{|\vartheta_T(s)| \leq m\}, \]

whence \( |\Delta Y_T^m(s)|^2 \leq m^2 d \cdot |\Delta Z_T(s)|^2 \) by Lemma 3 which together with equality 23 implies 14.

Similarly, by construction

\[ Y_T^l(t) - Y_T(t) = \int_0^t dZ_T(s) \otimes \vartheta_T(s) I\{|\vartheta_T(s)| > l\}, \]
whence
\[ \text{tr} \langle X^T_\vartheta - X_T \rangle (t) = \int_0^t |\vartheta_T(s)|^2 I\{|\vartheta_T(s)| > l\} \, d\text{tr}(Z_T)(s) \]
by Lemma \ref{lemma:5} and thus (24) implies (15).

In order to check (13), we represent \( z \in \mathbb{R}^{d^2} \) as \( z = (\zeta_1 \cdots \zeta_d) \), where
\[ \zeta_k = (z_{k1} \cdots z_{kd}) \in \mathbb{R}^{d^2}. \]
By construction, the vector \( X_T(t) \) consists of the columns
\[ \int_0^t dZ_1(s)\vartheta(s), \ldots, \int_0^t dZ_d(s)\vartheta(s) \]
written sequentially one by one, where \( Z_i \) is the component \( i \) of \( Z \). Hence
\[ zX_T(t) = \int_0^t d(\zeta_1 Z_1 + \cdots + \zeta_d Z_d)\vartheta(s) \]
or, which is equivalent, \( zX_T(t) = \int_0^t \psi(s) d\zeta(s), \) where \( \psi = \vartheta^\top \) and \( \zeta = \zeta_1 \zeta_2 \cdots \zeta_d \).

The same reasoning yields \( zX_m(t) = \int_0^t \psi(s)I\{|\vartheta_T(s)| \leq m\} d\zeta(s) \).
Now (13) follows from (2) and Corollary \ref{corollary:3}.

Since \( X_T = Y_T \), there exists an \( \mathcal{S}_+ \)-valued stochastic process \( H \) such that \( \langle X_T \rangle \xrightarrow{C} H \) (\( \langle G(t) \rangle \) and \( \langle H(t) \rangle \) have the same families of coordinates but they are grouped differently) in view of (25). Then Theorem \ref{theorem:3} implies that \( \langle X_T, \langle X_T \rangle \rangle \xrightarrow{C} \langle X, H \rangle \), where \( X \) is a continuous local martingale with \( \langle X \rangle = H \). By Lemma \ref{lemma:4} \( \langle Y_T, \langle Y_T \rangle \rangle \xrightarrow{C} \langle Y, G \rangle \), where \( Y \) is an \( \mathbb{R}^{d^2} \)-valued continuous local martingale such that \( Y = X \) and \( \langle Y \rangle = G \). This proves the first statement of the theorem. The second and third statements follow from Theorem \ref{theorem:3} and Remark \ref{remark:1}.

To understand better the probabilistic meaning of Theorem \ref{theorem:4} we provide another statement that does not require a separate proof. In this version of Theorem \ref{theorem:4} the processes \( \vartheta_T \) are scalar and thus any tensor construction disappear.

**Theorem 5.** For every \( T > 0 \), let the stochastic process \( Y_T \) be defined by equality
\[ Y_T(t) = \int_0^t \vartheta_T(s) dZ_T(s), \]
where \( Z_T \) is an \( \mathbb{R}^d \)-valued locally square integrable martingale with respect to the flow \( \mathbb{F}_T \) and where \( \vartheta_T \) is an \( \mathbb{R}^d \)-valued \( \mathbb{F}_T \)-predictable stochastic process satisfying condition (2). Assume that
1. for all positive \( t \) and \( \varepsilon \), conditions (23) and (24) hold;
2. the stochastic processes \( U_T \) defined by
\[ U_T(t) = \int_0^t \vartheta_T(s)^2 d\langle Z_T \rangle(s) \]
satisfy condition \( \mathcal{B} \);
3. there exists an \( \mathbb{R}^{d^2} \)-valued stochastic process \( U \) such that \( U_T \xrightarrow{C} U \).

Then
a) \( \langle Y_T, \langle Y_T \rangle \rangle \xrightarrow{C} \langle Y, \langle Y \rangle \rangle \), where \( Y \) is a continuous local martingale with initial value 0 and quadratic characteristic \( \langle Y \rangle = U \);
b) the process \( Y \) has conditionally independent increments with respect to \( U \);
c) for all \( t > s \geq 0 \) and \( z \in \mathbb{R}^{d^2} \),
\[ \mathbb{E} \left( e^{iz(Y(t) - Y(s))} \right) = \mathbb{E} \left( e^{-z(U(t) - U(s))z^2/2} \right). \]

Throughout below we consider the case where the probability space does not depend on \( T \).
Lemma 10. For every $T \in \mathbb{T}$, let the stochastic process $U_T$ be given by
\begin{align}
U_T(t) &= \frac{1}{T} \int_0^T \rho_T \left( \frac{s}{T} \right) d\Phi(s), \\
\Phi(t) &= \int_0^t dH(s) \otimes \nu(s)^2,
\end{align}
where $\rho_T$ and $\nu$ are measurable stochastic processes ($\mathbb{R}$-valued and $\mathbb{R}^d$-valued, respectively) and where $H$ is an increasing right continuous $\mathcal{G}$-valued stochastic process. We also assume that
1) \[ \lim_{L \to \infty} \lim_{T \to \infty} P \left\{ \int_0^T |\nu(s)|^2 d\text{tr} H(s) > TL \right\} = 0; \]
2) the processes $\rho_T$ are right continuous and with limits on the left;
3) for all $t > 0$ and $\varepsilon > 0$,
\[ \lim_{L \to \infty} \lim_{T \to \infty} P \left\{ \sup_{s \leq t} |\rho_T(s)| > L \right\} = 0, \]
4) there exists a net $(\Xi_T)$ of random tensors such that
\[ T^{-1} \Phi(Tt) - \Xi_T t \xrightarrow{p} O \]
for all $t > 0$.

Then relation (11) holds for all $t$, where $R_T(t) = \int_0^t \rho_T(\tau) d\tau$.

Proof. Let $0 = t_0 < \cdots < t_m = t$ be a partition of the interval $[0, t]$. Denote
\[ \psi_{T,m}(t_1, \ldots, t_m) = \frac{1}{T} \sum_{k=1}^m \int_{T_{k-1}}^{T_k} \left( \rho_T \left( \frac{s}{T} \right) - \rho_T(t_k) \right) d\Phi(s), \]
\[ \chi_{T,m}(t_1, \ldots, t_m) = \sum_{k=1}^m \rho_T(t_k) \left( T^{-1} \Phi(Tt_k) - \Xi_T t_k - (T^{-1} \Phi(Tt_{k-1}) - \Xi_T t_{k-1}) \right), \]
\[ \alpha_{T,m}(t_1, \ldots, t_m) = \max_{0 < k \leq m} \sup_{t_{k-1} \leq \tau < t_k} |\rho_T(\tau) - \rho_T(t_k)|, \]
\[ \beta_{T,m}(t_1, \ldots, t_m) = \sum_{k=1}^m \rho_T(t_k)(t_k - t_{k-1}) - R_T(t). \]

Considering equalities (26) and (27), we write
\[ U_T(t) - \Xi_T R_T(t) = \psi_{T,m}(t_1, \ldots, t_m) + \chi_{T,m}(t_1, \ldots, t_m) + \Xi_T \beta_{T,m}(t_1, \ldots, t_m). \]

In view of (31) and (29),
\[ \chi_{T,m}(t_1, \ldots, t_m) \xrightarrow{p} O. \]

By Lemma 2
\[ \left\| \int_{T_{k-1}}^{T_k} \left( \rho_T \left( \frac{s}{T} \right) - \rho_T(t_k) \right) dH(s) \otimes \nu(s)^2 \right\| \leq \int_{T_{k-1}}^{T_k} \left| \rho_T \left( \frac{s}{T} \right) - \rho_T(t_k) \right| |\nu(s)|^2 d\text{tr} H(s), \]
whence

\[(34) \quad |\psi_{T,m}(t_1, \ldots, t_m)| \leq \alpha_{T,m}(t_1, \ldots, t_m)V(Tt)/T,\]

where \( V(t) = \int_0^t |\varphi(s)|^2 \, d\text{tr}\, H(s). \)

By the same lemma, \( \|\Phi(t)\| \leq V(t) \), whence we obtain, in view of \( (28) \) and \( (31) \),

\[(35) \quad \lim_{N \to \infty} \lim_{T \to \infty} P\{\|\Xi_T\| > N\} = 0.\]

Now we conclude from \( (32) - (34) \) and from the following obvious inclusions

\[
\{\alpha_{T,m}(t_1, \ldots, t_m)V(Tt)/T \geq c\} \subset \{V(Tt) > TN\} \cup \{\alpha_{T,m}(t_1, \ldots, t_m) \geq c/N\},
\]

\[
\{\|\beta_{T,m}(t_1, \ldots, t_m)\|_T \geq c\} \subset \{\|\Xi_T\| > N\} \cup \{\|\beta_{T,m}(t_1, \ldots, t_m)/t \geq c/N\}
\]

that

\[
\lim_{T \to \infty} P\{\|U_T(t) - \Xi_T R_T(t)\| \geq \varepsilon\} \\
\leq \lim_{T \to \infty} P\{V(Tt) > TN\} + \lim_{T \to \infty} P\{\|\Xi_T\| > N\} \\
+ \lim_{T \to \infty} P\{\alpha_{T,m}(t_1, \ldots, t_m) \geq \varepsilon/3N\} + \lim_{T \to \infty} P\{\|\beta_{T,m}(t_1, \ldots, t_m)\|_T \geq \varepsilon/3N\}
\]

for all \( \varepsilon > 0 \). This together with \( (28) - (30) \) and \( (35) \) implies \( (11) \).

**Corollary 6** (of Lemmas\( [8] \) and \( [10] \)). Let all the assumptions of Lemma\( [10] \) hold. Further, let condition \( (10) \) hold for all \( t > 0 \), where \( R_T(t) = \int_0^t \rho_T(\tau) \, d\tau \). Then equality \( (12) \) holds for all \( t > s > 0 \) and \( \varepsilon > 0 \).

A stochastic process whose values are \( \mathcal{F}(0) \)-measurable random variables is called \( \mathbb{F}^0 \)-adapted.

**Theorem 6.** Let stochastic processes \( Y_T \) and \( R_T \) be given by the equalities

\[(36) \quad Y_T(t) = \frac{1}{\sqrt{T}} \int_0^T \sigma_T(s) \, dM(s) \otimes \varphi(s)\]

and

\[(37) \quad R_T(t) = \int_0^t \sigma_T(s)^2 \, ds,\]

respectively, where \( \sigma_T \) is an \( \mathbb{F}^0 \)-adapted \( \mathbb{R} \)-valued stochastic process whose trajectories belong to the space \( D \) and where \( M \) is an \( \mathbb{R}^d \)-valued locally square integrable martingale with respect to the flow \( \mathbb{F} = (\mathcal{F}(t), t \in \mathbb{R}_+) \). Here \( \varphi \) is an \( \mathbb{R}^d \)-valued \( \mathbb{F} \)-predictable stochastic process such that

\[(37) \quad \int_0^t |\varphi(s)|^2 \, d\text{tr}\,(M)(s) < \infty, \quad t \in \mathbb{R}_+.\]

Assume that

1) \( (38) \quad \lim_{L \to \infty} \lim_{T \to \infty} P\{\text{tr}(M)(T) > TL\} = 0, \)

2) \( \lim_{T \to \infty} T^{-1} \mathbb{E}_{\max_{s \leq T}} |\Delta M(s)|^2 = 0; \)

3) the net \( (\sigma_T^2) \) is relatively compact in the space \( C \);
4) there exists a net \((\Xi_T)\) of random tensors such that
\[
\frac{1}{T} \int_0^T d(M)(s) \otimes \varphi(s)^2 - \Xi_T \overset{\mathcal{P}}{\to} O
\]
for all \(t > 0\);
5) there exist a random tensor \(\Xi\) and a real-valued stochastic process \(R\) defined on a joint probability space such that
\[
(\Xi_T, R_T(t_1), \ldots, R_T(t_l)) \overset{d}{\to} (\Xi, R(t_1), \ldots, R(t_l))
\]
for all \(l \in \mathbb{N}\) and \(t_1, \ldots, t_l \in \mathbb{R}_+\) and
\[
P\{R(t) = 0\} = 0
\]
for all \(t > 0\).

Then
a) \((Y_T, \langle Y_T \rangle) \overset{C}{\to} (Y, \langle Y \rangle)\), where \(Y\) is a continuous local martingale with initial value \(O\) and quadratic characteristic \(\langle Y \rangle(t) = \Xi^2 R(t)\);

b) the process \(Y\) has conditionally independent increments with respect to \((\Xi, R(\cdot))\) (and thus the increments are conditionally independent with respect to \(\langle Y \rangle(\cdot), \text{too}\));

c) for all \(t > s \geq 0\) and \(z \in \mathbb{R}^{d^2}\),
\[
\mathbb{E} \left( e^{iz(\langle Y \rangle(t) - \langle Y \rangle(s))} \mid \langle Y \rangle(t), \cdot \right) = e^{-z(\langle Y \rangle(t) - \langle Y \rangle(s))z^T/2}.
\]

Proof. Equality \((36)\) is a partial case of \((1)\) corresponding to \(Z_T(s) = M(Ts)/\sqrt{T}\) and \(\vartheta_T(s) = \rho_T(s) \varphi(Ts)\). Thus the assumptions of Theorem 4 should be checked for such \(Z_T\) and \(\vartheta_T\) and for \(F_T(t) = \mathcal{F}(Ts)\).

Obviously, condition \((23)\) is equivalent to \((39)\) in this case.

Put \(\rho_T = \sigma_T^2\) and \(S_T^N(t) = \int_0^t |\vartheta_T(s)|^2 I\{|\vartheta_T(s)| > N\} \, d\text{tr}(Z_T(s)).\) By assumption, the processes \(\sigma_T\) do not have discontinuities of the second kind and thus are bounded on every finite interval. Hence \((37)\) implies \((2)\). The net \((\rho_T)\) is relatively compact in the space \(C\) and satisfies conditions \((29)\) and \((30)\) by Theorem 1. Substituting \(\vartheta_T(s)\) and \(Z_T(s)\) into the latter equation, we transform it to the form
\[
S_T^N(t) = \frac{1}{T} \int_0^T \left| \rho_T \left( \frac{T}{T} \right) \right| |\varphi(s)|^2 I \left\{ \left| \sigma_T \left( \frac{T}{T} \right) \right| |\varphi(s)| > N \right\} \, d\text{tr}(M)(s).
\]

Then we conclude that
\[
\left\{ |S_T^N(t)| > \varepsilon \right\} \subset \left\{ \sup_{s \leq t} |\rho_T(s)| > L \right\} \cup \left\{ \int_0^T |\varphi(s)|^2 I \left\{ |\varphi(s)| > \frac{N}{\sqrt{L}} \right\} \, d\text{tr}(M)(s) > T\varepsilon \right\},
\]
and thus \((24)\) follows from \((40)\) and \((29)\).

Now \((36)\) together with Lemma 5 implies that \(\langle Y_T \rangle = U_T^g\), where
\[
U_T(t) = \frac{1}{T} \int_0^T \rho_T \left( \frac{s}{T} \right) \, d(M)(s) \otimes \varphi(s)^{2}\.
\]

We are going to show that \(U_T \overset{C}{\to} \Xi R\) (this means that condition \((25)\) holds with \(G = \Xi^2 R\)).

Since \((\rho_T)\) is relatively compact in \(C\), the same property holds for the net \((R_T)\). Thus relation \((12)\) follows from \((11)\).
It is clear that $|\varphi(s)|^2 \leq |\varphi(s)|^2 I\{|\varphi(s)| > N\} + N^2$ for all $N > 0$. Thus
\[
\lim_{T \to \infty} \mathbb{P}\left\{ \int_0^T |\varphi(s)|^2 \, d\text{tr}(M)(s) > TL \right\}
\leq \mathbb{P}\{\text{tr}(M)(T) > TL/N^2\} + \lim_{T \to \infty} \mathbb{P}\left\{ \int_0^T |\varphi(s)|^2 I\{|\varphi(s)| > N\} \, d\text{tr}(M)(s) > T \right\}
\]
for $L > 1$, whence we derive (28) for $H = \langle M \rangle$ in view of (38) and (40). Conditions (29) and (30) are checked above, while condition (31) is equivalent to (11) in this case, since (44) means the same as (26) and (27) with $H = \langle M \rangle$. Now (11) follows from Lemma 10.

By Aleksandroff’s theorem, the relation $R_T(t) \xrightarrow{d} R(t)$ (being a part of condition (42)) implies that $\lim_{T \to \infty} \mathbb{P}\{R_T(t) \leq r\} \leq \mathbb{P}\{R(t) \leq r\}$, whence
\[
\lim_{r \to 0} \lim_{T \to \infty} \mathbb{P}\{R_T(t) \leq r\} \leq \mathbb{P}\{R(t) = 0\}
\]
which together with (43) yields (10). Now Corollary 6 proves relation (12) or, which is the same, relation (22) with
\[
\varphi_T^t(s,t,x) = g_r(R_T(t)) h_r(R_T(s)) x, \quad x \in \mathbb{R}^{d_4}.
\]
Since $\sigma_T$ is $\mathbb{F}^0$-measurable (this implies that $R_T$ also is $\mathbb{F}^0$-measurable), each of the random functions $\varphi_T^t$ is uniformly $(\mathbb{F}_T, \mathcal{B}(\mathbb{R}^{d_4}))$-adapted. Therefore, condition $\mathcal{B}$ holds, and Theorem 4 completes the proof.

Bibliography


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