Теорія Ймовір. та Матем. Статист.
Вип. 90, 2014

Theor. Probability and Math. Statist.
No. 90, 2015, Pages 207–221
http://dx.doi.org/10.1090/tpms/961
Article electronically published on August 11, 2015

CONVERGENCE OF STOCHASTIC INTEGRALS TO A CONTINUOUS LOCAL MARTINGALE WITH CONDITIONALLY INDEPENDENT INCREMENTS

UDC 519.21

ANDRIY YURACHKIVSKY

Abstract. For each $T > 0$, let a tensor-valued stochastic process Y_T be defined by

$$Y_T(t) = \int_0^t DZ_T(s) \otimes \vartheta_T(s),$$

where Z_T is an $\mathbf{R}^d$-valued locally square integrable martingale with respect to some filtration $\mathbb{F}_T$ and where ϑ_T is an $\mathbf{R}^d$-valued $\mathbb{F}_T$-predictable stochastic process such that $\int_0^t |\vartheta_T(s)|^2 D \operatorname{tr}\langle Z_T\rangle(s) < \infty$ for all t. In this paper, conditions are found for the convergence $(Y_T, \langle Y_T\rangle) \xrightarrow{\text{law}} (Y, \langle Y\rangle)$, where Y is a continuous local martingale with conditionally independent increments given $\langle Y\rangle$.

Introduction

The results on the convergence in distribution of a sequence of semimartingales to a stochastic process with independent increments is the classics of stochastic analysis [1, 2]. In the monographs [1, 2] as well as in a number of papers on this topic (see, for example, [3]), one of the main assumptions is that the predictable characteristics converge in probability. This assumption is restrictive and, moreover, simply unnatural if prelimit processes are defined on different probability spaces and predictable characteristics of the limit process are random. A sufficient condition is found in [4] for the asymptotic independence of increments of stochastic processes (possibly defined on different probability spaces) if their jumps are asymptotically small. However, that condition is not easy to check in general. Example 4.8 in [4] exhibits, nevertheless, the case where the condition is easy to check.

The aim of this paper is to find sufficient conditions for the asymptotic independence of increments for a subclass of locally square integrable martingales. The conditions below are more general than in Example 4.8 of [4] and, at the same time, can effectively be checked for the class under consideration. By using the results of [4], such a property of prelimit processes allows us to change the assumption on the convergence of quadratic characteristics in probability by that for the convergence in distribution.

Throughout below the symbol $\mathbb{T}$ stands for both sets of positive real numbers and positive integer numbers. Recall that an arbitrary function on a directed set is called a *net* ($\mathbb{T}$ plays the role of a directed set in this paper). A special case of a net is a sequence (or a function) on $\mathbf{N}$. Considering a net $(X_T, T \in \mathbb{T})$ of stochastic processes, we admit the possibility that the probability spaces are, generally speaking, different for

2010 *Mathematics Subject Classification.* Primary 60F17; Secondary 60G44.
Key words and phrases. Martingale, convergence, tensor.

different T. In view of this remark, any appearance of symbols like X_T or Y_T indicates that P and E should be understood as P_T and E_T, respectively.

All vectors are thought of, unless otherwise stated, as columns. All matrices are tacitly assumed square ones. We use the Euclidean norm $|\cdot|$ of vectors and the operator norm $\|\cdot\|$ of matrices. By $\mathbb{R}^d$ and $\mathbb{R}^{d*}$ we denote the spaces of d-dimensional column vectors and row vectors, respectively. $\mathfrak{S}$ and $\mathfrak{S}_+$ signify, respectively, the class of all symmetric square matrices of a fixed size (defined by context) with real entries and its subclass of nonnegative (in the spectral sense) matrices. The coordinates of row vectors are written with subscript, and those of column vectors with superscript. Since matrices of a fixed size are elements of a finite-dimensional vector space, we treat them, if necessary, as vectors, in which case we write $|\cdot|$ instead of $\|\cdot\|$. The same concerns the tensors considered below.

Some notions of the tensor algebra used throughout the paper can be found in § 24 of [5] or in § 23 and § 24 of [6]. The tensor multiplication is denoted by $\otimes$. In particular, $a \otimes b = ab^\top$ for $(0,1)$-tensors or, which is the same, for vector columns.

If a reader wants to concentrate on purely probabilistic aspects of the below results, then one can omit tensors by assuming that the dimensions d and p are equal to unity (then the tensor product is the product of real numbers). It is worth mentioning, however, that an estimator of a matrix parameter of a stochastic process is a random tensor and, as a function of the duration of observation, is a tensor valued stochastic process. Therefore if one wants function theorems to be applicable for an investigation of asymptotic properties of estimators of parameters (including matrix parameters) of stochastic processes, then the restriction to the scalar case is not appropriate.

Let X_T, $T \in \mathbb{T}$, and X be $\mathbf{R}^p$-valued or $\mathfrak{S}$-valued stochastic processes whose trajectories belong to the Skorokhod space D (that is, the trajectories are right continuous processes on $\mathbf{R}_+$ that have no discontinuities of the second kind). We write

$$X_T \xrightarrow{D} X$$

if the measures generated by the processes X_T in the Borel σ-algebra of D weakly converge as $T \to \infty$ to the measure generated by the process X. If, in addition, the process X is continuous, then we write $X_T \xrightarrow{C} X$.

A net (X_T) is called *relatively compact in D (in C)* if every its cofinal subnet contains a further cofinal net that converges in the appropriate sense.

The convergence in distribution in $\mathbf{R}^p$ is denoted by $\xrightarrow{\mathrm{d}}$.

For $f \in D$, let

$$\Delta f(t) = f(t) - f(t-);$$

the integral $\int_{t_1}^{t_2}$ is understood as $\int_{]t_1,t_2]}$. The parameter T is used to distinguish between the stochastic processes and does not correspond to the time variable. The time variable is written in parentheses as, for example, in equality (1) below. Each appearance of the symbol $\to$ corresponds to the passing to the limit as $T \to \infty$.

All locally square integrable martingales are assumed to be right continuous and such that the left limits exist. This assumption does not restrict the generality in view of Theorem 1.3.2 in [7].

The net (Y_T) of tensor valued stochastic processes for which we find conditions for the convergence to a process Y with conditionally independent increments is given by

$$Y_T(t) = \int_0^t dZ_T(s) \otimes \vartheta_T(s), \tag{1}$$

where, for each T, Z_T is an $\mathbf{R}^d$-valued locally square integrable martingale with respect to some filtration $\mathbb{F}_T$ on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, while ϑ_T is an $\mathbf{R}^d$-valued $\mathbb{F}_T$-predictable stochastic process such that

$$\int_0^t |\vartheta_T(s)|^2 \, d\,\mathrm{tr}\langle Z_T\rangle(s) < \infty, \qquad t \in \mathbf{R}_+. \tag{2}$$

The main results of this paper, Theorems 4–6, provide more information than simply the convergence $Y_T \xrightarrow{C} Y$. To investigate the asymptotic properties of estimators of parameters of stochastic processes, one needs to use all the information provided by Theorems 4–6 that does not use the unnatural assumption mentioned above, while the σ-algebra for which the increments of the limit process are conditionally independent arises as a result of a limit procedure and is a generalization of the σ-algebra of invariant sets well known in ergodic theory.

1. Some preliminary results

Lemma 1. *For each $d \times d$ matrix $B \in \mathfrak{S}_+$,*

$$\|B\| \leqslant \mathrm{tr}\, B \leqslant d \cdot \|B\|.$$

Proof. The norm of a real symmetric matrix, as is well known, equals the maximal absolute value of its eigen numbers. □

Corollary 1. *For each $d \times d$ matrix A,*

$$\mathrm{tr}\, AA^\top \leqslant d \cdot \|A\|^2.$$

In the space of tensors of order four, consider the norm such that $\|A \otimes B\| = \|A\|\,\|B\|$ for all square matrices A and B. Lemma 1 implies the following result.

Corollary 2. *Let A and B be two matrices of the same sizes, and let $B \in \mathfrak{S}_+$. Then $\|A \otimes B\| \leqslant \|A\|\,\mathrm{tr}\, B$.*

We write $A \leqslant B$ for two matrices A and B if $B - A \in \mathfrak{S}_+$.

Lemma 2. *Let $D_1(\cdot)$ and $D_2(\cdot)$ be measurable matrix valued functions on $]t_1, t_2]$. Assume that $D_2(\cdot)$ is increasing and right continuous. Then*

$$\left\| \int_{t_1}^{t_2} dD_2(s) \otimes D_1(s) \right\| \leqslant \int_{t_1}^{t_2} \|D_1(s)\| \; d\,\mathrm{tr}\, D_2(s)$$

if the right-hand side of the inequality is finite.

Proof. The result follows easily from Corollary 2 for piecewise constant functions $D_2(\cdot)$ (in this case, the integrals become the sums). The general case is derived from this particular one by using a standard procedure. □

Given an arbitrary matrix A, we denote by $\underline{A}$ the vector obtained by writing its columns one after another.

Lemma 3. *For each $d \times d$ matrix A, $|\underline{A}| \leqslant \sqrt{d}\,\|A\|$.*

Proof. Denoting by a_i the column i of the matrix, one can write $\sum |a_i|^2 = \mathrm{tr}\, AA^\top$. It remains to apply Corollary 1. □

In what follows we write $\underline{Y}_T$ rather than $\underline{Y_T}$.

Lemma 4. *For every $T > 0$, let Y_T be a $(0,2)$-tensor valued locally square integrable martingale. Assume that there exists a vector valued locally square integrable martingale X such that $(\underline{Y}_T, \langle \underline{Y}_T\rangle) \xrightarrow{D} (X, \langle X\rangle)$. Then $(Y_T, \langle Y_T\rangle) \xrightarrow{D} (Y, \langle Y\rangle)$, where Y is a $(0,2)$-tensor valued locally square integrable martingale such that $\underline{Y} = X$.*

Denote by $^\sharp$ the linear operation in the space of tensors of order four that acts as follows:

$$(a_1 \otimes a_2 \otimes a_3 \otimes a_4)^\sharp = a_1 \otimes a_3 \otimes a_2 \otimes a_4.$$

If Z_1 and Z_2 are tensor valued locally square integrable martingales, then we denote the compensator of the process $Z_1 \otimes Z_2$ by $\langle Z_1, Z_2 \rangle$. We write $\langle Z \rangle$ instead of $\langle Z, Z \rangle$.

Lemma 5. *Let Z be an $\mathbf{R}^d$-valued locally square integrable martingale with respect to the flow of the σ-algebras $\mathbb{F} = (\mathcal{F}(t), t \in \mathbf{R}_+)$, and let ϑ be an $\mathbf{R}^d$-valued $\mathbb{F}$-predictable stochastic process such that $\int_0^t |\vartheta(s)|^2 \, d\,\mathrm{tr}\langle Z \rangle(s) < \infty$ for all t. Then*

1) *for every t, the stochastic integral $Y(t) \equiv \int_0^t dZ(s) \otimes \vartheta(s)$ is well defined;*
2) *Y is a locally square integrable martingale with quadratic characteristic*

$$\langle Y \rangle(t) = \left(\int_0^t d\langle Z \rangle(s) \otimes \vartheta(s)^{\otimes 2} \right)^\sharp; \tag{3}$$

3) $\mathrm{tr}\, \langle \underline{Y} \rangle\, (t) = \int_0^t |\vartheta(s)|^2 \, d\,\mathrm{tr}\langle Z \rangle(s)$.

Proof. The first two statements for the case of $d = 1$ are contained in Theorem I.4.40 of [2]. It is obvious that the first statement remains valid for the general case as well. It is also obvious that the process Y is a locally square integrable martingale for $d > 1$, since this property holds for $d = 1$.

The second statement holds in the case of a general d for integrands

$$\upsilon(t) = \alpha_0 I_{\{0\}}(t) + \sum_{i=1}^m \alpha_i I_{]s_{i-1}, s_i]}(t),$$

where $0 = s_0 < \cdots < s_m$ and α_i is an $\mathbf{R}^d$-valued $\mathcal{F}(s_{(i-1)+})$-measurable random variable for each $i \in \{0, \ldots, m\}$. Indeed, in this case

$$Y(t) = \sum_{i=1}^m (Z(s_i \wedge t) - Z(s_{i-1} \wedge t)) \otimes \alpha_i,$$

$$Y^{\otimes 2}(t) = \left(\sum_{i,j=1}^m \big((Z(s_i \wedge t) - Z(s_{i-1} \wedge t)) \otimes (Z(s_j \wedge t) - Z(s_{j-1} \wedge t)) \big) \otimes \alpha_i \otimes \alpha_j \right)^\sharp,$$

whence

$$\langle \Upsilon \rangle(t) = \left(\sum_{i=1}^m (\langle Z \rangle(s_i \wedge t) - \langle Z \rangle(s_{i-1} \wedge t)) \otimes \alpha_i^{\otimes 2} \right)^\sharp.$$

The latter equality is nothing else as (3). The general case follows from this particular one in a standard way.

By construction, the vector $\underline{Y}(t)$ consists of columns

$$\int_0^t \vartheta(s) \, dZ_1(s), \quad \ldots, \quad \int_0^t \vartheta(s) \, dZ_d(s)$$

written one after another, where Z_i is the component i of the vector process Z. Thus the $d^2 \times d^2$ matrix $\langle \underline{Y} \rangle\, (t)$ consists of d^2 blocks $\int_0^t \vartheta(s)^{\otimes 2} \, d\langle Z_i, Z_j \rangle(s)$ of size $d \times d$. Hence,

$$\mathrm{tr}\, \langle \underline{Y} \rangle\, (t) = \sum_{i=1}^d \int_0^t \mathrm{tr}\, \upsilon(s)^{\otimes 2} \;\, d\langle Z_i \rangle(s).$$

It remains to note that $\mathrm{tr}\, x^{\otimes 2} = |x|^2$ for an arbitrary $x \in \mathbf{R}^d$ to establish the third statement of the lemma. $\square$

For all $a, b \in \mathbf{R}^d$ and $u, v \in \mathbf{R}^{d*}$, we have

$$(ua)(vb) = uab^\top v^\top = u(a \otimes b)v^\top.$$

Repeating the proof of Lemma 5 with obvious changes, we obtain the following result.

Lemma 6. *Let ζ be an $\mathbf{R}^d$-valued locally square integrable martingale with respect to the flow of σ-algebras $\mathbb{F} = (\mathcal{F}(t), t \in \mathbf{R}_+)$, and let ψ be an $\mathbf{R}^{d*}$-valued $\mathbb{F}$-predictable stochastic process such that $\int_0^t |\psi(s)|^2 \, d\operatorname{tr}\langle\zeta\rangle(s) < \infty$ for all t. Then*

1) *for all t, the stochastic integral $\Psi(t) \equiv \int_0^t \psi(s)\, d\zeta(s)$ is well defined;*
2) *Ψ is a locally square integrable martingale with characteristic*

$$\langle\Psi\rangle(t) = \int_0^t \psi(s)\, d\langle\zeta\rangle(s)\psi(s)^\top.$$

Corollary 3. *Let ζ, ψ, and Ψ be the same as in Lemma 6. Also let*

$$\eta(t) = \int_0^t \chi(s)\psi(s)\, d\zeta(s),$$

where χ is an $\mathbb{F}$-predictable $[0, 1]$-valued stochastic process. Then

$$\langle\eta\rangle(t_2) - \langle\eta\rangle(t_1) \leqslant \langle\Psi\rangle(t_2) - \langle\Psi\rangle(t_1)$$

for all $t_2 > t_1 \geqslant 0$.

Put $\Pi(t, c) = \{(t_1, t_2) \colon (t_2 - c)_+ \leqslant t_1 < t_2 \leqslant t\}$, and recall the well known criterion of the relative compactness in the space C (see, for example Proposition VI.3.26 in [2] and Remarks VI.3.25 and VI.3.9 therein).

Theorem 1. *A net $(\xi_T, T \in \mathbb{T})$ of stochastic processes whose trajectories belong to the space D is relatively compact in the space C if and only if*

$$\lim_{L\to\infty} \overline{\lim_{T\to\infty}} \mathsf{P}\left\{\sup_{s\leqslant t} |\xi_T(s)| > L\right\} = 0,$$

$$\lim_{c\to 0} \overline{\lim_{T\to\infty}} \mathsf{P}\left\{\sup_{(t_1,t_2)\in\Pi(t,c)} |\xi_T(t_2) - \xi_T(t_1)| > \varepsilon\right\} = 0$$

for all positive t and ε.

Theorem 1 is usually stated in the literature for the case of $\mathbb{T} = \mathbf{N}$ which obviously implies the case of $\mathbb{T} = \{T \in \mathbf{R} \colon T > 0\}$.

Corollary 4 (of Theorem 1 and Lemma 1)**.** *The relative compactness in the space C of the net (V_T) of right continuous increasing $\mathfrak{S}$-valued stochastic processes is equivalent to the relative compactness of the net $(\operatorname{tr} V_T)$.*

Theorem 1, Lemma 2, and an obvious inequality

$$|\varphi_T(s)|^2 \leqslant N^2 + |\varphi_T(s)|^2 I\{|\varphi_T(s)| > N\}$$

allow one to state the following result.

Corollary 5. *For every $T \in \mathbb{T}$, let a stochastic process U_T be defined by*

$$U_T(t) = \int_0^t dH_T(s) \otimes \varphi_T(s)^{\otimes 2}, \tag{4}$$

where H_T is a right-continuous-increasing and $\mathfrak{S}$ valued stochastic process and where φ_T is an $\mathbf{R}^d$-valued measurable stochastic process. Assume that a net $(\operatorname{tr} H_T)$ is relatively compact in the space C and

$$\lim_{N\to\infty}\overline{\lim_{T\to\infty}}\ \mathsf{P}\left\{\int_0^t |\varphi_T(s)|^2 I\{|\varphi_T(s)|>N\}\ d\operatorname{tr} H_T(s)>\varepsilon\right\}=0 \tag{5}$$

for all positive ε and t. Then the net (U_T) also is relatively compact in the space C.

Put $S=\{(s,t)\in\mathbf{R}^2\colon 0<s<t\}$. Let $(\Theta,\mathcal{E})$ be a measurable space, and let

$$\mathbb{H}\equiv(\mathcal{H}(t),t\in\mathbf{R}_+)$$

be a filtration in the probability space $(\Omega,\mathcal{F},\mathsf{P})$ (the filtration may depend on T). We say that a Θ-valued random function $\varkappa$ defined on $S\times\Theta$ is *uniformly* $(\mathbb{H},\mathcal{E})$*-adapted* if, for all $t>s>0$, the random function $\varkappa(s,t,\cdot)$ of the variable $\theta\in\Theta$ is $\mathcal{H}(s)\otimes\mathcal{E}$-measurable with respect to $(\omega,\theta)\in\Omega\times\Theta$.

The Borel σ-algebra in the metric space $\mathfrak{X}$ is denoted by $\mathcal{B}(\mathfrak{X})$.

Lemma 7. *For every $T\in\mathbb{T}$, let Q_T be an $\mathbb{F}_T$-adapted stochastic process assuming values in a metric space $(\mathfrak{X},\varrho)$. Assume that there exists a family $\{\varkappa_T^r, r>0, T\in\mathbb{T}\}$ of $\mathfrak{X}$-valued random functions defined on $S\times\mathfrak{X}$ such that each $\varkappa_T^r$ is uniformly $(\mathbb{F}_T,\mathcal{B}(\mathfrak{X}))$-adapted and*

$$\lim_{r\to 0}\overline{\lim_{T\to\infty}}\ \mathsf{P}\left\{\varrho\big(Q_T(t),\varkappa_T^r(s,t,Q_T(s))\big)>\varepsilon\right\}=0 \tag{6}$$

for all $t>s>0$ and $\varepsilon>0$. Then

$$\mathsf{E}(f(Q_T(t))\mid\mathcal{F}_T(s))-f(Q_T(t))\xrightarrow{\mathsf{P}}0 \tag{7}$$

for all $t>s>0$ and all bounded uniformly continuous functions $f\colon\mathfrak{X}\to\mathbf{R}$.

Proof. Fix $t>s>0$ and a bounded uniformly continuous function defined on $\mathfrak{X}$. Put $q_T^r=\varkappa_T^r(s,t,Q_T(s))$ and $\gamma_T^r=f(Q_T(t))-f(q_T^r)$. Equality (6) together with the obvious inequality

$$\mathsf{E}\gamma_T^r\leqslant 2\|f\|_\infty\mathsf{P}\{\varrho(Q_T(t),q_T^r)>\varepsilon\}+\sup_{\varrho(x,y)\leqslant\varepsilon}|f(x)-f(y)|$$

implies that

$$\lim_{r\to 0}\overline{\lim_{T\to\infty}}\ \mathsf{E}\,|\gamma_T^r|=0. \tag{8}$$

Moreover

$$\lim_{r\to 0}\overline{\lim_{T\to\infty}}\ \mathsf{E}\,|\mathsf{E}\,(\gamma_T^r\mid\mathcal{G}_T)|=0$$

for all σ-algebras $\mathcal{G}_T\subset\mathcal{F}_T$. Since Q_T is $\mathbb{F}_T$-adapted and $\varkappa_T^r$ is uniformly $(\mathbb{F}_T,\mathcal{B}(\mathfrak{X}))$-adapted, the random variable q_T^r is $\mathcal{F}_T(s)$-measurable. The latter relation with

$$\mathcal{G}_T=\mathcal{F}_T(s)$$

is rewritten as follows:

$$\lim_{r\to 0}\overline{\lim_{T\to\infty}}\ \mathsf{E}\big|\mathsf{E}(f(Q_T(t))\mid\mathcal{F}_T(s))-f(q_T^r)\big|=0.$$

Comparing with (8) we conclude that

$$\lim_{T\to\infty}\mathsf{E}\big|\mathsf{E}(f(Q_T(t))\mid\mathcal{F}_T(s))-f(Q_T(t))\big|=0,$$

which is equivalent to (7). □

Put $g_r(x)=(|x|\vee r)\operatorname{sgn}x$ and $h_r(x)=(|x|\vee r)^{-1}\operatorname{sgn}x$, $x\in\mathbf{R}$, $r>0$.

Lemma 8. 1) *Let* (R_T) *and* (U_T) *be two nets of* $\mathbf{R}$*-valued and* $\mathbf{R}^p$*-valued stochastic processes, respectively; and*

2) *let* (Ξ_T) *be a net of* $\mathbf{R}^p$*-valued random variables such that*

$$\lim_{L\to\infty}\overline{\lim_{T\to\infty}}\,\mathsf{P}\{|R_T(t)|>L\}=0, \tag{9}$$

$$\lim_{r\to 0}\overline{\lim_{T\to\infty}}\,\mathsf{P}\{|R_T(t)|\leqslant r\}=0, \tag{10}$$

$$U_T(t)-\Xi_T R_T(t)\xrightarrow{\mathsf{P}}0 \tag{11}$$

for all $t>0$.

Then

$$\lim_{r\to 0}\overline{\lim_{T\to\infty}}\,\mathsf{P}\left\{\left|U_T(t)-g_r(R_T(t))h_r(R_T(s))U_T(s)\right|>\varepsilon\right\}=0 \tag{12}$$

for all $t>s>0$ *and* $\varepsilon>0$.

Proof. Put $J_T(t)=U_T(t)/R_T(t)$ (if $R_T(t)\neq 0$). It is clear that

$$U_T(t)-g_r(R_T(t))h_r(R_T(s))U_T(s)=R_T(t)(J_T(t)-J_T(s))\quad\text{for } |R_T(t)|\wedge|R_T(s)|\geqslant r.$$

Hence

$$\begin{aligned}&\{|U_T(t)-g_r(R_T(t))h_r(R_T(s))U_T(s)|>\varepsilon\}\\&\qquad\subset\{|R_T(t)|\leqslant r\}\cup\{|R_T(s)|\leqslant r\}\cup\{|R_T(t)|\;|J_T(t)-J_T(s)|>\varepsilon\}\cap\{|R_T(t)|>r\}\\&\qquad\quad\cap\{|R_T(s)|>r\}.\end{aligned}$$

It is also clear that

$$\{|R_T(t)|\cdot|J_T(t)-J_T(s)|>\varepsilon\}\subset\{|R_T(t)|>L\}\cup\{|J_T(t)-J_T(s)|>\varepsilon/L\}$$

for all $L>0$. Thus

$$\begin{aligned}&\mathsf{P}\{|U_T(t)-g_r(R_T(t))h_r(R_T(s))U_T(s)|>\varepsilon\}\\&\qquad\leqslant\mathsf{P}\{|R_T(t)|\leqslant r\}+\{|R_T(s)|\leqslant r\}+\mathsf{P}\{|R_T(t)|>L\}\\&\qquad\quad+\mathsf{P}\{|J_T(t)-\Xi_T|>\varepsilon/2L,|R_T(t)|>r\}\\&\qquad\quad+\mathsf{P}\{|J_T(s)-\Xi_T|>\varepsilon/2L,|R_T(s)|>r\}\end{aligned}$$

for all $t>s>0$ and positive ε, r, and L. In view of (9) and (10), this means that the result desired follows from

$$\lim_{T\to\infty}\mathsf{P}\{|J_T(t)-\Xi_T|>\delta,\,|R_T(t)|>r\}=0$$

for all positive t, δ, and r. In turn, this property follows from (11) and an obvious inclusion $\{|R_T(t)|>r\}\cap\{|J_T(t)-\Xi_T|>\delta\}\subset\{|U_T(t)-\Xi_T R_T(t)|>r\delta\}$. □

The following result is proved similarly.

Lemma 9. 1) *Let* (R_T) *and* (U_T) *be two nets of* $\mathbf{R}$*-valued and* $\mathbf{R}^p$*-valued stochastic processes, respectively;*

2) *let* (Ξ_T) *be a net of* $\mathbf{R}^p$*-valued random variables such that relations* (9) *and* (11) *hold for all* $t>0$*; and*
3) *let* $R_T(t)\neq 0$ *for all* $t>0$ *and sufficiently large* $T\in\mathbb{T}$.

Then

$$U_T(t)-R_T(t)R_T(s)^{-1}U_T(s)\xrightarrow{\mathsf{P}}0$$

for all $t>s>0$ *and* $\varepsilon>0$.

2. General results on the asymptotic conditional independence of increments of locally square integrable martingales

Two theorems of the paper [4] are presented below in a more convenient form. The proof of the main results uses only Theorem 3, while Theorem 2 helps to better understand the key assumption **A** given below:

A. there exists a family $\{\phi_T^r, r > 0, T \in \mathbb{T}\}$ of $\mathfrak{S}_+$-valued random functions on $S \times \mathfrak{S}_+$ such that every ϕ_T^r is uniformly $(\mathbb{F}_T, \mathcal{B}(\mathfrak{S}_+))$-adapted and

$$\lim_{r\to 0} \overline{\lim_{T\to\infty}} \mathsf{P}\{\|\langle X_T\rangle(t) - \phi_T^r(s,t,\langle X_T\rangle(s))\| > \varepsilon\} = 0$$

for all $t > s > 0$ and $\varepsilon > 0$.

Theorem 2. *For every $T \in \mathbb{T}$, let $X_T, X_T^1, X_T^2, \dots$ be $\mathbf{R}^p$-valued locally square integrable martingales with respect to a flow $\mathbb{F}_T = (\mathcal{F}_T(t), t \in \mathbf{R}_+)$. Assume that*

1) *condition* **A** *holds and that a net $(\mathrm{tr}\langle X_T\rangle, T \in \mathbb{T})$ is relatively compact in the space C;*
2)
$$\langle zX_T^m\rangle(t_2) - \langle zX_T^m\rangle(t_1) \le \langle zX_T\rangle(t_2) - \langle zX_T\rangle(t_1) \tag{13}$$
for all $t_2 > t_1 \geqslant 0$, $m \in \mathbf{N}$, and $z \in \mathbf{R}^{p*}$;
3)
$$\lim_{T\to\infty} \mathsf{E} \max_{s\leqslant t} |\Delta X_T^m(s)|^2 = 0 \tag{14}$$
and
$$\lim_{l\to\infty} \overline{\lim_{T\to\infty}} \mathsf{P}\left\{\mathrm{tr}\left\langle X_T^l - X_T\right\rangle(t) > \varepsilon\right\} = 0 \tag{15}$$
for all $t > 0$, $\varepsilon > 0$ and $m \in \mathbf{N}$.

Then

$$\begin{aligned} &\mathsf{E}\left(\exp\left\{i\sum_{j=1}^n z_j\left(X_T(t_j) - X_T(t_{j-1})\right)\right\} \,\middle|\, \mathcal{F}_T(s)\right) \\ &\qquad - \exp\left\{-\frac{1}{2}\sum_{j=1}^n z_j\left(\langle X_T\rangle(t_j) - \langle X_T\rangle(t_{j-1})\right) z_j^\top\right\} \xrightarrow{\mathsf{P}} 0 \end{aligned} \tag{16}$$

for all $n \in \mathbf{N}$, $z_1, \dots, z_n \in \mathbf{R}^{p*}$, and $t_n > \dots > t_1 > t_0 \geqslant s > 0$.

Proof. According to Corollary 4, the net $(\langle X_T\rangle, T \in \mathbb{T})$ is relatively compact in C, since $(\mathrm{tr}\langle X_T\rangle, T \in \mathbb{T})$ possesses this property. In view of Lemma 7, condition **A** implies that

$$\mathsf{E}(F(\langle zX_T\rangle(t))|\mathcal{F}_T(s)) - F(\langle zX_T\rangle(t)) \xrightarrow{\mathsf{P}} 0 \tag{17}$$

for all $t > s > 0$ and $z \in \mathbf{R}^{d*}$ and for a bounded uniformly continuous function F. Thus for the case of $\mathbb{T} = \mathbf{N}$ Theorem 2 follows from Theorem 4.6 of [4].

From the part proved above, we conclude that relation (16) holds for every increasing to infinity sequence $(T_k, k \in \mathbf{N})$ of positive real functions and for all $n \in \mathbf{N}$ if T approaches infinity in the set $\{T_k\}$ rather than in $\mathbf{N}$, where $z_1, \dots, z_n \in \mathbf{R}^{p*}$ and $t_n > \dots > t_1 > t_0 \geqslant s > 0$. This implies Theorem 2 for the case of $\mathbb{T} = \{T \in \mathbf{R} \colon T > 0\}$. □

Theorem 3. *For every $T \in \mathbb{T}$, let $X_T, X_T^1, X_T^2, \dots$ be $\mathbf{R}^p$-valued locally square integrable martingales with respect to the flow $\mathbb{F}_T$. Assume that*

1) *condition* **A** *holds and relations* (13)–(15) *are satisfied;*

2)
$$\lim_{L\to\infty}\sup_{l}\overline{\lim_{T\to\infty}}\,\mathsf{P}\left\{\left|X_T^l(0)\right|>L\right\}=0;$$

3) *for all* $\varepsilon>0$ *and* $t>0$,
$$\lim_{l\to\infty}\overline{\lim_{T\to\infty}}\,\mathsf{P}\left\{\left|X_T^l(0)-X_T(0)\right|>\varepsilon\right\}=0;$$

4) *there exist an* $\mathbf{R}^p$*-random variable* $\mathring{X}$ *and an* $\mathfrak{S}_+$*-valued stochastic process* H *defined on a common probability space and such that*

$$(X_T(0),\langle X_T\rangle)\xrightarrow{C}\left(\mathring{X},H\right). \tag{18}$$

Then $(X_T,\langle X_T\rangle)\xrightarrow{C}(X,H)$*, where* X *is a continuous local martingale with the initial value* $\mathring{X}$ *and quadratic characteristic* H *such that*

(19)
$$\mathsf{E}\left(\exp\left\{i\sum_{j=1}^n z_j(X(t_j)-X(t_{j-1}))\right\}\,\middle|\,\mathring{X},H(\cdot)\right)=\exp\left\{-\frac{1}{2}\sum_{j=1}^n z_j(H(t_j)-H(t_{j-1}))z_j^\top\right\}$$

for all $n\in\mathbf{N}$*, where* $t_n>\dots>t_0\geqslant 0$ *and* $z_1,\dots,z_n\in\mathbf{R}^{p*}$.

Proof. Rewriting inequality (13) as
$$z\left(\langle X_T^m\rangle(t_2)-\langle X_T^m\rangle(t_1)\right)z^\top\leqslant z(\langle X_T\rangle(t_2)-\langle X_T\rangle(t_1))z^\top,$$
fixing an orthonormal basis $e^1,\dots,e^p$ in $\mathbf{R}^{p*}$, letting sequentially $z=e^1,\ \dots,\ z=e^p$, and adding together these p inequalities, we get

$$\operatorname{tr}\left(\langle X_T^m\rangle(t_2)-\langle X_T^m\rangle(t_1)\right)\leqslant\operatorname{tr}(\langle X_T\rangle(t_2)-\langle X_T\rangle(t_1)). \tag{20}$$

We further deal with the case of $\mathbb{T}=\mathbf{N}$ (the case of $\mathbb{T}=\{T\in\mathbf{R}\colon T>0\}$ is reduced to $\mathbb{T}=\mathbf{N}$ in the same way as in the proof of Theorem 2).

According to Lemma 7, condition **A** implies convergence (17).

The sequence $(\langle X_T\rangle,T\in\mathbf{N})$ is relatively compact in the space C by assumption (18), whence
$$\lim_{L\to\infty}\sup_{l}\overline{\lim_{T\to\infty}}\,\mathsf{P}\left\{\operatorname{tr}\left\langle X_T^l\right\rangle(t)>L\right\}=0,$$

$$\lim_{c\to 0}\sup_{l}\overline{\lim_{T\to\infty}}\,\mathsf{P}\left\{\sup_{(t_1,t_2)\in\Pi(t,c)}\operatorname{tr}\left(\left\langle X_T^l\right\rangle(t_2)-\left\langle X_T^l\right\rangle(t_1)\right)>\varepsilon\right\}=0$$

for all positive t and ε by inequality (20) and Theorem 1.

Therefore all the assumptions of Theorem 5.6 in [4] hold (three of them are just checked, while the rest are assumed to hold in Theorem 3). This completes the proof of Theorem 3. □

Remark 1. It is clear that property (19) for an $\mathbf{R}^p$-valued process X is equivalent to the following pair of properties:

(i) X has conditionally independent $H(\cdot)$ increments with respect to $\mathring{X}$;
(ii) for all $t>s\geqslant 0$ and $z\in\mathbf{R}^{p*}$,
$$\mathsf{E}\left(e^{iz(X(t)-X(s))}\,\middle|\,\mathring{X},H(\cdot)\right)=e^{-z(\langle X\rangle(t)-\langle X\rangle(s))z^\top/2}.$$

3. Main results

In Theorem 4 below, we use the following notation

$$U_T(t) = \int_0^t d\langle Z_T\rangle(s) \otimes \vartheta_T(s)^{\otimes 2}. \tag{21}$$

Also, condition **A** changes as follows:

B. There exists a family $\{\varkappa_T^r, r > 0, T \in \mathbb{T}\}$ of $\mathbf{R}^{d^4}$-valued random functions such that every $\varkappa_T^r$ is uniformly $(\mathbb{F}_T, \mathcal{B}(\mathbf{R}^{d^4}))$-adapted and

$$\lim_{r\to 0} \overline{\lim_{T\to\infty}} \mathsf{P}\left\{\left|U_T(t) - \varkappa_T^r(s,t,U_T(s))\right| > \varepsilon\right\} = 0 \tag{22}$$

for all $t > s > 0$ and $\varepsilon > 0$.

Theorem 4. *For every $T > 0$, let a stochastic process Y_T be defined by equality* (1), *where Z_T is an $\mathbf{R}^d$-valued locally square integrable martingale with respect to the flow $\mathbb{F}_T$ and where ϑ_T is an $\mathbf{R}^d$-valued $\mathbb{F}_T$-predictable stochastic process satisfying condition* (2). *Assume also that*

$$\lim_{T\to\infty} \mathsf{E} \max_{s\leqslant t} |\Delta Z_T(s)|^2 = 0, \tag{23}$$

$$\lim_{N\to\infty} \overline{\lim_{T\to\infty}} \mathsf{P}\left\{\int_0^t |\vartheta_T(s)|^2 I\{|\vartheta_T(s)| > N\}\, d\operatorname{tr}\langle Z_T\rangle(s) > \varepsilon\right\} = 0 \tag{24}$$

for all positive t and ε. Finally, assume that the stochastic processes U_T defined by equality (21) *satisfy condition* **B** *and there exists an $\mathbf{R}^{d^4}$-valued stochastic process G such that*

$$\langle Y_T\rangle \xrightarrow{C} G. \tag{25}$$

Then

1) *$(Y_T, \langle Y_T\rangle) \xrightarrow{C} (Y, \langle Y\rangle)$, where Y is a continuous local martingale with initial value O (O denotes the zero tensor) and with quadratic characteristic $\langle Y\rangle = G$;*
2) *the process Y has conditionally independent increments with respect to G;*
3) *for all $t > s \geqslant 0$ and $z \in \mathbf{R}^{d^2*}$,*

$$\mathsf{E}\left(e^{iz(\underline{Y}(t)-\underline{Y}(s))} \,\middle|\, \langle Y\rangle(\cdot)\right) = e^{-z(\langle \underline{Y}\rangle(t)-\langle \underline{Y}\rangle(s))z^\top/2}.$$

Proof. Put

$$Y_T^m(t) = \int_0^t dZ_T(s) \otimes \vartheta_T(s) I\{|\vartheta_T(s)| \leqslant m\},$$

$X_T = \underline{Y}_T$, and $X_T^m = \underline{Y}_T^m$. By Lemma 5, $\langle Y_T\rangle = U_T^\sharp$ and thus **B** implies **A**. We are going to show that the rest of the assumptions of Theorem 3 hold for $X_T, X_T^1, X_T^2, \ldots$.

Two assumptions concerning the initial values are trivial in the case under consideration, since all processes start from zero:

By construction,

$$\Delta Y_T^m(s) = \Delta Z_T(s) \otimes \vartheta_T(s) I\{|\vartheta_T(s)| \leqslant m\},$$

whence $|\Delta Y_T^m(s)|^2 \leqslant m^2 d \cdot |\Delta Z_T(s)|^2$ by Lemma 3 which together with equality (23) implies (14).

Similarly, by construction

$$Y_T^l(t) - Y_T(t) = \int_0^t dZ_T(s) \otimes \vartheta_T(s) I\{|\vartheta_T(s)| > l\},$$

whence

$$\operatorname{tr}\left\langle X_T^l - X_T\right\rangle(t) = \int_0^t |\vartheta_T(s)|^2 I\{|\vartheta_T(s)| > l\}\, d\operatorname{tr}\langle Z_T\rangle(s)$$

by Lemma 5 and thus (24) implies (15).

In order to check (13), we represent $z \in \mathbf{R}^{d^2*}$ as $z = (\mathfrak{z}_1 \cdots \mathfrak{z}_d)$, where

$$\mathfrak{z}_k = (z_{k1} \cdots z_{kd}) \in \mathbf{R}^{d*}.$$

By construction, the vector $X_T(t)$ consists of the columns

$$\int_0^t dZ_1(s)\vartheta(s), \quad \ldots, \quad \int_0^t dZ_d(s)\vartheta(s)$$

written sequentially one by one, where Z_i is the component i of Z. Hence

$$zX_T(t) = \int_0^t d(\mathfrak{z}_1 Z_1 + \cdots + \mathfrak{z}_d Z_d)\vartheta(s)$$

or, which is equivalent, $zX_T(t) = \int_0^t \psi(s)\, d\zeta(s)$, where $\psi = \vartheta^\top$ and $\zeta = \mathfrak{z}_1^\top Z_1 + \cdots + \mathfrak{z}_d^\top Z_d$. The same reasoning yields $zX_T^m(t) = \int_0^t \psi(s) I\{|\vartheta_T(s)| \leqslant m\}\, d\zeta(s)$. Now (13) follows from (2) and Corollary 3.

Since $X_T = \underline{Y}_T$, there exists an $\mathfrak{S}_+$-valued stochastic process H such that $\langle X_T\rangle \xrightarrow{C} H$ ($G(t)$ and $H(t)$ have the same families of coordinates but they are grouped differently) in view of (25). Then Theorem 3 implies that $(X_T, \langle X_T\rangle) \xrightarrow{C} (X, H)$, where X is a continuous local martingale with $\langle X\rangle = H$. By Lemma 4, $(Y_T, \langle Y_T\rangle) \xrightarrow{C} (Y, G)$, where Y is an $\mathbf{R}^{d^2}$-valued continuous local martingale such that $\underline{Y} = X$ and $\langle Y\rangle = G$. This proves the first statement of the theorem. The second and third statements follow from Theorem 3 and Remark 1. □

To understand better the probabilistic meaning of Theorem 4, we provide another statement that does not require a separate proof. In this version of Theorem 4, the processes ϑ_T are scalar and thus any tensor construction disappear.

Theorem 5. *For every $T > 0$, let the stochastic process Y_T be defined by equality $Y_T(t) = \int_0^t \vartheta_T(s)\, dZ_T(s)$, where Z_T is an $\mathbf{R}^d$-valued locally square integrable martingale with respect to the flow $\mathbb{F}_T$ and where ϑ_T is an $\mathbf{R}$-valued $\mathbb{F}_T$-predictable stochastic process satisfying condition* (2). *Assume that*

1) *for all positive t and ε, conditions* (23) *and* (24) *hold;*
2) *the stochastic processes U_T defined by*

$$U_T(t) = \int_0^t \vartheta_T(s)^2\, d\langle Z_T\rangle(s)$$

satisfy condition **B***;*

3) *there exists an $\mathbf{R}^{d^2}$-valued stochastic process U such that $U_T \xrightarrow{C} U$.*

Then

a) *$(Y_T, \langle Y_T\rangle) \xrightarrow{C} (Y, \langle Y\rangle)$, where Y is a continuous local martingale with initial value* 0 *and quadratic characteristic $\langle Y\rangle = U$;*
b) *the process Y has conditionally independent increments with respect to U;*
c) *for all $t > s \geqslant 0$ and $z \in \mathbf{R}^{d*}$,*

$$\mathsf{E}\left(e^{iz(Y(t)-Y(s))}\,\middle|\, U(\cdot)\right) = e^{-z(U(t)-U(s))z^\top/2}.$$

Throughout below we consider the case where the probability space does not depend on T.

Lemma 10. *For every $T \in \mathbb{T}$, let the stochastic process U_T be given by*

$$U_T(t) = \frac{1}{T}\int_0^{Tt} \rho_T\left(\frac{s}{T}\right)\, d\Phi(s), \tag{26}$$

$$\Phi(t) = \int_0^t dH(s) \otimes \varphi(s)^{\otimes 2}, \tag{27}$$

where ρ_T and φ are measurable stochastic processes ($\mathbf{R}$-valued and $\mathbf{R}^d$-valued, respectively) and where H is an increasing right continuous $\mathfrak{S}$-valued stochastic process. We also assume that

1)

$$\lim_{L\to\infty} \overline{\lim_{T\to\infty}}\, \mathsf{P}\left\{\int_0^T |\varphi(s)|^2\, d\operatorname{tr} H(s) > TL\right\} = 0; \tag{28}$$

2) *the processes ρ_T are right continuous and with limits on the left;*

3) *for all $t > 0$ and $\varepsilon > 0$,*

$$\lim_{L\to\infty} \overline{\lim_{T\to\infty}}\, \mathsf{P}\left\{\sup_{s\leqslant t} |\rho_T(s)| > L\right\} = 0, \tag{29}$$

$$\lim_{c\to 0} \overline{\lim_{T\to\infty}}\, \mathsf{P}\left\{\sup_{(t',t'')\in\Pi(t,c)} |\rho_T(t') - \rho_T(t'')| > \varepsilon\right\} = 0; \tag{30}$$

4) *there exists a net (Ξ_T) of random tensors such that*

$$T^{-1}\Phi(Tt) - \Xi_T t \xrightarrow{\mathsf{P}} O \tag{31}$$

for all $t > 0$.

Then relation (11) *holds for all t, where $R_T(t) = \int_0^t \rho_T(\tau)\, d\tau$.*

Proof. Let $0 = t_0 < \dots < t_m = t$ be a partition of the interval $[0, t]$. Denote

$$\psi_{T,m}(t_1,\dots,t_m) = \frac{1}{T}\sum_{k=1}^m \int_{Tt_{k-1}}^{Tt_k} \left(\rho_T\left(\frac{s}{T}\right) - \rho_T(t_k)\right)\, d\Phi(s),$$

$$\chi_{T,m}(t_1,\dots,t_m) = \sum_{k=1}^m \rho_T(t_k)\left(T^{-1}\Phi(Tt_k) - \Xi_T t_k - \left(T^{-1}\Phi(Tt_{k-1}) - \Xi_T t_{k-1}\right)\right),$$

$$\alpha_{T,m}(t_1,\dots,t_m) = \max_{0<k\leqslant m} \sup_{t_{k-1}\leqslant\tau<t_k} |\rho_T(\tau) - \rho_T(t_k)|,$$

$$\beta_{T,m}(t_1,\dots,t_m) = \sum_{k=1}^m \rho_T(t_k)(t_k - t_{k-1}) - R_T(t).$$

Considering equalities (26) and (27), we write

$$U_T(t) - \Xi_T R_T(t) = \psi_{T,m}(t_1,\dots,t_m) + \chi_{T,m}(t_1,\dots,t_m) + \Xi_T\beta_{T,m}(t_1,\dots,t_m). \tag{32}$$

In view of (31) and (29),

$$\chi_{T,m}(t_1,\dots,t_m) \xrightarrow{\mathsf{P}} O. \tag{33}$$

By Lemma 2,

$$\left\|\int_{Tt_{k-1}}^{Tt_k} \left(\rho_T\left(\frac{s}{T}\right) - \rho_T(t_k)\right)\, dH(s)\otimes\varphi(s)^{\otimes 2}\right\|$$
$$\leqslant \int_{Tt_{k-1}}^{Tt_k} \left|\rho_T\left(\frac{s}{T}\right) - \rho_T(t_k)\right|\, |\varphi(s)|^2\, d\operatorname{tr} H(s),$$

whence

$$|\psi_{T,m}(t_1,\dots,t_m)| \leqslant \alpha_{T,m}(t_1,\dots,t_m)V(Tt)/T, \tag{34}$$

where $V(t) = \int_0^t |\varphi(s)|^2\, d\,\mathrm{tr}\, H(s)$.

By the same lemma, $\|\Phi(t)\| \leqslant V(t)$, whence we obtain, in view of (28) and (31),

$$\lim_{N\to\infty} \overline{\lim_{T\to\infty}}\ \mathsf{P}\,\{\|\Xi_T\| > N\} = 0. \tag{35}$$

Now we conclude from (32)–(34) and from the following obvious inclusions

$$\{\alpha_{T,m}(t_1,\dots,t_m)V(Tt)/T \geqslant c\} \subset \{V(Tt) > TN\} \cup \{\alpha_{T,m}(t_1,\dots,t_m) \geqslant c/N\},$$
$$\{\|\beta_{T,m}(t_1,\dots,t_m)\Xi_T\| \geqslant c\} \subset \{\|\Xi_T\| > N\} \cup \{/\beta_{T,m}(t_1,\dots,t_m)/ \geqslant c/N\}$$

that

$$\begin{aligned}\overline{\lim_{T\to\infty}}\ &\mathsf{P}\{\|U_T(t) - \Xi_T R_T(t)\| \geqslant \varepsilon\}\\ &\leqslant \overline{\lim_{T\to\infty}}\ \mathsf{P}\{V(Tt) > TN\} + \overline{\lim_{T\to\infty}}\ \mathsf{P}\,\{\|\Xi_T\| > N\}\\ &\quad + \overline{\lim_{T\to\infty}}\ \mathsf{P}\{\alpha_{T,m}(t_1,\dots,t_m) \geqslant \varepsilon/3N\} + \overline{\lim_{T\to\infty}}\ \mathsf{P}\{\beta_{T,m}(t_1,\dots,t_m) \geqslant \varepsilon/3N\}\end{aligned}$$

for all $\varepsilon > 0$. This together with (28)–(30) and (35) implies (11). □

Corollary 6 (of Lemmas 8 and 10)**.** *Let all the assumptions of Lemma* 10 *hold. Further, let condition* (10) *hold for all* $t > 0$*, where* $R_T(t) = \int_0^t \rho_T(\tau)\, d\tau$*. Then equality* (12) *holds for all* $t > s > 0$ *and* $\varepsilon > 0$*.*

A stochastic process whose values are $\mathcal{F}(0)$-measurable random variables is called $\mathbb{F}^0$-adapted.

Theorem 6. *Let stochastic processes* Y_T *and* R_T *be given by the equalities*

$$Y_T(t) = \frac{1}{\sqrt{T}} \int_0^{Tt} \sigma_T\left(\frac{s}{T}\right)\, dM(s) \otimes \varphi(s) \tag{36}$$

and

$$R_T(t) = \int_0^t \sigma_T(s)^2\, ds,$$

respectively, where σ_T *is an* $\mathbb{F}^0$*-adapted* $\mathbf{R}$*-valued stochastic process whose trajectories belong to the space* D *and where* M *is an* $\mathbf{R}^d$*-valued locally square integrable martingale with respect to the flow* $\mathbb{F} = (\mathcal{F}(t), t \in \mathbf{R}_+)$*. Here* φ *is an* $\mathbf{R}^d$*-valued* $\mathbb{F}$*-predictable stochastic process such that*

$$\int_0^t |\varphi(s)|^2\ d\,\mathrm{tr}\langle M\rangle(s) < \infty, \qquad t \in \mathbf{R}_+. \tag{37}$$

Assume that

1)

$$\lim_{L\to\infty} \overline{\lim_{T\to\infty}}\ \mathsf{P}\,\{\mathrm{tr}\langle M\rangle(T) > TL\} = 0, \tag{38}$$

$$\lim_{T\to\infty} T^{-1}\mathsf{E} \max_{s\leqslant T} |\Delta M(s)|^2 = 0; \tag{39}$$

2) *for all* $\varepsilon > 0$*,*

$$\lim_{N\to\infty} \overline{\lim_{T\to\infty}}\ \mathsf{P}\left\{\int_0^T |\varphi(s)|^2 I\{|\varphi(s)| > N\}\ d\,\mathrm{tr}\langle M\rangle(s) > T\varepsilon\right\} = 0; \tag{40}$$

3) *the net* (σ_T^2) *is relatively compact in the space* C*;*

4) *there exists a net* (Ξ_T) *of random tensors such that*

$$\frac{1}{T}\int_0^{Tt} d\langle M\rangle(s)\otimes\varphi(s)^{\otimes 2}-\Xi_T t \xrightarrow{\mathsf{P}} O \tag{41}$$

for all $t>0$*;*

5) *there exist a random tensor* Ξ *and a real-valued stochastic process* R *defined on a joint probability space such that*

$$(\Xi_T, R_T(t_1),\dots,R_T(t_l)) \xrightarrow{\mathrm{d}} (\Xi, R(t_1),\dots,R(t_l)) \tag{42}$$

for all $l\in\mathbf{N}$ *and* $t_1,\dots,t_l\in\mathbf{R}_+$ *and*

$$\mathsf{P}\{R(t)=0\}=0 \tag{43}$$

for all $t>0$*.*

Then

a) $(Y_T,\langle Y_T\rangle)\xrightarrow{C}(Y,\langle Y\rangle)$*, where* Y *is a continuous local martingale with initial value* O *and quadratic characteristic* $\langle Y\rangle(t)=\Xi^\sharp R(t)$*;*

b) *the process* Y *has conditionally independent increments with respect to* $(\Xi,R(\cdot))$ *(and thus the increments are conditionally independent with respect to* $\langle Y\rangle(\cdot)$*, too);*

c) *for all* $t>s\geqslant 0$ *and* $z\in\mathbf{R}^{d^2*}$*,*

$$\mathsf{E}\left(e^{iz(\underline{Y}(t)-\underline{Y}(s))}\;\middle|\;\langle Y\rangle(\cdot)\right)=e^{-z(\langle\underline{Y}\rangle(t)-\langle\underline{Y}\rangle(s))z^\top/2}.$$

Proof. Equality (36) is a partial case of (1) corresponding to $Z_T(s)=M(Ts)/\sqrt{T}$ and $\vartheta_T(s)=\rho_T(s)\varphi(Ts)$. Thus the assumptions of Theorem 4 should be checked for such Z_T and ϑ_T and for $\mathcal{F}_T(t)=\mathcal{F}(Tt)$.

Obviously, condition (23) is equivalent to (39) in this case.

Put $\rho_T=\sigma_T^2$ and $S_T^N(t)=\int_0^t|\vartheta_T(s)|^2 I\{|\vartheta_T(s)|>N\}\,d\operatorname{tr}\langle Z_T\rangle(s)$. By assumption, the processes σ_T do not have discontinuities of the second kind and thus are bounded on every finite interval. Hence (37) implies (2). The net (ρ_T) is relatively compact in the space C and satisfies conditions (29) and (30) by Theorem 1. Substituting $\vartheta_T(s)$ and $Z_T(s)$ into the latter equation, we transform it to the form

$$S_T^N(t)=\frac{1}{T}\int_0^{Tt}\left|\rho_T\left(\frac{\tau}{T}\right)\right||\varphi(s)|^2 I\left\{\left|\sigma_T\left(\frac{\tau}{T}\right)\right||\varphi(s)|>N\right\}\,d\operatorname{tr}\langle M\rangle(s).$$

Then we conclude that

$$\left\{\left|S_T^N(t)\right|>\varepsilon\right\}\subset\left\{\sup_{s\leqslant t}|\rho_T(s)|>L\right\}\cup\left\{\int_0^{Tt}|\varphi(s)|^2 I\left\{|\varphi(s)|>\frac{N}{\sqrt{L}}\right\}\,d\operatorname{tr}\langle M\rangle(s)>T\varepsilon\right\},$$

and thus (24) follows from (40) and (29).

Now (36) together with Lemma 5 implies that $\langle Y_T\rangle=U_T^\sharp$, where

$$U_T(t)=\frac{1}{T}\int_0^{Tt}\rho_T\left(\frac{s}{T}\right)\,d\langle M\rangle(s)\otimes\varphi(s)^{\otimes 2}. \tag{44}$$

We are going to show that $U_T\xrightarrow{C}\Xi R$ (this means that condition (25) holds with $G=\Xi^\sharp R$).

Since (ρ_T) is relatively compact in C, the same property holds for the net (R_T). Thus relation (42) follows from (11).

It is clear that $|\varphi(s)|^2 \leqslant |\varphi(s)|^2 I\{|\varphi(s)| > N\} + N^2$ for all $N > 0$. Thus

$$\overline{\lim_{T\to\infty}}\ \mathsf{P}\left\{\int_0^T |\varphi(s)|^2\ d\,\mathrm{tr}\langle M\rangle(s) > TL\right\}$$
$$\leqslant \mathsf{P}\{\mathrm{tr}\langle M\rangle(T) > TL/N^2\} + \overline{\lim_{T\to\infty}}\ \mathsf{P}\left\{\int_0^T |\varphi(s)|^2 I\{|\varphi(s)| > N\}\ d\,\mathrm{tr}\langle M\rangle(s) > T\right\}$$

for $L > 1$, whence we derive (28) for $H = \langle M\rangle$ in view of (38) and (40). Conditions (29) and (30) are checked above, while condition (31) is equivalent to (41) in this case, since (44) means the same as (26) and (27) with $H = \langle M\rangle$. Now (11) follows from Lemma 10.

By Aleksandroff's theorem, the relation $R_T(t) \xrightarrow{\mathrm{d}} R(t)$ (being a part of condition (42)) implies that $\overline{\lim}_{T\to\infty} \mathsf{P}\{R_T(t) \leqslant r\} \leqslant \mathsf{P}\{R(t) \leqslant r\}$, whence

$$\lim_{r\to 0}\ \overline{\lim_{T\to\infty}}\ \mathsf{P}\{R_T(t) \leqslant r\} \leqslant \mathsf{P}\{R(t) = 0\}$$

which together with (43) yields (10). Now Corollary 6 proves relation (12) or, which is the same, relation (22) with

$$\varkappa_T^r(s,t,x) = g_r(R_T(t))h_r(R_T(s))x, \qquad x \in \mathbf{R}^{d^4}.$$

Since σ_T is $\mathbb{F}^0$-measurable (this implies that R_T also is $\mathbb{F}^0$-measurable), each of the random functions $\varkappa_T^r$ is uniformly $(\mathbb{F}_T, \mathcal{B}(\mathbf{R}^{d^4}))$-adapted. Therefore, condition **B** holds, and Theorem 4 completes the proof. □

Bibliography

1. R. Sh. Liptser and A. N. Shiryaev, *Theory of Martingales*, "Nauka", Moscow, 1986; English transl., Kluwer Academic, Dordrecht, 1989. MR1022664 (90j:60046)
2. J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer Verlag, Berlin, 1987. MR959133 (89k:60044)
3. A. Touati, *On the functional convergence in distribution of sequences of semimartingales to a mixture of Brownian motions*, Teor. Veroyatnost. Primenen. **36** (1991), no. 4, 744–763; English transl. in Theory Probab. Appl. **36** (1991), no. 4, 752–771. MR1147174 (93f:60033)
4. A. Yurachkivsky, *Convergence of locally square integrable martingales to a continuous local martingale*, J. Probab. Statist. **2011** (2011), Article ID 580292. (`http://dx.doi.org/10.1155/2011/580292`) MR2854289
5. B. L. van der Waerden, *Algebra*, vol. I, Springer-Verlag, Berlin–Heidelberg–New York, 1971.
6. I. M. Gel'fand, *Lectures on Linear Algebra*, "Nauka", Moscow, 1971; English transl., Dover Publications, New York, 1989. MR1034245 (90j:15001)
7. I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations and their Applications*, "Naukova dumka", Kiev, 1982. (Russian) MR678374 (84j:60003)

Department of Operations Research, Faculty for Cybernetics, National Taras Shevchenko University, Volodymyrs'ka Street, 64, Kyiv 01601, Ukraine

E-mail address: `andriy.yurachkivsky@gmail.com`

Received 12/JUNE/2012

Translated by N. SEMENOV