Теорія Ймовір. та Матем. Статист.
Вип. 91, 2014

Theor. Probability and Math. Statist.
No. 91, 2015, Pages 43–60
http://dx.doi.org/10.1090/tpms/965
Article electronically published on February 3, 2016

LIPSCHITZ CONDITIONS FOR STOCHASTIC PROCESSES IN THE BANACH SPACES $\mathbb{F}_\psi(\Omega)$ OF RANDOM VARIABLES

UDC 519.21

D. V. ZATULA AND YU. V. KOZACHENKO

Abstract. The Lipschitz continuity is studied for stochastic processes

$$X = (X(t), t \in \mathbb{T})$$

belonging to the Banach spaces $\mathbb{F}_\psi(\Omega)$, where $(\mathbb{T}, \rho)$ is a metric space. Some bounds for the distributions of the norms of stochastic processes in the Lipschitz spaces are also obtained.

1. Introduction

Let $(\mathbb{T}, \rho)$ be a metric space. Below we find some conditions under which sample paths of stochastic processes $X = (X(t), t \in \mathbb{T})$ satisfy the Lipschitz condition. In particular, we

(1) determine the moduli of continuity, that is, the functions f such that

$$\limsup_{\varepsilon \downarrow 0} \frac{\sup_{0<\rho(t,s)\le\varepsilon} |X(t) - X(s)|}{f(\varepsilon)} \le 1$$

with probability one, and

(2) estimate the probability

$$\mathsf{P}\left\{ \sup_{0<\rho(t,s)\le v} \frac{|X(t) - X(s)|}{f(\rho(t,s))} > x \right\}$$

for stochastic processes belonging to the spaces $\mathbb{F}_\psi(\Omega)$ of random variables, that is, to the Banach spaces equipped with the norm

$$\|\xi\|_\psi = \sup_{u\ge1} \frac{(\mathsf{E}\,|\xi|^u)^{1/u}}{\psi(u)},$$

where $\psi(u) > 0$ is an increasing function. These spaces are introduced in [6] and the properties of random variables and stochastic processes belonging to the spaces $\mathbb{F}_\psi(\Omega)$ are studied in [9].

Similar problems for Gaussian stochastic processes are considered in [3]. The results of [3] are generalized in [2, 7, 8, 15] for some classes of processes belonging to the Orlicz spaces. The Lipschitz continuity property for φ-sub-Gaussian processes is studied in [11]; some estimates for the distribution of the Lipschitz norms of such processes are also obtained in [11].

2010 *Mathematics Subject Classification.* Primary 60G07; Secondary 60G17.

Key words and phrases. Banach spaces $\mathbb{F}_\psi(\Omega)$, stochastic processes, Lipschitz conditions, continuity modulus, metric massiveness.

Many analytic properties of sample paths are considered recently for non-Gaussian (sub-Gaussian or Orlicz, for example) stochastic processes and random fields. In particular, the convergence of weighted sums of φ-sub-Gaussian dependent random variables is investigated in [5], some applications to random Fourier series of φ-sub-Gaussian random variables are considered in [4], necessary and sufficient conditions for a symmetric infinitely divisible stochastic process to have sample paths belonging to the Orlicz space L_ψ are found in [1], where the function ψ satisfies the so-called Δ_2-conditions. Various properties of exponential type Orlicz spaces and Fenchel–Orlicz spaces are obtained in [12]. More general classes of stochastic processes with values belonging to the Orlicz spaces and their properties are studied in [13]. Stochastic processes whose values belong to the exponential Orlicz spaces are considered in [14].

The paper is organized as follows. We introduce the so-called condition **A** in Section 2. This condition is needed for the proof of the main theorem. Section 3 is devoted to the statement and proof of the main result, Theorem 3.1. The conditions under which sample paths of stochastic processes of the spaces $\mathbb{F}_\psi(\Omega)$ possess the Lipschitz property are given in Section 4. Section 5 contains a number of examples of applications for specific functions $\psi(u)$ and $\sigma(h)$.

2. Definitions and condition A for the space $\mathbb{F}_\psi(\Omega)$

Below we provide some definitions and auxiliary results to be used in the proof of the main results. The so-called condition **A** is also introduced.

Definition 2.1. Let $\psi(u) > 0, u \geq 1$ be some increasing function such that $\psi(u) \to \infty$ as $u \to \infty$. We say that a random variable ξ belongs to the space $\mathbb{F}_\psi(\Omega)$ (see [9]) if

$$\sup_{u\geq 1} \frac{(\mathsf{E}\,|\xi|^u)^{1/u}}{\psi(u)} < \infty. \tag{1}$$

It is proved in the paper [6] (also see [9]) that $\mathbb{F}_\psi(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{(\mathsf{E}\,|\xi|^u)^{1/u}}{\psi(u)}. \tag{2}$$

Theorem 2.1 ([9])**.** *If a random variable ξ belongs to the space $\mathbb{F}_\psi(\Omega)$, then*

$$\mathsf{P}\{|\xi| > x\} \leq \inf_{u\geq 1} \frac{\|\xi\|_\psi^u \cdot (\psi(u))^u}{x^u} \tag{3}$$

for all $x > 0$.

Let $\xi_1, \ldots, \xi_n$ be random variables belonging to the space $\mathbb{F}_\psi(\Omega)$. Put

$$\eta = \max_{1\leq k\leq n} |\xi_k|, \qquad a = \max_{1\leq k\leq n} \|\xi_k\|_\psi.$$

Condition A. Given a function $z(x) > 0$, an increasing function $U(n)$, and a real number $x_0 > 0$, let

$$\mathsf{P}\{\eta > x \cdot a \cdot U(n)\} \leq \frac{1}{n} \exp\{-z(x)\} \tag{4}$$

for all $x > x_0$.

Remark 2.1. If condition **A** holds for a space $\mathbb{F}_\psi(\Omega)$, then

$$\mathsf{P}\{\eta > x \cdot a \cdot U(M)\} \leq \frac{1}{M} \exp\{-z(x)\}$$

for $M > n$ and for all $x > x_0$.

Below are some examples of the spaces $\mathbb{F}_\psi(\Omega)$ for which condition **A** holds and find for them the corresponding functions $U(n)$ and $z(x)$.

Theorem 2.2. *Let $\psi(u) = u^\alpha$, $\alpha > 0$. Then*

$$\mathsf{P}\left\{\eta > x \cdot a \cdot (\ln(n+2))^\alpha\right\} \le \frac{1}{n} \exp\left\{-\frac{\alpha}{e} x^{1/\alpha}\right\} \tag{5}$$

for all $x > \max\left\{(\ln 3)^{-\alpha}, (2e\ln 3/(\alpha(\ln 3 - 1)))^\alpha\right\}$.

Proof. Inequality (3) can be rewritten in this case as follows:

$$\mathsf{P}\{|\xi| > x\} \le \exp\left\{-\frac{\alpha}{e}\left(\frac{x}{\|\xi\|_\psi}\right)^{1/\alpha}\right\}$$

for $x > \|\xi\|_\psi$ (see [9]). Then

$$\begin{aligned}
\mathsf{P}\{\eta > x \cdot a \cdot (\ln(n+2))^\alpha\} &= \mathsf{E}\mathbb{1}\{\omega\colon \eta > x \cdot a \cdot (\ln(n+2))^\alpha\} \\
&\le \sum_{k=1}^n \mathsf{E}\mathbb{1}\{\eta = |\xi_k|\} \cdot \mathbb{1}\{\omega\colon |\xi_k| > x \cdot a \cdot (\ln(n+2))^\alpha\} \\
&\le n \cdot \max_{1\le k\le n} \mathsf{P}\{|\xi_k| > x \cdot a \cdot (\ln(n+2))^\alpha\} \\
&\le n \cdot \max_{1\le k\le n} \exp\left\{-\frac{\alpha}{e}\left(\frac{x\cdot a}{\|\xi_k\|_\psi}\right)^{1/\alpha} \cdot \ln(n+2)\right\} \\
&\le n \cdot \max_{1\le k\le n} \exp\left\{-\frac{\alpha}{e}\left(\frac{x\cdot a}{a}\right)^{1/\alpha} \cdot \ln(n+2)\right\} = \frac{1}{n}\cdot n^2 \cdot \exp\left\{-\frac{\alpha}{e} x^{1/\alpha} \cdot \ln(n+2)\right\}
\end{aligned}$$

for all $x > (\ln 3)^{-\alpha}$.

Denoting $d_n = \frac{\alpha}{e} x^{1/\alpha} \ln(n+2)$ we obtain the equality

$$n^2 \cdot \exp\{-d_n\} = \exp\{2\ln n - d_n\}.$$

It is easy to see that if

$$\frac{2\ln n}{1 - \frac{1}{\ln 3}} \le d_n,$$

that is, $x \ge (2e\ln 3/(\alpha(\ln 3 - 1)))^\alpha$, then

$$2\ln n - d_n \le -d_n \cdot \frac{1}{\ln(n+2)}.$$

Indeed,

$$\begin{aligned}
x \ge \left(\frac{2e\ln 3}{\alpha(\ln 3 - 1)}\right)^\alpha &\Leftrightarrow \frac{2}{1 - \frac{1}{\ln 3}} \le \frac{\alpha}{e} x^{1/\alpha} \Rightarrow \frac{2}{1 - \frac{1}{\ln(n+2)}} \le \frac{\alpha}{e} x^{1/\alpha} \\
&\Leftrightarrow \frac{2\ln n}{1 - \frac{1}{\ln(n+2)}} \le \frac{\alpha}{e} x^{1/\alpha} \ln n \Rightarrow \frac{2\ln n}{1 - \frac{1}{\ln(n+2)}} \le \frac{\alpha}{e} x^{1/\alpha} \ln(n+2)
\end{aligned}$$

for all $n \ge 1$. Thus

$$\begin{aligned}
\mathsf{P}\{\eta > x \cdot a \cdot (\ln(n+2))^\alpha\} &\le \frac{1}{n}\exp\left\{-\frac{\alpha}{e} x^{1/\alpha} \ln(n+2) \cdot \frac{1}{\ln(n+2)}\right\} \\
&= \frac{1}{n}\exp\left\{-\frac{\alpha}{e} x^{1/\alpha}\right\}
\end{aligned}$$

for all $x > \max\left\{(\ln 3)^{-\alpha}, (2e\ln 3/(\alpha(\ln 3 - 1)))^\alpha\right\}$. $\square$

Theorem 2.3. *Let $\psi(u) = (\ln(u+1))^\lambda$, $\lambda > 0$. Then the inequality*

$$\mathsf{P}\left\{\eta > x \cdot a \cdot \left(\ln\ln\left(n+1+e^{\lambda/2}\right)^{2/\lambda}\right)^\lambda\right\} \le \frac{1}{n}\exp\left\{-\lambda\left(\exp\left\{\frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e}\right\}-1\right)\right\} \tag{6}$$

holds for all

$$x \ge \left(e \cdot \ln\frac{2\ln(2+e^{\lambda/2})}{\lambda(\ln(2+e^{\lambda/2})-1)} \cdot \left(\ln\frac{2}{\lambda}\right)^{-1}\right)^\lambda.$$

Proof. Inequality (3) can be rewritten in this case as follows:

$$\mathsf{P}\{|\xi| > x\} \le e^\lambda \cdot \exp\left\{-\lambda \cdot \exp\left\{\left(\frac{x}{\|\xi\|_\psi}\right)^{1/\lambda}\frac{1}{e}\right\}\right\}$$

for $x > 0$ (see [9]). Put $U(n) = \left(\ln\ln\left(n+1+e^{\lambda/2}\right)^{2/\lambda}\right)^\lambda$. Then

$$\begin{aligned}
\mathsf{P}\{\eta > x\cdot a\cdot U(n)\} &= \mathsf{E}\,\mathbb{1}\{\omega\colon \eta > x\cdot a\cdot U(n)\}\\
&\le \sum_{k=1}^n \mathsf{E}\,\mathbb{1}\{\eta = |\xi_k|\}\cdot\mathbb{1}\{\omega\colon |\xi_k| > x\cdot a\cdot U(n)\}\\
&\le n\cdot\max_{1\le k\le n}\mathsf{P}\{|\xi_k| > x\cdot a\cdot U(n)\}\\
&\le ne^\lambda\cdot\max_{1\le k\le n}\exp\left\{-\lambda\cdot\exp\left\{\left(\frac{x\cdot a\cdot U(n)}{\|\xi_k\|_\psi}\right)^{1/\lambda}\frac{1}{e}\right\}\right\}\\
&\le ne^\lambda\cdot\max_{1\le k\le n}\exp\left\{-\lambda\cdot\exp\left\{\left(\frac{x\cdot a\cdot U(n)}{a}\right)^{1/\lambda}\frac{1}{e}\right\}\right\}\\
&= \frac{e^\lambda}{n}\cdot n^2\cdot\exp\left\{-\lambda\cdot\exp\left\{\frac{x^{1/\lambda}}{e}\cdot\ln\ln\left(n+1+e^{\lambda/2}\right)^{2/\lambda}\right\}\right\}
\end{aligned}$$

for all $x > 0$. Set

$$d_n = \lambda\cdot\exp\left\{\frac{x^{1/\lambda}}{e}\cdot\ln\ln\left(n+1+e^{\lambda/2}\right)^{2/\lambda}\right\}.$$

Then

$$n^2\cdot\exp\{-d_n\} = \exp\{2\ln n - d_n\}.$$

It is easy to see that if

$$\frac{2\ln n}{1-\frac{1}{\ln(2+e^{\lambda/2})}} \le d_n,$$

then

$$2\ln n - d_n \le -d_n\cdot\frac{1}{(\ln(n+1+e^{\lambda/2}))^{\frac{x^{1/\lambda}}{e}}}$$

for all

$$x \ge \left(e \cdot \ln\frac{2\ln(2+e^{\lambda/2})}{\lambda(\ln(2+e^{\lambda/2})-1)} \cdot \left(\ln\frac{2}{\lambda}\right)^{-1}\right)^\lambda.$$

Indeed,

$$x \geq \left(e \cdot \ln \frac{2\ln(2+e^{\lambda/2})}{\lambda(\ln(2+e^{\lambda/2})-1)} \cdot \left(\ln \frac{2}{\lambda}\right)^{-1} \right)^{\lambda} \Leftrightarrow \frac{x^{1/\lambda}}{e} \ln \frac{2}{\lambda} \geq \ln \frac{2}{\lambda\left(1 - \frac{1}{\ln(2+e^{\lambda/2})}\right)}$$

$$\Rightarrow \frac{2\ln n}{1 - \frac{1}{\ln(2+e^{\lambda/2})}} \leq \lambda \cdot \left(\frac{2}{\lambda}\right)^{\frac{x^{1/\lambda}}{e}} \cdot \ln(n+1+e^{\lambda/2})$$

$$\Leftrightarrow \frac{2\ln n}{1 - \frac{1}{\ln(2+e^{\lambda/2})}} \leq \lambda \cdot \exp\left\{ \frac{x^{1/\lambda}}{e} \left(\ln \frac{2}{\lambda} + \ln\ln(n+1+e^{\lambda/2}) \right) \right\}$$

$$\Rightarrow \frac{2\ln n}{1 - \frac{1}{\ln(n+1+e^{\lambda/2})}} \leq \lambda \cdot \exp\left\{ \frac{x^{1/\lambda}}{e} \cdot \ln\ln(n+1+e^{\lambda/2})^{\frac{2}{\lambda}} \right\}$$

for all $n \geq 1$. Then, for all

$$x \geq \left(e \cdot \ln \frac{2\ln(2+e^{\lambda/2})}{\lambda(\ln(2+e^{\lambda/2})-1)} \cdot \left(\ln \frac{2}{\lambda}\right)^{-1} \right)^{\lambda},$$

we have

$$\mathsf{P}\left\{ \eta > xa\left(\ln\ln\left(n+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} \right\} \leq \frac{e^{\lambda}}{n} \exp\left\{ -d_n \frac{1}{(\ln(n+1+e^{\lambda/2}))^{\frac{x^{1/\lambda}}{e}}} \right\}$$

$$= \frac{e^{\lambda}}{n} \exp\left\{ -\lambda \exp\left\{ \frac{x^{1/\lambda}}{e} \ln\ln\left(n+1+e^{\lambda/2}\right)^{2/\lambda} \right\} \frac{1}{(\ln(n+1+e^{\lambda/2}))^{\frac{x^{1/\lambda}}{e}}} \right\}$$

$$= \frac{e^{\lambda}}{n} \exp\left\{ -\lambda \exp\left\{ \frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e} + \ln\left(\ln\left(n+1+e^{\frac{\lambda}{2}}\right)\right)^{\frac{x^{1/\lambda}}{e}} \right\} \right.$$

$$\left. \times \frac{1}{(\ln(n+1+e^{\frac{\lambda}{2}}))^{\frac{x^{1/\lambda}}{e}}} \right\}$$

$$= \frac{e^{\lambda}}{n} \exp\left\{ -\lambda \exp\left\{ \frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e} \right\} \left(\ln\left(n+1+e^{\frac{\lambda}{2}}\right)\right)^{\frac{x^{1/\lambda}}{e}} \frac{1}{\left(\ln\left(n+1+e^{\frac{\lambda}{2}}\right)\right)^{\frac{x^{1/\lambda}}{e}}} \right\}$$

$$= \frac{e^{\lambda}}{n} \exp\left\{ -\lambda \exp\left\{ \frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e} \right\} \right\} = \frac{1}{n} \exp\left\{ -\lambda\left(\exp\left\{ \frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e} \right\} - 1 \right) \right\}. \qquad \square$$

Definition 2.2 ([2]). Let $(\mathbb{T}, \rho)$ be a metric space. The metric massiveness

$$N_{(\mathbb{T},\rho)}(u) := N(u)$$

of the space $(\mathbb{T}, \rho)$ is the minimal number of closed balls that cover $\mathbb{T}$ and that have radiuses which do not exceed u.

Definition 2.3 ([9]). We say that a stochastic process $X = (X(t), t \in \mathbb{T})$ belongs to the space $\mathbb{F}_{\psi}(\Omega)$ if random variables $X(t)$ belong to $\mathbb{F}_{\psi}(\Omega)$ for all $t \in \mathbb{T}$.

Definition 2.4 ([2, 11]). A function $q = \{q(t), t \in \mathbb{R}\}$ is called a modulus of continuity if $q(t) \geq 0$, $q(0) = 0$, and $q(t+s) \leq q(t) + q(s)$ for $t > 0$ and $s > 0$.

Definition 2.5 ([2, 11])**.** Let $(\mathbb{T},\rho)$ be a metric space and let q be a modulus of continuity. A family of functions $\{x(t), t\in\mathbb{T}\}$ such that

$$\sup_{\substack{t,s\in\mathbb{T}\\ t\neq s}} \frac{|x(t)-x(s)|}{q(\rho(t,s))} < \infty$$

(or $\sup_{\rho(t,s)\le h}|x(t)-x(s)| = o(q(h))$ as $h\to 0$), is called the Lipschitz space $\Lambda_q(\mathbb{T},\rho)$ (or $\Lambda_q^0(\mathbb{T},\rho)$).

3. Moduli of continuity for stochastic processes belonging to the spaces $\mathbb{F}_\psi(\Omega)$ of random variables

This section contains the statement and proof of the main result of the paper, the theorem on the moduli of continuity of stochastic processes belonging to the spaces $\mathbb{F}_\psi(\Omega)$ of random variables.

Theorem 3.1. *Let $(\mathbb{T},\rho)$ be some compact metric space. Consider a separable stochastic process $X=(X(t),t\in\mathbb{T})$ belonging to the Banach space $\mathbb{F}_\psi(\Omega)$ where condition **A** holds with some functions $U(n)$ and $z(x)$ and with some $x_0>0$.*

Assume that there exists an increasing continuous function

$$\sigma = \{\sigma(h), h\ge 0\}$$

such that $\sigma(0)=0$ and

$$\sup_{\rho(t,s)\le h} \|X(t)-X(s)\|_\psi \le \sigma(h). \tag{7}$$

Let $N(\varepsilon)=N_\rho(\mathbb{T},\varepsilon)$ be the metric massiveness of the space $(\mathbb{T},\rho)$ and let

$$\varepsilon_0 = \sigma^{(-1)}\left(\sup_{t,s\in\mathbb{T}}\rho(t,s)\right)$$

and

$$g_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U\left(B^2N^2\left(\sigma^{(-1)}(t)\right)\right)dt < \infty, \qquad \varepsilon>0.$$

Then

$$\begin{aligned} &\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon} \frac{|X(t)-X(s)|}{(6+4\sqrt{2})f_B(\rho(t,s))+(5+2\sqrt{6})g_B(\rho(t,s))} > x\right\}\\ &\qquad\le \frac{2B^2+B}{(B^2-1)N(\varepsilon)}\cdot\exp\{-z(x)\} \end{aligned} \tag{8}$$

for $x>x_0$ and $\varepsilon\in(0,\varepsilon_0)$, where $B>1$ is a certain number, and

$$f_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U\left(BN\left(\sigma^{(-1)}(t)\right)\right)dt, \qquad \varepsilon>0.$$

Proof. Let $r\in(0,1)$ and let $\{\nu_k, k=0,1,2,\dots\}$ be some sequence such that $\nu_0 = \sup_{t,s\in\mathbb{T}}\rho(t,s)$, $\nu_{k+1}=\min\{r\nu_k,\delta_k\}$, where

$$\delta_k = A\inf\left\{\nu\colon N\left(\sigma^{(-1)}(\nu)\right) < BN\left(\sigma^{(-1)}(\nu_k)\right)\right\}, \tag{9}$$

$\sigma^{(-1)}(\nu)$ denotes the inverse function to σ, $B>1$, and where $A>1$ is a number such that $Ar<1$. Then $\{\nu_k, k=0,1,2,\dots\}$ satisfies the property

$$\nu_{k+1}\le r\nu_k, \qquad k=0,1,2,\dots, \tag{10}$$

or

$$\nu_k \le \frac{1}{1-r}(\nu_k - \nu_{k+1}). \tag{11}$$

Inequalities (9) and (10) imply

$$N\left(\sigma^{(-1)}(\nu_{k+2})\right) \ge N\left(\sigma^{(-1)}(r\nu_{k+1})\right) \ge N\left(\sigma^{(-1)}(r\delta_k)\right) \ge BN\left(\sigma^{(-1)}(\nu_k)\right).$$

Thus

$$N\left(\sigma^{(-1)}(\nu_k)\right) \ge BN\left(\sigma^{(-1)}(\nu_{k-2})\right) \ge B^2 N\left(\sigma^{(-1)}(\nu_{k-4})\right) \ge \dots. \tag{12}$$

Let $\varepsilon_0 = \sigma^{(-1)}(\nu_0), \dots, \varepsilon_k = \sigma^{(-1)}(\nu_k)$. Also let V_{ε_k}, $k = 0,1,2,\dots$ be the set of centers of closed balls with radiuses ε_k that form a minimal covering of the space $(\mathbb{T},\rho)$. The total number of points in V_{ε_k} equals $N(\varepsilon_k)$. Let $V_0 = \bigcup_{k=0}^{\infty} V_{\varepsilon_k}$. Inequality (7) together with the Chebyshev inequality implies that the process X is continuous in probability. Thus V_0 is a set of separability of the process X. Let α_n be a mapping $V_0 \to V_{\varepsilon_n}$, where $\alpha_n(t) = t$ if $t \in V_{\varepsilon_n}$; otherwise $\alpha_n(t)$ is a point of V_{ε_n} such that $\rho(t,\alpha_n(t)) < \varepsilon_n$. Inequality (2) yields that, for all $\xi \in \mathbb{F}_\psi(\Omega)$,

$$\frac{\left(\mathsf{E}\,|\xi|^2\right)^{1/2}}{\psi(2)} \le \|\xi\|_\psi \iff \mathsf{E}\,\xi^2 \le \|\xi\|_\psi^2 \cdot (\psi(2))^2. \tag{13}$$

Now the Chebyshev inequality together with (13) and (10) implies

$$\begin{aligned}
\mathsf{P}\left\{|X(t) - X(\alpha_n(t))| > r^{n/2}\right\} &\le \frac{\mathsf{E}\big(X(t) - X(\alpha_n(t))\big)^2}{r^n} \le \frac{\|X(t) - X(\alpha_n(t))\|_\psi^2 \cdot (\psi(2))^2}{r^n} \\
&\le \frac{\big(\sigma(\rho(t,\alpha_n(t)))\big)^2 \cdot (\psi(2))^2}{r^n} \le \frac{(\sigma(\varepsilon_n))^2 \cdot (\psi(2))^2}{r^n} \\
&= \frac{\nu_n^2 \cdot (\psi(2))^2}{r^n} \le \frac{r^{2n}\nu_0^2(\psi(2))^2}{r^n} = \nu_0^2 r^n (\psi(2))^2.
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \mathsf{P}\left\{|X(t) - X(\alpha_n(t))| > r^{n/2}\right\} < \infty.$$

The Borel–Cantelli lemma gives $X(\alpha_n(t)) \to X(t)$ with probability one as $n \to \infty$. Since the set V_0 is countable, $X(\alpha_n(t)) \to X(t)$ as $n \to \infty$ with probability one for all $t \in V_0$.

Take $0 < \varepsilon \le \varepsilon_0$ and choose m such that $\varepsilon_{m+1} < \varepsilon \le \varepsilon_m$. Since V_0 is a set of separability of the process X,

$$\sup_{\substack{\rho(t,s)<\varepsilon \\ t,s\in\mathbb{T}}} |X(t) - X(s)| = \sup_{\substack{\rho(t,s)<\varepsilon \\ t,s\in V_0}} |X(t) - X(s)| \tag{14}$$

with probability one.

Let $t,s \in V_0$ and $\rho(t,s) < \varepsilon$. Consider $k > m+1$. Set $t_k = \alpha_k(t)$, $t_{k-1} = \alpha_{k-1}(t_k)$, $\dots$, $t_m = \alpha_m(t_{m+1})$; $s_k = \alpha_k(s)$, $s_{k-1} = \alpha_{k-1}(s_k), \dots, s_m = \alpha_m(s_{m+1})$. Then

$$\begin{aligned}
X(t) - X(s) &= (X(t) - X(t_k)) + \sum_{l=m+2}^{k} (X(t_l) - X(t_{l-1})) - (X(s) - X(s_k)) \\
&\quad - \sum_{l=m+2}^{k} (X(s_l) - X(s_{l-1})) + (X(t_{m+1}) - X(s_{m+1}))
\end{aligned} \tag{15}$$

for all t and s such that $\rho(t,s) < \varepsilon$.

Inequality (15) yields

$$X(t_{m+1}) - X(s_{m+1}) = (X(t) - X(s)) - (X(t) - X(t_k)) + (X(s) - X(s_k)) - \sum_{l=m+2}^{k} (X(t_l) - X(t_{l-1})) + \sum_{l=m+2}^{k} (X(s_l) - X(s_{l-1}))$$

and

$$\begin{aligned} \|X(t_{m+1}) - X(s_{m+1})\|_\psi &\leq \|X(t) - X(s)\|_\psi + \|X(t) - X(t_k)\|_\psi \\ &\quad + \|X(s) - X(s_k)\|_\psi + \sum_{l=m+2}^{k} \|X(t_l) - X(t_{l-1})\|_\psi \\ &\quad + \sum_{l=m+2}^{k} \|X(s_l) - X(s_{l-1})\|_\psi \\ &\leq \sigma(\rho(t,s)) + \sigma(\rho(t,t_k)) + \sigma(\rho(s,s_k)) \\ &\quad + \sum_{l=m+2}^{k} \sigma(\rho(t_l, t_{l-1})) + \sum_{l=m+2}^{k} \sigma(\rho(s_l, s_{l-1})) \\ &\leq \sigma(\varepsilon) + 2\sigma(\varepsilon_k) + 2 \sum_{l=m+2}^{k} \sigma(\varepsilon_{l-1}) \\ &\leq \sigma(\varepsilon) + 2 \sum_{l=m+2}^{\infty} \sigma(\varepsilon_{l-1}) = \sigma(\varepsilon) + 2 \sum_{l=m+2}^{\infty} \nu_{l-1} \\ &\leq \sigma(\varepsilon) + 2 \sum_{l=1}^{\infty} \nu_{m+l} \leq \sigma(\varepsilon) + 2 \sum_{l=1}^{\infty} \nu_{m+1} r^{l-1} \\ &= \sigma(\varepsilon) + \nu_{m+1} \frac{2}{1-r} \leq \sigma(\varepsilon) \left(1 + \frac{2}{1-r}\right) = \sigma(\varepsilon) \frac{3-r}{1-r}. \end{aligned} \tag{16}$$

Then we conclude from inequalities (15) and (16) that

$$\begin{aligned} |X(t) - X(s)| \leq{}& \sum_{l=m+2}^{k} |X(t_l) - X(t_{l-1})| + \sum_{l=m+2}^{k} |X(s_l) - X(s_{l-1})| + |X(t) - X(t_k)| \\ &+ |X(s) - X(s_k)| + |X(t_{m+1}) - X(s_{m+1})| \\ \leq{}& 2 \sum_{l=m+2}^{k} \max_{p \in V_{\varepsilon_l}} |X(p) - X(\alpha_{l-1}(p))| + |X(t) - X(t_k)| + |X(s) - X(s_k)| \\ &+ \max_{\substack{v,w \in V_{\varepsilon_{m+1}}: \\ \|X(v) - X(w)\|_\psi \leq \sigma(\varepsilon)\frac{3-r}{1-r}}} |X(v) - X(w)| \end{aligned}$$

for all $t, s \in \mathbb{T}$ such that $\rho(t,s) < \varepsilon$.

Passing to the limit as $k \to \infty$ we get

$$|X(t) - X(s)| \leq 2 \sum_{l=m+2}^{\infty} \max_{p \in V_{\varepsilon_l}} |X(p) - X(\alpha_{l-1}(p))| + \max_{\substack{v,w \in V_{\varepsilon_{m+1}}: \\ \|X(v) - X(w)\|_\psi \leq \sigma(\varepsilon)\frac{3-r}{1-r}}} |X(v) - X(w)|.$$

Now we establish from equality (14) that

$$\begin{aligned}\sup_{\substack{\rho(t,s)<\varepsilon\\ t,s\in\mathbb{T}}}|X(t)-X(s)| &= \sup_{\substack{\rho(t,s)<\varepsilon\\ t,s\in V_0}}|X(t)-X(s)|\\ &\le 2\sum_{k=m+2}^{\infty}\max_{p\in V_{\varepsilon_k}}|X(p)-X(\alpha_{k-1}(p))|\\ &\quad+\max_{\substack{v,w\in V_{\varepsilon_{m+1}}:\\ \|X(v)-X(w)\|_\psi\le\sigma(\varepsilon)\frac{3-r}{1-r}}}|X(v)-X(w)|.\end{aligned}\tag{17}$$

Set

$$c_k=\sigma(\varepsilon_{k-1})\cdot U(N(\varepsilon_k)),$$

$$b_m(\varepsilon)=\frac{3-r}{1-r}\sigma(\varepsilon)\cdot U\left(N^2(\varepsilon_{m+1})\right),$$

$$\xi_k=\max_{t\in V_{\varepsilon_k}}|X(t)-X(\alpha_{k-1}(t))|,$$

$$\eta_m(\varepsilon)=\max_{\substack{v,w\in V_{\varepsilon_{m+1}}:\\ \|X(v)-X(w)\|_\psi\le\sigma(\varepsilon)\frac{3-r}{1-r}}}|X(v)-X(w)|.$$

Let $\{G(\varepsilon),\varepsilon\ge 0\}$ be an increasing function such that

$$G(\varepsilon)\ge 2\sum_{k=m+2}^{\infty}c_k+b_m(\varepsilon),$$

where m is a number for which $\varepsilon_{m+1}<\varepsilon\le\varepsilon_m$. Then

$$\begin{aligned}\sup_{0<\rho(t,s)\le\varepsilon}\frac{|X(t)-X(s)|}{G(\rho(t,s))} &\le \sup_{0<y\le\varepsilon}\left[\sup_{0<\rho(t,s)\le y}\frac{|X(t)-X(s)|}{G(y)}\right]\\ &\le \sup_{l\ge m+1}\ \sup_{\varepsilon_{l+1}<y\le\varepsilon_l}\frac{2\sum_{p=l+1}^{\infty}\xi_p+\eta_l(y)}{2\sum_{p=l+1}^{\infty}c_p+b_l(y)}.\end{aligned}$$

The latter inequality implies

$$\begin{aligned}\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon}\frac{|X(t)-X(s)|}{G(\rho(t,s))}>x\right\} &\le \sum_{k=m+2}^{\infty}\mathsf{P}\left\{\frac{\xi_k}{c_k}>x\right\}\\ &\quad+\sum_{l=m+1}^{\infty}\mathsf{P}\left\{\sup_{\varepsilon_{l+1}<v\le\varepsilon_l}\frac{\eta_l(v)}{b_l(v)}>x\right\}.\end{aligned}\tag{18}$$

Now we estimate from above the expression $2\sum_{k=m+2}^{\infty}c_k+b_m(\varepsilon)$. First we split the sum into two parts:

$$\sum_{k=m+2}^{\infty}c_k=\sum_{k=m+2}^{\infty}\nu_{k-1}\cdot U(N(\varepsilon_k))=A_1+A_2,$$

where

$$A_1=\sum_{k\in D_1(m)}\nu_{k-1}\cdot U(N(\varepsilon_k)),\qquad A_2=\sum_{k\in D_2(m)}\nu_{k-1}\cdot U(N(\varepsilon_k)),$$

$$D_1(m)=\{k\ge m+2\colon \nu_k=r\nu_{k-1}\},\qquad D_2(m)=\{k\ge m+2\colon \nu_k=\delta_{k-1}\}.$$

It follows from inequalities (10) and (11) that

$$
\begin{aligned}
A_1 &= \frac{1}{r} \sum_{k\in D_1(m)} \nu_k \cdot U\left(N\left(\sigma^{(-1)}(\nu_k)\right)\right) \\
&\le \frac{1}{r(1-r)} \sum_{k=m+2}^{\infty} (\nu_k - \nu_{k+1}) \cdot U\left(N\left(\sigma^{(-1)}(\nu_k)\right)\right) \\
&\le \frac{1}{r(1-r)} \sum_{k=m+2}^{\infty} \int_{\nu_{k+1}}^{\nu_k} U\left(N\left(\sigma^{(-1)}(t)\right)\right) dt \\
&= \frac{1}{r(1-r)} \int_0^{\nu_{m+2}} U\left(N\left(\sigma^{(-1)}(t)\right)\right) dt.
\end{aligned} \tag{19}
$$

Since $N\big(\sigma^{(-1)}(\delta_k)\big) < BN\big(\sigma^{(-1)}(\nu_k)\big)$, we deduce from inequalities (10) and (11) that

$$
\begin{aligned}
A_2 = &\sum_{k\in D_2(m)} \nu_{k-1} \cdot U\left(N\left(\sigma^{(-1)}(\delta_{k-1})\right)\right) \le \sum_{k\in D_2(m)} \nu_{k-1} \cdot U\left(BN\left(\sigma^{(-1)}(\nu_{k-1})\right)\right) \\
&\le \frac{1}{1-r} \sum_{k=m+2}^{\infty} (\nu_{k-1} - \nu_k) \cdot U\left(BN\left(\sigma^{(-1)}(\nu_{k-1})\right)\right) \\
&\le \frac{1}{1-r} \int_0^{\nu_{m+1}} U\left(BN\left(\sigma^{(-1)}(t)\right)\right) dt.
\end{aligned} \tag{20}
$$

By $\nu_{m+2} < \nu_{m+1} < \sigma(\varepsilon)$, inequalities (19) and (20) imply

$$
2 \sum_{k=m+2}^{\infty} c_k \le \frac{2(1+r)}{r(1-r)} \int_0^{\sigma(\varepsilon)} U\left(BN\left(\sigma^{(-1)}(t)\right)\right) dt.
$$

Now we estimate from above the term $b_m(\varepsilon)$. Since $\nu_{m+1} = \min\{r\nu_m, \delta_m\}$, we consider the two cases, namely $\nu_{m+1} = r\nu_m$ and $\nu_{m+1} = \delta_m$. If $\nu_{m+1} = r\nu_m$, then

$$
\begin{aligned}
\sigma(\varepsilon) \cdot U\left(N^2\left(\sigma^{(-1)}(\nu_{m+1})\right)\right) &= \sigma(\varepsilon) \cdot U\left(N^2\left(\sigma^{(-1)}(r\nu_m)\right)\right) \\
&\le \sigma(\varepsilon) \cdot U\left(N^2\left(\sigma^{(-1)}(r\sigma(\varepsilon))\right)\right) \\
&\le \int_0^{\sigma(\varepsilon)} U\left(N^2\left(\sigma^{(-1)}(rv)\right)\right) dv \\
&= \frac{1}{r} \int_0^{r\sigma(\varepsilon)} U\left(N^2\left(\sigma^{(-1)}(t)\right)\right) dt \\
&\le \frac{1}{r} \int_0^{\sigma(\varepsilon)} U\left(N^2\left(\sigma^{(-1)}(t)\right)\right) dt
\end{aligned}
$$

for $\varepsilon_{m+1} < \varepsilon \le \varepsilon_m$ $(\nu_{m+1} < \sigma(\varepsilon) \le \nu_m)$.

If $\nu_{m+1} = \delta_m$, then we derive from inequality (9) that

$$
\begin{aligned}
\sigma(\varepsilon) \cdot U\left(N^2\left(\sigma^{(-1)}(\nu_{m+1})\right)\right) &= \sigma(\varepsilon) \cdot U\left(N^2\left(\sigma^{(-1)}(\delta_m)\right)\right) \\
&\le \sigma(\varepsilon) \cdot U\left(B^2N^2\left(\sigma^{(-1)}(\nu_m)\right)\right) \\
&\le \sigma(\varepsilon) \cdot U\left(B^2N^2\left(\sigma^{(-1)}(\sigma(\varepsilon))\right)\right) \\
&\le \int_0^{\sigma(\varepsilon)} U\left(B^2N^2\left(\sigma^{(-1)}(t)\right)\right) dt.
\end{aligned}
$$

Therefore

$$b_m(\varepsilon) \leq \frac{3-r}{r(1-r)} \int_0^{\sigma(\varepsilon)} U\left(B^2 N^2\left(\sigma^{(-1)}(t)\right)\right)\, dt,$$

whence

$$\begin{aligned} 2\sum_{k=m+2}^{\infty} c_k + b_m(\varepsilon) \leq{}& \frac{2(1+r)}{r(1-r)} \int_0^{\sigma(\varepsilon)} U\left(BN\left(\sigma^{(-1)}(t)\right)\right)\, dt \\ &+ \frac{3-r}{r(1-r)} \int_0^{\sigma(\varepsilon)} U\left(B^2 N^2\left(\sigma^{(-1)}(t)\right)\right)\, dt. \end{aligned} \tag{21}$$

As a result,

$$\begin{aligned} &\mathsf{P}\left\{\sup_{0<\rho(t,s)\leq\varepsilon} \frac{|X(t)-X(s)|}{\frac{1}{r(1-r)}(2(1+r)f_B(\rho(t,s)) + (3-r)g_B(\rho(t,s)))} > x\right\} \\ &\quad\leq \sum_{k=m+2}^{\infty} \mathsf{P}\left\{\frac{\xi_k}{c_k} > x\right\} + \sum_{l=m+1}^{\infty} \mathsf{P}\left\{\sup_{\varepsilon_{l+1}<v\leq\varepsilon_l} \frac{\eta_l(v)}{b_l(v)} > x\right\}, \end{aligned} \tag{22}$$

where

$$f_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U\left(BN\left(\sigma^{(-1)}(t)\right)\right)\, dt \quad \text{and} \quad g_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U\left(B^2 N^2\left(\sigma^{(-1)}(t)\right)\right)\, dt.$$

Now we estimate the probability on the right hand side of (22). According to condition **A**, for all $x > x_0$,

$$\begin{aligned} \mathsf{P}\left\{\frac{\xi_k}{c_k} > x\right\} &= \mathsf{P}\left\{\max_{t\in V_{\varepsilon_k}} |X(t) - X(\alpha_{k-1}(t))| > x\sigma(\varepsilon_{k-1})U(N(\varepsilon_k))\right\} \\ &\leq \frac{\exp\{-z(x)\}}{N(\varepsilon_k)} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\mathsf{P}\left\{\sup_{\varepsilon_{l+1}<v\leq\varepsilon_l} \frac{\eta_l(v)}{b_l(v)} > x\right\} \\ &\quad= \mathsf{P}\left\{\sup_{\varepsilon_{l+1}<v\leq\varepsilon_l} \frac{\max\limits_{\substack{v,w\in V_{\varepsilon_{l+1}}:\\ \|X(v)-X(w)\|_\psi \leq \sigma(\varepsilon)\frac{3-r}{1-r}}} |X(v)-X(w)|}{\frac{3-r}{1-r}\sigma(\varepsilon)\cdot U(N^2(\varepsilon_{l+1}))} > x\right\} \\ &\quad= \mathsf{P}\left\{\max_{\substack{v,w\in V_{\varepsilon_{l+1}}:\\ \|X(v)-X(w)\|_\psi \leq \sigma(\varepsilon)\frac{3-r}{1-r}}} |X(v)-X(w)| > x\cdot\frac{3-r}{1-r}\sigma(\varepsilon)\cdot U(N^2(\varepsilon_{l+1}))\right\} \\ &\quad\leq \frac{\exp\{-z(x)\}}{N^2(\varepsilon_{l+1})}. \end{aligned} \tag{24}$$

This yields

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon}\frac{|X(t)-X(s)|}{\frac{1}{r(1-r)}(2(1+r)f_B(\rho(t,s))+(3-r)g_B(\rho(t,s)))}>x\right\}$$
$$\le\sum_{k=m+2}^{\infty}\frac{1}{N(\varepsilon_k)}\cdot\exp\{-z(x)\}+\sum_{l=m+1}^{\infty}\frac{1}{N^2(\varepsilon_{l+1})}\cdot\exp\{-z(x)\}$$
$$:=R(m)\cdot\exp\{-z(x)\},$$

where

$$f_B(\varepsilon)=\int_0^{\sigma(\varepsilon)}U\left(BN\left(\sigma^{(-1)}(t)\right)\right)dt \quad\text{and}\quad g_B(\varepsilon)=\int_0^{\sigma(\varepsilon)}U\left(B^2N^2\left(\sigma^{(-1)}(t)\right)\right)dt.$$

Inequality (12) implies

$$R(m)=\sum_{k=m+2}^{\infty}\frac{1}{N(\varepsilon_k)}+\sum_{l=m+1}^{\infty}\frac{1}{N^2(\varepsilon_{l+1})}\le\frac{1}{N(\varepsilon_{m+2})}\sum_{p=0}^{\infty}\frac{1}{B^p}+\frac{1}{N^2(\varepsilon_{m+2})}\sum_{p=0}^{\infty}\frac{1}{B^{2p}}$$
$$=\frac{B}{(B-1)N(\varepsilon_{m+2})}+\frac{B^2}{(B^2-1)N^2(\varepsilon_{m+2})}\le\frac{1}{N(\varepsilon)}\cdot\left(\frac{B}{B-1}+\frac{B^2}{B^2-1}\right)$$
$$=\frac{2B^2+B}{(B^2-1)N(\varepsilon)}.$$

Since $\inf_{0<r<1}\frac{2(1+r)}{r(1-r)}=6+4\sqrt{2}$ and $\inf_{0<r<1}\frac{3-r}{r(1-r)}=5+2\sqrt{6}$,

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon}\frac{|X(t)-X(s)|}{(6+4\sqrt{2})f_B(\rho(t,s))+(5+2\sqrt{6})g_B(\rho(t,s))}>x\right\}$$
$$\le\frac{2B^2+B}{(B^2-1)N(\varepsilon)}\cdot\exp\{-z(x)\}$$

for $x>x_0$. □

Definition 3.1. We say that condition **B** holds for a space $\mathbb{F}_\psi(\Omega)$ if condition **A** holds with some functions $z(x)$ and $U(n)$ for $x>x_0$ and if there exists a constant $b_0>1$ such that

$$U(n^2)\le b_0U(n)$$

for all $n\ge 1$.

Corollary 3.1. *Let all the assumptions of Theorem* 3.1 *hold and let condition **B** hold for a space $\mathbb{F}_\psi(\Omega)$. Then*

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon}\frac{|X(t)-X(s)|}{\left(6+4\sqrt{2}+b_0(5+2\sqrt{6})\right)f_B(\rho(t,s))}>x\right\}\le\frac{2B^2+B}{(B^2-1)N(\varepsilon)}\cdot\exp\{-z(x)\}$$

for $x>x_0$, $\varepsilon\in(0,\varepsilon_0)$, and $B>1$, where

$$f_B(\varepsilon)=\int_0^{\sigma(\varepsilon)}U\left(BN\left(\sigma^{(-1)}(t)\right)\right)dt.$$

4. Lipschitz conditions for stochastic processes belonging to the spaces $\mathbb{F}_\psi(\Omega)$ of random variables

The results of this section follow from Theorem 3.1.

Theorem 4.1. *Let all the assumptions of Theorem* 3.1 *hold. Then*

$$\limsup_{\varepsilon\downarrow 0} \frac{\Delta(X;\varepsilon)}{(6+4\sqrt{2})f_B(\varepsilon)+(5+2\sqrt{6})g_B(\varepsilon)} \le 1 \tag{25}$$

with probability one, where $\varepsilon > 0$,

$$\Delta(X;\varepsilon) = \sup_{\substack{t,s\in\mathbb{T}\\ 0<\rho(t,s)\le\varepsilon}} |X(t)-X(s)|,$$

$$f_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U\left(BN\left(\sigma^{(-1)}(t)\right)\right)\,dt$$

and

$$g_B(\varepsilon) = \int_0^{\sigma(\varepsilon)} U\left(B^2N^2\left(\sigma^{(-1)}(t)\right)\right)\,dt < \infty.$$

Proof. Inequality (17) implies that

$$\sup_{\substack{\rho(t,s)<v\\ t,s\in\mathbb{T}}} |X(t)-X(s)| \le 2\sum_{l=m+2}^{\infty} \xi_l + \eta_m(v) \tag{26}$$

with probability one. Then we derive from inequality (23) that $\xi_l \le xc_l$ with probability one for sufficiently large l and for $x > x_0$. Then it follows from (24) that $\eta_m(v) \le xb_m(v)$ with probability one for sufficiently large m and for $x > x_0$. Thus if l is sufficiently large (or if v is sufficiently small) we get

$$\sup_{\substack{\rho(t,s)<v\\ t,s\in\mathbb{T}}} |X(t)-X(s)| \le x\left(2\sum_{l=m+2}^{\infty} c_l + b_m(v)\right) \tag{27}$$

with probability one for $x > x_0$.

Using inequality (21) we obtain

$$\sup_{\substack{\rho(t,s)\le v\\ t,s\in\mathbb{T}}} |X(t)-X(s)| \le \big(6+4\sqrt{2}\big)f_B(v) + \big(5+2\sqrt{6}\big)g_B(v)$$

with probability one for sufficiently small v. □

Corollary 4.1. *Let all the assumptions of Theorem* 4.1 *hold. Then*

$$\sup_{\rho(t,s)\le v} |X(t)-X(s)| \le \big(6+4\sqrt{2}\big)f_B(v) + \big(5+2\sqrt{6}\big)g_B(v)$$

with probability one if v is sufficiently small.

5. Examples

Below are some examples of applications of the above results for specific functions $\psi(u)$ and $\sigma(h)$.

Example 5.1. Let $\psi(u)=u^\alpha$, $\alpha>0$, and let $\sigma(h)=dh^c$, $h,c,d>0$.

The inverse function to $\sigma(h)$ is $\sigma^{(-1)}(h)=\sqrt[c]{h/d}$. According to Theorem 2.2, condition **A** holds for the space $\mathbb{F}_\psi(\Omega)$ with the functions $z(x)=\frac{\alpha}{e}x^{1/\alpha}$, $U(n)=(\ln(n+2))^\alpha$, and with

$$x_0=\max\left\{\frac{1}{(\ln 3)^\alpha},\left(\frac{2e\ln 3}{\alpha(\ln 3-1)}\right)^\alpha\right\}.$$

Hence $f_B(\varepsilon)$ and $g_B(\varepsilon)$ can be rewritten as follows:

$$f_B(\varepsilon)=\int_0^{d\varepsilon^c}U\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)\right)dt=\int_0^{d\varepsilon^c}\left(\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\right)^\alpha dt;$$

$$g_B(\varepsilon)=\int_0^{d\varepsilon^c}U\left(B^2N^2\left(\sqrt[c]{\frac{t}{d}}\right)\right)dt=\int_0^{d\varepsilon^c}\left(\ln\left(B^2N^2\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\right)^\alpha dt.$$

Since

$$\ln\left(B^2N^2\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\le\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)^2=2\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)+2\right),$$

we conclude that

$$\begin{aligned}g_B(\varepsilon)&=\int_0^{d\varepsilon^c}\left(\ln\left(B^2N^2\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\right)^\alpha dt\\&\le\int_0^{d\varepsilon^c}\left(2\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\right)^\alpha dt=2^\alpha f_B(\varepsilon).\end{aligned}$$

Thus condition **B** with $b_0=2^\alpha$ holds for the space $\mathbb{F}_\psi(\Omega)$. According to Corollary 3.1,

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon}\frac{|X(t)-X(s)|}{\gamma_B(\rho(t,s))}>x\right\}\le\frac{2B^2+B}{(B^2-1)N(\varepsilon)}\cdot\exp\left\{-\frac{\alpha}{e}x^{1/\alpha}\right\}$$

for $\varepsilon\in(0,\varepsilon_0)$, $\varepsilon_0=\sigma^{(-1)}\left(\sup_{t,s\in\mathbb{T}}\rho(t,s)\right)$, and for all

$$x>\max\left\{\frac{1}{(\ln 3)^\alpha},\left(\frac{2e\ln 3}{\alpha(\ln 3-1)}\right)^\alpha\right\},$$

$B>1$, and $N(\varepsilon)\ge 2$, where

$$\gamma_B(\varepsilon)=\left(6+4\sqrt{2}+\left(5+2\sqrt{6}\right)\cdot 2^\alpha\right)\int_0^{d\varepsilon^c}\left(\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\right)^\alpha dt.$$

Moreover,

$$\limsup_{\varepsilon\downarrow 0}\frac{\sup_{0<\rho(t,s)\le\varepsilon}|X(t)-X(s)|}{(6+4\sqrt{2}+(5+2\sqrt{6})\cdot 2^\alpha)\int_0^{d\varepsilon^c}\left(\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\right)^\alpha dt}\le 1$$

with probability one by Theorem 4.1.

Next we consider the space $\mathbb{T}=[0,T]$. Since the metric massiveness $N(u)$ is the minimal number of elements in a u-covering of the space $\mathbb{T}$ (of the interval $[0,T]$ in the case under consideration), $\frac{T}{2u}\le N(u)\le\frac{T}{2u}+1$. In other words,

$$N\left(\sqrt[c]{\frac{u}{d}}\right)=N\left(\sigma^{(-1)}(u)\right)\le\frac{T}{2\sigma^{(-1)}(u)}+1=\frac{T}{2\sqrt[c]{\frac{u}{d}}}+1=\frac{T}{2}\sqrt[c]{\frac{d}{u}}+1.$$

This means that the function $f_B(\varepsilon)$ is bounded from above by

$$f_B(\varepsilon) = \int_0^{d\varepsilon^c} \left(\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right)+2\right)\right)^{\alpha} dt \le \int_0^{d\varepsilon^c} \left(\ln\left(B\cdot\left(\frac{T}{2}\sqrt[c]{\frac{d}{t}}+1\right)+2\right)\right)^{\alpha} dt.$$

According to Corollary 3.1,

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon} \frac{|X(t)-X(s)|}{\gamma_{1,B}(\rho(t,s))} > x\right\} \le \frac{2\varepsilon\left(2B^2+B\right)}{T(B^2-1)}\cdot\exp\left\{-\frac{\alpha}{e}x^{1/\alpha}\right\}$$

for all

$$x > \max\left\{\frac{1}{(\ln 3)^{\alpha}}, \left(\frac{2e\ln 3}{\alpha(\ln 3-1)}\right)^{\alpha}\right\},$$

$B>1$, and $\varepsilon \in \left(0, \min\left\{\varepsilon_0, \frac{T}{2}\right\}\right)$, where

$$\gamma_{1,B}(\varepsilon) = \left(6+4\sqrt{2}+\left(5+2\sqrt{6}\right)\cdot 2^{\alpha}\right)\int_0^{d\varepsilon^c}\left(\ln\left(B\cdot\left(\frac{T}{2}\sqrt[c]{\frac{d}{t}}+1\right)+2\right)\right)^{\alpha} dt.$$

Moreover,

$$\limsup_{\varepsilon\downarrow 0} \frac{\sup_{0<\rho(t,s)\le\varepsilon}|X(t)-X(s)|}{\left(6+4\sqrt{2}+\left(5+2\sqrt{6}\right)\cdot 2^{\alpha}\right)\int_0^{d\varepsilon^c}\left(\ln\left(B\cdot\left(\frac{T}{2}\sqrt[c]{\frac{d}{t}}+1\right)+2\right)\right)^{\alpha} dt} \le 1$$

with probability one by Theorem 4.1.

Example 5.2. Let $\psi(u) = (\ln(u+1))^{\lambda}$, $\lambda > 0$, and let

$$\sigma(h) = \frac{1}{\ln\left(\frac{1}{h}+1\right)}, \qquad h > 0.$$

The inverse function to $\sigma(h)$ is $\sigma^{(-1)}(h) = \left(e^{1/h}-1\right)^{-1}$. According to Theorem 2.3, condition **A** with

$$z(x) = \lambda\left(\exp\left\{\frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e}\right\}-1\right), \qquad U(n) = \left(\ln\ln\left(n+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda}$$

holds for the space $\mathbb{F}_{\psi}(\Omega)$ if

$$x_0 = \left(e\cdot\ln\frac{2\ln\left(2+e^{\lambda/2}\right)}{\lambda\left(\ln\left(2+e^{\lambda/2}\right)-1\right)}\cdot\left(\ln\frac{2}{\lambda}\right)^{-1}\right)^{\lambda}.$$

Thus the functions can be rewritten as follows:

$$f_B(\varepsilon) = \int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(BN\left(\frac{1}{e^{1/t}-1}\right)+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt;$$

$$g_B(\varepsilon) = \int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(B^2N^2\left(\frac{1}{e^{1/t}-1}\right)+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt.$$

According to Theorem 3.1,

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon}\frac{|X(t)-X(s)|}{\gamma_B(\rho(t,s))} > x\right\} \le \frac{2B^2+B}{(B^2-1)N(\varepsilon)}\exp\left\{-\lambda\left(\exp\left\{\frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e}\right\}-1\right)\right\},$$

for all

$$x > \left(e\cdot\ln\frac{2\ln\left(2+e^{\lambda/2}\right)}{\lambda\left(\ln\left(2+e^{\lambda/2}\right)-1\right)}\cdot\left(\ln\frac{2}{\lambda}\right)^{-1}\right)^{\lambda},$$

$B > 1$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 = \sigma^{(-1)}\left(\sup_{t,s\in\mathbb{T}} \rho(t,s)\right)$, and $N(\varepsilon) \ge 2$, where

$$\gamma_B(\varepsilon) = \left(6+4\sqrt{2}\right)\int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(BN\left(\frac{1}{e^{1/t}-1}\right)+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt$$
$$+\left(5+2\sqrt{6}\right)\int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(B^2N^2\left(\frac{1}{e^{1/t}-1}\right)+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt.$$

Moreover,

$$\limsup_{\varepsilon\downarrow 0} \frac{\sup_{0<\rho(t,s)\le\varepsilon} |X(t)-X(s)|}{\gamma_B(\varepsilon)} \le 1$$

with probability one by Theorem 4.1.

Now let $\mathbb{T} = [0,T]$. Then

$$N\left(\frac{1}{e^{1/u}-1}\right) = N\left(\sigma^{(-1)}(u)\right) \le \frac{T}{2\sigma^{(-1)}(u)} + 1 = \frac{T\left(e^{1/u}-1\right)}{2} + 1.$$

This allows us to estimate the functions $f_B(\varepsilon)$ and $g_B(\varepsilon)$:

$$f_B(\varepsilon) = \int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(BN\left(\frac{1}{e^{1/t}-1}\right)+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt$$
$$\le \int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(B\cdot\left(\frac{T\left(e^{1/t}-1\right)}{2}+1\right)+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt;$$

$$g_B(\varepsilon) \le \int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(B^2\left(\frac{T\left(e^{1/t}-1\right)}{2}+1\right)^2+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt.$$

Hence, for all

$$x > \left(e\cdot\ln\frac{2\ln(2+e^{\lambda/2})}{\lambda\left(\ln\left(2+e^{\lambda/2}\right)-1\right)}\cdot\left(\ln\frac{2}{\lambda}\right)^{-1}\right)^{\lambda},$$

$B > 1$, and $\varepsilon \in \left(0, \min\left\{\varepsilon_0, \frac{T}{2}\right\}\right)$, we have

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon} \frac{|X(t)-X(s)|}{\gamma_{1,B}(\rho(t,s))} > x\right\} \le \frac{2\varepsilon\left(2B^2+B\right)}{T(B^2-1)}\cdot\exp\left\{-\lambda\left(\exp\left\{\frac{x^{1/\lambda}\ln\frac{2}{\lambda}}{e}\right\}-1\right)\right\}$$

by Theorem 3.1, where

$$\gamma_{1,B}(\varepsilon) = \left(6+4\sqrt{2}\right)\int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(B\cdot\left(\frac{T\left(e^{1/t}-1\right)}{2}+1\right)+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt$$
$$+\left(5+2\sqrt{6}\right)\int_0^{\left(\ln\left(\frac{1}{\varepsilon}+1\right)\right)^{-1}} \left(\ln\ln\left(B^2\left(\frac{T\left(e^{1/t}-1\right)}{2}+1\right)^2+1+e^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda} dt.$$

Moreover,

$$\limsup_{\varepsilon\downarrow 0} \frac{\sup_{0<\rho(t,s)\le\varepsilon} |X(t)-X(s)|}{\gamma_{1,B}(\varepsilon)} \le 1$$

with probability one by Theorem 4.1.

Theorem 5.1. *Let $X = (X(t), t \in \mathbb{T})$ be a stochastic process and let all the assumptions of Theorem 3.1 hold. If $(6 + 4\sqrt{2}) f_B(\varepsilon) + (5 + 2\sqrt{6}) g_B(\varepsilon) \le q_B(\varepsilon)$ in a neighborhood of zero, where q_B is a modulus of continuity, then X belongs to the space $\Lambda_q(\mathbb{T}, \rho)$ with probability one and*

$$\mathsf{P}\left\{\sup_{0<\rho(t,s)\le\varepsilon} \frac{|X(t) - X(s)|}{q_B(\rho(t,s))} > x\right\} \le \frac{2B^2 + B}{(B^2 - 1)N(\varepsilon)} \cdot \exp\{-z(x)\}. \tag{28}$$

If $(6 + 4\sqrt{2}) f_B(\varepsilon) + (5 + 2\sqrt{6}) g_B(\varepsilon) = o(q_B(\varepsilon))$, then the stochastic process X belongs to the space $\Lambda_q^0(\mathbb{T}, \rho)$ with probability one and inequality (28) holds for sufficiently small numbers $\varepsilon > 0$.

Theorem 5.1 follows from Theorem 3.1.

Example 5.3. Let the functions $\psi(u)$ and $\sigma(h)$ be the same as in Example 5.1. Also let all the assumptions of Theorem 3.1 hold for a stochastic process $X = (X(t), t \in \mathbb{T})$ and q_B be a modulus of continuity. If

$$\int_0^{d\varepsilon_0^c} \frac{\left(\ln\left(B^2N^2\left(\sqrt[c]{\frac{t}{d}}\right) + 2\right)\right)^\alpha}{q_B(t)}\, dt < \infty,$$

then X belongs to the space $\Lambda_q^0(\mathbb{T}, \rho)$ with probability one.

In this case,

$$f_B(\varepsilon) = \int_0^{d\varepsilon^c} \left(\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right) + 2\right)\right)^\alpha dt = \int_0^{d\varepsilon^c} q_B(t) \cdot \frac{\left(\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right) + 2\right)\right)^\alpha}{q_B(t)}\, dt$$

$$\le q_B(\varepsilon) \cdot \int_0^{d\varepsilon^c} \frac{\left(\ln\left(BN\left(\sqrt[c]{\frac{t}{d}}\right) + 2\right)\right)^\alpha}{q_B(t)}\, dt = o(q_B(\varepsilon)), \qquad \varepsilon \to 0.$$

Similarly, $g_B(\varepsilon) = o(q_B(\varepsilon))$ as $\varepsilon \to 0$ and the above result follows from Theorem 5.1.

6. Concluding remarks

In this paper, we found sufficient conditions under which the sample paths of a stochastic process

$$X = (X(t), t \in \mathbb{T})$$

belonging to the space $\mathbb{F}_\psi(\Omega)$ of random variables satisfy the Lipschitz condition. We also obtained some upper bounds for the distributions of the norms of sample paths of stochastic processes in the Lipschitz space. Some examples of applications of general results are given for specific cases. Moduli of continuity for stochastic processes belonging to the spaces $\mathbb{F}_\psi(\Omega)$ on infinite intervals will be considered elsewhere.

Bibliography

1. M. Braverman and G. Samorodnitsky, *Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes*, Stoch. Process. Appl. **78** (1998), no. 1, 1–26. MR1653284 (99h:60009)
2. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, "TViMS", Kyiv; English transl., American Mathematical Society, Providence, RI, 2000. MR1743716 (2001g:60089)
3. R. M. Dudley, *Sample functions of the Gaussian processes*, Ann. Probab. **1** (1973), no. 1, 3–68. MR0346884 (49:11605)
4. Antonini R. Giuliano, T.-C. Hu, Yu. Kozachenko, and A. Volodin, *An application of φ-subgaussian technique to Fourier analysis*, J. Math. Anal. Appl. **408** (2013), 114–124. MR3079951

5. Antonini R. Giuliano, Yu. Kozachenko, and A. Volodin, *Convergence of series of dependent ϕ-sub-Gaussian random variables*, J. Math. Anal. Appl. **338** (2008), 1188–1203. MR2386491 (2008m:60050)
6. S. V. Ermakov and E. I. Ostrovskiĭ, *Conditions for the continuity, exponential estimates, and central limit theorem for random fields*, Dep. VINITI, #3752-B.86.0, 1986. (Russian)
7. Yu. V. Kozachenko, *Random processes in Orlicz spaces. I*, Teor. Veroytnost. Matem. Statist. **30** (1984), 92–107; English transl. in Theory Probab. Math. Stat. **30** (1985), 103–117. MR800835 (86m:60111)
8. Yu. V. Kozachenko, *Random processes in Orlicz spaces. II*, Teor. Veroytnost. Matem. Statist. **31** (1984), 44–50; English transl. in Theory Probab. Math. Stat. **31** (1985), 51–58. MR816125 (87b:60063)
9. Yu. V. Kozachenko and Yu. Yu. Mlavets', *The Banach spaces $\mathbb{F}_\psi(\Omega)$ of random variables*, Teor. Imovirnost. Matem. Statyst. **86** (2012), 92–107; English transl. in Theory Probab. Math. Stat. **86** (2013), 105–121. MR2986453
10. Yu. V. Kozachenko and I. V. Rozora, *Accuracy and reliability of models of stochastic processes of the space $Sub_\varphi(\Omega)$*, Teor. Imovirnost. Matem. Statyst. **71** (2005), 93–104; English transl. in Theor. Probability and Math. Statist. **71** (2005), 105–117. MR2144324 (2005m:60077)
11. Yu. Kozachenko, T. Sottinen, and O. Vasylyk, *Lipschitz conditions for $Sub_\varphi(\Omega)$-processes and applications to weakly self-similar processes with stationary increments*, Teor. Imovirnost. Matem. Statyst. **82** (2010), 67–81; English transl. in Theor. Probability and Math. Statist. **82** (2011), 57–73. MR2790484 (2011m:60109)
12. A. C. Krinik and R. J. Swift, *Stochastic Processes and Functional Analysis: A Volume of Recent Advances in Honor of M. M. Rao*, Lecture Notes in Pure and Applied Mathematics, CRC Press, New York, 2004. MR2060706 (2004k:60004)
13. M. M. Rao and Z. D. Ren, *Applications of Orlicz Spaces*, Pure and Applied Mathematics, CRC Press, New York, 2002. MR1890178 (2003e:46041)
14. M. Weber, *Stochastic processes with value in exponential type Orlicz spaces*, Ann. Probab. **16** (1988), no. 3, 1365–1371. MR942774 (89k:60054)
15. D. V. Zatula, *Moduli of continuity of stochastic processes belonging to Orlicz spaces of random variables defined on an interval*, Visti Taras Shevchenko Nat. Univer. Ser. Physics Matem. (2013), no. 2, 23–28.

Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue, 6, Kyiv 03127, Ukraine

E-mail address: dm_zatula@mail.ru

Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue, 6, Kyiv 03127, Ukraine

E-mail address: yvk@univ.kiev.ua

Received 23/JUNE/2014

Translated by S. KVASKO