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ASYMPTOTIC NORMALITY OF THE CORRELOGRAM ESTIMATOR OF THE COVARIANCE FUNCTION OF A RANDOM NOISE IN THE NONLINEAR REGRESSION MODEL

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Abstract. The asymptotic behavior of the correlogram estimator of the covariance function of a random noise is studied for the nonlinear regression model. A functional theorem on the asymptotic normality of the estimator is proved in the space of continuous functions.

1. Introduction

Assume that the observations are described by the model

$$X(t) = g(t,\theta) + \varepsilon(t), \qquad t \in [0,\infty),$$

where $g\colon [0,\infty) \times \Theta_\gamma \to \mathbb{R}^1$ is a continuous function that depends on an unknown parameter $\theta = (\theta_1, \dots, \theta_q) \in \Theta \subset \mathbb{R}^q$, Θ is a bounded open convex set,

$$\Theta_\gamma = \bigcup_{\|a\|<1} (\Theta + \gamma a),$$

$\gamma > 0$ is a certain number, and $\varepsilon(t)$, $t \in \mathbb{R}^1$, is a random noise satisfying the following condition.

A1. $\varepsilon = \varepsilon(t)$, $t \in \mathbb{R}^1$, is a real mean square continuous as well as almost surely continuous stationary Gaussian process defined on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ with zero mean and integrable covariance function $\mathbf{B} = \left\{\mathbf{B}(t), t \in \mathbb{R}^1\right\}$.

It is obvious that if $\mathbf{B} \in L_1\left(\mathbb{R}^1\right)$, then $\mathbf{B} \in L_2\left(\mathbb{R}^1\right)$ and the process ε has a bounded and continuous spectral density $f = \left\{f(\lambda), \lambda \in \mathbb{R}^1\right\}$; that is, $f \in L_2\left(\mathbb{R}^1\right)$. The Plancherel theorem implies that

$$\|\mathbf{B}\|_2^2 = \int_{-\infty}^{\infty} \mathbf{B}^2(t)\,dt = 2\pi \int_{-\infty}^{\infty} f^2(\lambda)\,d\lambda = 2\pi \|f\|_2^2.$$

If $\mathbf{B}$ is unknown, then one faces a problem of estimation of the covariance function from observations $\{X(t), t \in [0,\infty)\}$. Then θ becomes a nuisance parameter in this problem.

Any random vector $\widehat{\theta}_T = \widehat{\theta}_T\left(X(t), t \in [0,T]\right) = (\widehat{\theta}_{1T}, \dots, \widehat{\theta}_{qT}) \in \Theta^c$ (where Θ^c is the closure of Θ) such that

$$L_T(\widehat{\theta}_T) = \min_{\tau \in \Theta^c} L_T(\tau), \qquad L_T(\tau) = \int_0^T [X(t) - g(t,\tau)]^2\,dt,$$

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is called the least squares estimator of the unknown parameter $\theta \in \Theta$ constructed from observations in the interval $[0, T]$.

As an estimator of $\mathbf{B}$ related to $\widehat{\theta}_T$, one can take the correlogram constructed from the deviations $\widehat{X}(t) = X(t) - g(t, \widehat{\theta}_T)$, $t \in [0, T+H]$; namely

$$\mathbf{B}_T\big(z, \widehat{\theta}_T\big) = T^{-1} \int_0^T \widehat{X}(t+z)\widehat{X}(t)\, dt, \qquad z \in [0, H], \tag{1}$$

where $H > 0$ is a fixed number.

Note that $\mathbf{B}_T(0, \widehat{\theta}_T) = T^{-1} L_T(\widehat{\theta}_T)$ is the least squares estimator of the variance $\sigma^2 = \mathbf{B}(0)$ of the random process ε, while

$$\mathbf{B}_T(z, \theta) = \mathbf{B}_T(z) = T^{-1} \int_0^T \varepsilon(t+z)\varepsilon(t)\, dt, \qquad t \in [0, H], \tag{2}$$

is the correlogram of the process ε.

A stochastic asymptotic expansion and corresponding asymptotic expansion of moments of the estimator $\mathbf{B}_T(z, \widehat{\theta}_T)$ are obtained in [11, 12]. The asymptotic normality of the correlogram estimator $\mathbf{B}_T(z, \widehat{\theta}_T)$ is proved in the current paper.

Condition **A1** implies that the integrals (1) and (2) can be viewed as Riemann's integrals of the trajectories of the corresponding processes. Moreover, these integrals, as functions of the argument z, are almost surely continuous in the interval $[0, H]$ (see, for example, [3, 4, 9]).

2. Conditions and results

Consider the process

$$X_T(z) = T^{1/2} \Big(\mathbf{B}_T\big(z, \widehat{\theta}_T\big) - \mathbf{B}(z) \Big) = Y_T(z) + R_T(z), \qquad z \in [0, H],$$

where

$$Y_T(z) = T^{1/2} \left(\mathbf{B}_T(z) - \mathbf{B}(z) \right),$$
$$R_T(z) = T^{-1/2} I_{2T}(z) + T^{-1/2} I_{3T}(z) + T^{-1/2} I_{4T}(z),$$
$$I_{2T}(z) = \int_0^T \Big(g\big(t+z, \widehat{\theta}_T\big) - g(t+z, \theta) \Big) \Big(g\big(t, \widehat{\theta}_T\big) - g(t, \theta) \Big)\, dt,$$
$$I_{3T}(z) = \int_0^T \varepsilon(t+z) \Big(g\big(t, \widehat{\theta}_T\big) - g(t, \theta) \Big)\, dt,$$
$$I_{4T}(z) = \int_0^T \varepsilon(t) \Big(g\big(t+z, \widehat{\theta}_T\big) - g(t+z, \theta) \Big)\, dt.$$

The processes Y_T, I_{2T}, I_{3T}, I_{4T} as well as X_T and R_T are viewed as random elements in the measurable space $(C[0, H], \mathfrak{B})$, where $\mathfrak{B}$ is the σ-algebra of Borel subsets of the space $C[0, H]$ of continuous functions in the interval $[0, H]$. Let Z be any of the above random elements. The distribution of Z is a probability measure $P = PZ^{-1}$ on $(C[0, H], \mathfrak{B})$.

We say that a family $\{U_T\}$ of random elements converges in distribution as $T \to \infty$ to a random element U in the space $C[0, H]$ if the distributions P_T of elements U_T weakly converge as $T \to \infty$ to the distribution P of the element U (see [1]). We write $U_T \xrightarrow{\mathcal{D}} U$ in this case and use the same notation for the convergence in distribution of random variables.

Consider the following pseudometrics [3]:

$$\rho(z_1, z_2) = \left(\int_{-\infty}^{\infty} f^2(\lambda) \sin^2 \frac{\lambda(z_1 - z_2)}{2}\, d\lambda\right)^{1/2},$$

$$\sqrt{\rho}(z_1, z_2) = \sqrt{\rho(z_1, z_2)}, \qquad z_1, z_2 \in \mathbb{R}^1.$$

Let $H_{\sqrt{\rho}}(\varepsilon)$, $\varepsilon > 0$, be the metric entropy of the interval $[0, 1]$ with respect to the pseudometric $\sqrt{\rho}$.

A2. $\displaystyle\int_{0+} H_{\sqrt{\rho}}(\varepsilon)\, d\varepsilon < \infty$.

If $f \in L_2(\mathbb{R}^1)$, then

$$\mathsf{E}\, Y_T(z_1) Y_T(z_2) \to b(z_1, z_2) = 4\pi \int_{-\infty}^{\infty} f^2(\lambda) \cos \lambda z_1 \cos \lambda z_2\, d\lambda$$

as $T \to \infty$ for all $z_1, z_2 \geq 0$.

Let $Y = \{Y(z), z \in [0, H]\}$ be a real centered Gaussian process with the covariance function $b(z_1, z_2)$, $z_1, z_2 \in [0, H]$. Then all finite-dimensional distributions of the processes Y_T weakly converge as $T \to \infty$ to the corresponding finite-dimensional distributions of the Gaussian process Y (see [2]).

The following result is proved in [3, Theorem 6.4.1].

Theorem 2.1. *Let conditions* **A1** *and* **A2** *hold. Then, for all* $H > 0$,

(i) $Y \in C[0, H]$ *almost surely*;

(ii) $Y_T \in C[0, H]$ *almost surely*, $T > 0$;

(iii) $Y_T \xrightarrow{\mathcal{D}} Y$.

The convergence in distribution of normalized correlograms of random fields is studied in [9] for the space $C[0, H]$.

Using Theorem 2.1 we state the following result, which is important for our consideration below. In fact, Lemma 2.1 is Theorem 4.1 of [1] stated for the space $C[0, H]$.

For functions $a(z)$, $z \in [0, H]$, we write $\|a\| = \sup_{z \in [0,H]} |a(z)|$.

Lemma 2.1. *Let* $Y_T \xrightarrow{\mathcal{D}} Y$ *and*

$$\|R_T\| \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty. \tag{3}$$

Then $X_T \xrightarrow{\mathcal{D}} Y$.

Thus relation (3) is sufficient for a functional theorem describing the asymptotic normality of the process X_T.

Now we introduce some necessary conditions. Assume that the regression function $g(t, \tau)$ is twice continuously differentiable with respect to $\tau \in \Theta_\gamma$ for every $t \geq 0$. We also assume that these derivatives are continuous with respect to all arguments.

Put

$$g_j(t, \tau) = \frac{\partial}{\partial \tau_j} g(t, \tau), \qquad g_{ij}(t, \tau) = \frac{\partial^2}{\partial \tau_i \partial \tau_j} g(t, \tau),$$

$$d_{iT}^2(\tau) = \int_0^T g_i^2(t, \tau)\, dt, \qquad d_{ij,T}^2(\tau) = \int_0^T g_{ij}^2(t, \tau)\, dt,$$

$$d_T(\tau) = \operatorname{diag}\left(d_{iT}(\tau), i = 1, \ldots, q\right).$$

We assume throughout the paper that

$$\varliminf_{T \to \infty} T^{-1/2} d_{iT}(\tau) > 0, \qquad \tau \in \Theta, \quad i = 1, \ldots, q.$$

AB. $d_T(\theta)\big(\widehat{\theta}_T-\theta\big)\xrightarrow{\mathcal{D}}\xi\sim N(0,K)$.

Sufficient conditions for the asymptotic normality of the least squares estimator can be found in [8, 12].

B1.(i) $\sup_{\tau\in\Theta^c}|g_j(t,\tau)|\le G_j(t)$ for some continuous functions $G_j(t)$, $t\ge 0$. Moreover,

$$\frac{D_{jT}}{d_{jT}(\tau)}\le c_j(\tau)<\infty,\qquad \tau\in\Theta,$$

where $D_{jT}^2=\int_0^T G_j^2(t)\,dt$, $j=1,\dots,q$;

(ii) $\frac{d_{j,T+H}(\tau)}{d_{jT}(\tau)}\to 1$ as $T\to\infty$ for $\tau\in\Theta$ and all fixed $H>0$.

B2. $\sup_{\tau\in\Theta^c}|g_{ij}(t,\tau)|\le G_{ij}(t)$ for some continuous functions $G_{ij}(t)$, $t\ge 0$, and

$$\frac{D_{ij,T}}{d_{iT}(\tau)d_{jT}(\tau)}\le c_{ij}(\tau)T^{-1/2},\qquad \tau\in\Theta,$$

where $D_{ij,T}^2=\int_0^T G_{ij}^2(t)\,dt$, $i,j=1,\dots,q$.

Put $g_{jT}(\lambda,\tau)=\int_0^T e^{i\lambda t}g_j(t,\tau)\,dt$, $\lambda\in\mathbb{R}^1$. By the Plancherel theorem,

$$\int_{-\infty}^{\infty}|g_{jT}(\lambda,\tau)|^2\,d\lambda=2\pi d_{jT}^2(\theta).$$

Consider the following family of measures on $\big(\mathbb{R}^1,\mathfrak{B}^1\big)$:

$$\mu_{jT}(d\lambda,\tau)=\frac{|g_{jT}(\lambda,\tau)|^2\,d\lambda}{\int_{-\infty}^{\infty}|g_{jT}(\lambda,\tau)|^2\,d\lambda},\qquad T>0.$$

B3. The family of measures $\mu_{jT}(\tau)$ weakly converges as $T\to\infty$ to the measure $\mu_j(\tau)$, $\tau\in\Theta$, $j=1,\dots,q$.

Then $\mu_j(\tau)=\mu_j(d\lambda,\tau)$ is called the spectral measure of the function $g_j(t,\tau)$ (see, for example, [6, 9]).

A3. $\sup_{\lambda\in\mathbb{R}^1}|\lambda|^{1+\delta}f(\lambda)\le c(\delta)<\infty$ for some $\delta\in(0,1]$.

Note that condition **A3** is sufficient for condition **A2** (see §6.4 in [3]). Put

$$\Psi_{jT}(z_1,z_2;\tau)=\int_0^T\left(g_j(t+z_1,\tau)-g_j(t+z_2,\tau)\right)^2\,dt.$$

B4. $d_{jT}^{-2}\Psi_{jT}(z_1,z_2;\tau)\le\overline{c}_j(\tau)\,|z_1-z_2|^2$ for $z_1,z_2\in[0,H]$; $\overline{c}_j(\tau)<\infty$ for $\tau\in\Theta$ and $j=1,\dots,q$.

Below we assume that conditions **B1**(**i**), **B2**, and **B4** hold for sufficiently large T, say for $T>T_0$.

Theorem 2.2. *Let conditions* **A1**, **A3**, **AB**, *and* **B1**–**B4** *hold. Then*

$$X_T(\cdot)=T^{1/2}\Big(\mathbf{B}_T\big(\cdot,\widehat{\theta}_T\big)-\mathbf{B}(\cdot)\Big)\xrightarrow{\mathcal{D}}Y.$$

In view of Lemma 2.1, the proof of Theorem 2.2 consists of three steps.

Lemma 2.2. *Let conditions* **AB** *and* **B1** *hold. Then*

$$T^{-1/2}\left\|I_{2T}\right\|\xrightarrow{\mathsf{P}}0,\qquad T\to\infty.$$

Proof. Introducing the notation

$$\Phi_T(\tau_1,\tau_2)=\int_0^T\left(g(t,\tau_1)-g(t,\tau_2)\right)^2\,dt,$$

we get

$$T^{-1/2}\,\|I_{2T}\| \le T^{-1/2}\Phi_T^{1/2}\big(\widehat{\theta}_T,\theta\big)\Phi_{T+H}^{1/2}\big(\widehat{\theta}_T,\theta\big),$$

$$\begin{aligned} T^{-1/2}\Phi_T^{1/2}\big(\widehat{\theta}_T,\theta\big) &\le T^{-1/2}\left(\sum_{i,j=1}^{q}\int_0^T G_i(t)G_j(t)\,dt\big|\widehat{\theta}_{iT}-\theta_i\big|\big|\widehat{\theta}_{jT}-\theta_j\big|\right)^{1/2} \\ &\le \sum_{j=1}^{q}\frac{D_{jT}}{d_{jT}(\theta)}\Big(T^{-1/2}d_{jT}(\theta)\big|\widehat{\theta}_{jT}-\theta_j\big|\Big) \xrightarrow{\mathsf{P}} 0, \qquad T\to\infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Phi_{T+H}^{1/2}\big(\widehat{\theta}_T,\theta\big) &\le \sum_{j=1}^{q}\frac{D_{j,T+H}}{d_{j,T+H}(\theta)}\cdot\frac{d_{j,T+H}(\theta)}{d_{jT}(\theta)}\Big|d_{jT}(\theta)\big(\widehat{\theta}_{jT}-\theta_j\big)\Big| \\ &\le (1+\Delta)\|c(\theta)\|\,\Big\|d_T\big(\widehat{\theta}_T-\theta\big)\Big\| \end{aligned}$$

for all $\Delta>0$ and $T>T_0(\Delta)$, where $c(\theta)=(c_j(\theta))_{j=1}^q$. □

Lemma 2.3. *Let conditions* **A1**, **A3**, **AB**, *and* **B1**–**B3** *hold. Then*

$$T^{-1/2}\,\|I_{3T}\| \xrightarrow{\mathsf{P}} 0, \qquad T\to\infty.$$

Proof. Conditions **B1(i)** and **B2(i)** allow one to apply Taylor's formula to the integral $T^{-1/2}I_{3T}(z)$, so that

$$\begin{aligned} T^{-1/2}I_{3T}(z) &= \sum_{j=1}^{q} d_{jT}^{-1}(\theta)\int_0^T \varepsilon(t+z)g_j(t,\theta)\,dt\,\Big(T^{-1/2}d_{jT}(\theta)\big(\widehat{\theta}_{jT}-\theta_j\big)\Big) \\ &\quad+\frac{1}{2}\sum_{i,j=1}^{q} d_{iT}^{-1}(\theta)d_{jT}^{-1}(\theta)\int_0^T \varepsilon(t+z)g_{ij}(t,\theta_T^*)\,dt \\ &\qquad\times d_{iT}(\theta)\big(\widehat{\theta}_{iT}-\theta_i\big)\Big(T^{-1/2}d_{jT}(\theta)\big(\widehat{\theta}_{jT}-\theta_j\big)\Big) \\ &= \sum_{1T}(z)+\sum_{2T}(z), \qquad \|\theta_T^*-\theta\|\le\|\widehat{\theta}_T-\theta\|. \end{aligned} \tag{4}$$

Consider the following stochastic processes:

$$\xi_{jT}(z)=d_{jT}^{-1}(\theta)\int_0^T \varepsilon(t+z)g_j(t,\theta)\,dt, \qquad z\in[0,H],\ T>0,\ j=1,\dots,q.$$

Note that the $\xi_{jT}(z)$ are almost surely continuous.

Condition **B3** implies that, as $T\to\infty$,

$$\begin{aligned} \mathbf{B}_{jT}(z_1-z_2) &= \mathsf{E}\,\xi_{jT}(z_1)\xi_{jT}(z_2) \\ &= d_{jT}^{-2}(\theta)\int_0^T\int_0^T \mathbf{B}(t-s+z_1-z_2)g_j(t,\theta)g_j(s,\theta)\,dt\,ds \\ &= 2\pi\int_{-\infty}^{\infty} e^{i\lambda(z_1-z_2)}f(\lambda)\,\mu_{jT}(d\lambda,\theta) \\ &\to 2\pi\int_{-\infty}^{\infty}\cos\lambda(z_1-z_2)f(\lambda)\,\mu_j(d\lambda,\theta)=\mathbf{B}_j(z_1-z_2), \qquad z_1,z_2\in[0,H]. \end{aligned}$$

Therefore all finite-dimensional distributions of the Gaussian stationary process ξ_{jT} converge to the corresponding finite-dimensional distributions of the Gaussian process $\xi_j=\{\xi_j(z), z\in[0,H]\}$ with the covariance function $\mathbf{B}_j(z)$. We assume that the processes ξ_j, $j=1,\dots,q$, are separable.

Since

$$\mathsf{E}\left(\xi_j(z_1)-\xi_j(z_2)\right)^2 = 2\left(\mathbf{B}_j(0)-\mathbf{B}_j(z_1-z_2)\right) \le 2^{1-\delta}\pi c(\delta)|z_1-z_2|^{1+\delta}, \quad z_1, z_2 \in [0,H],$$

for $j = 1,\dots,q$ by condition **A3**, the processes ξ_j are almost surely continuous in view of Kolmogorov's theorem (see, for example, [5]).

On the other hand,

$$\begin{aligned}\mathsf{E}\left(\xi_{jT}(z_1)-\xi_{jT}(z_2)\right)^2 &= 2\left(\mathbf{B}_{jT}(0)-\mathbf{B}_{jT}(z_1-z_2)\right)\\ &\le 2^{2-\delta}\pi\left(\int_{-\infty}^{\infty}|\lambda|^{1+\delta}f(\lambda)\,\mu_j(d\lambda,\theta)+1\right)|z_1-z_2|^{1+\delta}\end{aligned}$$

for $\delta \in (0,1]$ by assumptions of the lemma for $T > T_0$

Therefore, $\xi_{jT} \xrightarrow{\mathcal{D}} \xi_j$, $j = 1,\dots,q$, in the space $C[0,H]$, and, for all continuous in $C[0,H]$ functionals φ, the distribution of $\varphi(\xi_{jT})$ converges to the distribution of $\varphi(\xi_j)$. In particular, $\|\xi_{jT}\| \xrightarrow{\mathcal{D}} \|\xi_j\|$, $j = 1,\dots,q$, and (see (4))

$$\|\Sigma_{1T}\| \le \sum_{j=1}^{q}\|\xi_{jT}\|\left(T^{-1/2}d_{jT}(\theta)\left|\widehat{\theta}_{jT}-\theta_{jT}\right|\right) \xrightarrow{\mathsf{P}} 0. \tag{5}$$

Each term of the sum $\sum_{2T}(z)$, $z \in [0,H]$, is bounded from above by

$$\sqrt{2}\frac{T^{1/2}D_{ij,T}}{d_{iT}(\theta)d_{jT}(\theta)}\left((2T)^{-1}\int_0^{2T}\varepsilon^2(t)\,dt\right)^{1/2}\left|d_{iT}(\theta)\left(\widehat{\theta}_{iT}-\theta_i\right)\right|\left(T^{-1/2}d_{jT}(\theta)\left|\widehat{\theta}_{jT}-\theta_j\right|\right),$$

where

$$\frac{T^{1/2}D_{ij,T}}{d_{iT}(\theta)d_{jT}(\theta)} \le c_{ij}(\theta)$$

by condition **B2**. In addition,

$$(2T)^{-1}\int_0^{2T}\varepsilon^2(t)\,dt \xrightarrow{\mathsf{P}} \mathbf{B}(0), \qquad T\to\infty,$$

since

$$\mathsf{E}\left(T^{-1}\int_0^T\left(\varepsilon^2(t)-\mathbf{B}(0)\right)\,dt\right)^2 = 2T^{-2}\int_0^T\int_0^T\mathbf{B}^2(t-s)\,dt\,ds = O\left(T^{-1}\right).$$

Hence

$$\|\Sigma_{2T}\| \xrightarrow{\mathsf{P}} 0, \qquad T\to\infty, \tag{6}$$

and Lemma 2.3 follows from (4)–(6). □

Lemma 2.4. *Let conditions* **A1**, **AB**, **B1**, **B2**, *and* **B4** *hold. Then*

$$T^{-1/2}\|I_{4T}\| \xrightarrow{\mathsf{P}} 0, \qquad T\to\infty.$$

Proof. We have

$$\begin{aligned}T^{-1/2}I_{4T}(z) &= \sum_{j=1}^{q} d_{jT}^{-1}(\theta)\int_0^T \varepsilon(t)g_j(t+z,\theta)\,dt\left(T^{-1/2}d_{jT}(\theta)\big(\widehat{\theta}_{jT}-\theta_j\big)\right)\\ &\quad+\frac{1}{2}\sum_{i,j=1}^{q} d_{iT}^{-1}(\theta)d_{jT}^{-1}(\theta)\int_0^T \varepsilon(t)g_{ij}(t+z,\theta_T^*)\,dt\\ &\quad\times\left(d_{iT}(\theta)\big(\widehat{\theta}_{iT}-\theta_i\big)\right)\left(T^{-1/2}d_{jT}(\theta)\big(\widehat{\theta}_{jT}-\theta_j\big)\right)\\ &= \Sigma_{3T}(z)+\Sigma_{4T}(z).\end{aligned}$$

Consider the following almost surely continuous Gaussian processes:

$$\eta_{jT}(z) = d_{jT}^{-1}(\theta)\int_0^T \varepsilon(t)g_j(t+z,\theta)\,dt,\qquad z\in[0,H],\ T>0,\ j=1,\dots,q.$$

Then, for $z_1, z_2 \in [0,H]$,

$$\begin{aligned}\mathsf{E}\,\eta_{jT}(z_1)\eta_{jT}(z_2) &= d_{jT}^{-2}(\theta)\int_0^T\int_0^T \mathbf{B}(t-s)g_j(t+z_1,\theta)g_j(s+z_2,\theta)\,dt\,ds\\ &= d_{jT}^{-2}(\theta)\int_{z_1}^{T+z_1}\int_{z_2}^{T+z_2}\mathbf{B}(t-s+z_2-z_1)g_j(t,\theta)g_j(s,\theta)\,dt\,ds\end{aligned}$$

and

$$\int_{z_1}^{T+z_1}\int_{z_2}^{T+z_2} = \left(\int_0^T+\int_T^{T+z_1}-\int_0^{z_1}\right)\left(\int_0^T+\int_T^{T+z_2}-\int_0^{z_2}\right).$$

Next we estimate the integral

$$\begin{aligned}\left|d_{jT}^{-2}(\theta)\int_0^T\int_T^{T+z_2}\right| &\le \left(\int_T^{T+z_2}\int_0^T \mathbf{B}^2(t-s+z_2-z_1)\,dt\,ds\right)^{1/2}\\ &\quad\times\left(d_{jT}^{-2}(\theta)\int_T^{T+z_2} g_j^2(s,\theta)\,ds\right)^{1/2}\\ &\le H^{1/2}\|\mathbf{B}\|_2\left(d_{jT}^{-2}(\theta)\left(d_{j,T+H}^2(\theta)-d_{jT}^2(\theta)\right)\right)^{1/2}\to 0\end{aligned}$$

by condition **B1**(**ii**).

Similarly,

$$d_{jT}^{-2}(\theta)\int_0^{T+z_1}\int_0^T \to 0.$$

In the same way as above, we get

$$\left|d_{jT}^{-2}(\theta)\int_0^T\int_0^{z_2}\right| \le H^{1/2}\|\mathbf{B}\|_2 d_{jH}(\theta)d_{jT}^{-1}(\theta)\to 0,\qquad d_{jT}^{-2}\int_0^{z_1}\int_0^T \to 0.$$

Further,

$$\left|d_{jT}^{-2}(\theta)\int_T^{T+z_1}\int_T^{T+z_2}\right| \le H\mathbf{B}(0)d_{jT}^{-2}(\theta)\int_T^{T+H} g_j^2(t,\theta)\,dt\to 0,$$

$$\left|d_{jT}^{-2}(\theta)\int_0^{z_1}\int_T^{T+z_2}\right| \le H\mathbf{B}(0)d_{jH}(\theta)d_{jT}^{-1}(\theta)\left(d_{jT}^{-2}(\theta)\int_T^{T+H} g_j^2(s,\theta)\,ds\right)^{1/2}\to 0,$$

$$d_{jT}^{-2}(\theta)\int_0^{T+z_1}\int_0^{z_2}\to 0,\qquad d_{jT}^{-2}(\theta)\int_0^{z_1}\int_0^{z_2}\to 0.$$

Thus

$$\mathsf{E}\,\eta_{jT}(z_1)\eta_{jT}(z_2) = \mathbf{B}_{jT}(z_1,z_2) + o_{jT}(1), \qquad o_{jT}(1)\to 0,\ z_1,z_2\in[0,H],$$

and all finite-dimensional distributions of the processes η_{jT} converge to the corresponding finite-dimensional distributions of the almost surely continuous processes ξ_j, $j=1,\dots,q$ (see the proof of Lemma 2.3).

Now conditions **A1** and **B4** imply that

$$\begin{aligned}
&\mathsf{E}\left(\eta_{jT}(z_1)-\eta_{jT}(z_2)\right)^2 \\
&\quad = d_{jT}^{-2}(\theta)\int_0^T\int_0^T \mathbf{B}(t-s)\left(g_j(t+z_1,\theta)-g_j(t+z_2,\theta)\right) \\
&\qquad\qquad\qquad\qquad \times \left(g_j(s+z_1,\theta)-g_j(s+z_2,\theta)\right)\,dt\,ds \\
&\quad \le d_{jT}^{-2}(\theta)\int_0^T\int_0^T |\mathbf{B}(t-s)|\left(g_j(t+z_1,\theta)-g_j(t+z_2,\theta)\right)^2\,dt\,ds \\
&\quad \le \|\mathbf{B}\|_1 d_{jT}^{-2}(\theta)\Psi_{jT}(z_1,z_2;\theta)\le \overline{c}_j(\theta)\|\mathbf{B}\|_1|z_1-z_2|^2,
\end{aligned}$$

$z_1,z_2\in[0,H]$, $j=1,\dots,q$, where

$$\|\mathbf{B}\|_1=\int_{-\infty}^{\infty}|\mathbf{B}(t)|\,dt<\infty.$$

We proved that $\|\eta_{jT}\|\xrightarrow{\mathcal{D}}\|\xi_j\|$, $j=1,\dots,q$. Condition **AB** yields $\|\sum_{3T}\|\xrightarrow{\mathsf{P}}0$ as $T\to\infty$.

Each term of the sum $\Sigma_{4T}(z)$, $z\in[0,H]$, is estimated from above by

$$\begin{aligned}
&\left(T^{-1}\int_0^T\varepsilon^2(t)\,dt\right)^{1/2}\frac{D_{ij,T+H}(T+H)^{1/2}}{d_{i,T+H}(\theta)d_{j,T+H}(\theta)}\frac{T^{1/2}}{(T+H)^{1/2}}\left(\frac{d_{i,T+H}(\theta)}{d_{iT}(\theta)}\right)\left(\frac{d_{j,T+H}(\theta)}{d_{jT}(\theta)}\right) \\
&\qquad \times\left|d_{iT}(\theta)\big(\widehat{\theta}_{iT}-\theta_i\big)\right|\left|T^{-1/2}d_{jT}(\theta)\big(\widehat{\theta}_{jT}-\theta_j\big)\right|\xrightarrow{\mathsf{P}}0,\qquad T\to\infty,
\end{aligned}$$

in view of conditions **A1**, **AB**, **B1**, and **B2**. Therefore, $\|\sum_{4T}\|\xrightarrow{\mathsf{P}}0$, and Lemma 2.4 is proved. □

3. Example

Let $g(t,\theta)=A\cos\varphi t+B\sin\varphi t$, $\theta=(\theta_1,\theta_2,\theta_3)=(A,B,\varphi)\in\Theta$, and let Θ^c be a compact subset of $(0,\infty)^3$. Then

$$d_{1T}^2(\theta)=\frac{T}{2}+O(1),\qquad d_{2T}^2(\theta)=\frac{T}{2}+O(1),\qquad d_{3T}^2(\theta)=\frac{A^2+B^2}{6}T^3+O(T^2).$$

One can take $G_1(t)=G_2(t)=1$ and $G_3(t)=(\overline{A}+\overline{B})t$ in condition **B1**(**i**), where $\overline{A}$ and $\overline{B}$ are upper bounds for the parameters A and B, respectively. Then $D_{1T}^2=D_{2T}^2=T$ and $D_{3T}^2=\frac{1}{3}\big(\overline{A}+\overline{B}\big)^2T^3$, whence condition **B1**(**i**) follows. Condition **B1**(**ii**) is obvious.

It is also clear that

$$\begin{gathered}
g_{11}=g_{22}=g_{12}=0,\qquad g_{13}=-t\sin\varphi t,\qquad g_{23}=t\cos\varphi t,\\
g_{33}=-At^2\cos\varphi t-Bt^2\sin\varphi t,\\
G_{13}(t)=G_{23}(t)=t,\qquad G_{33}(t)=\big(\overline{A}+\overline{B}\big)\,t^2,
\end{gathered}$$

and

$$D_{13,T}^2\ =\ D_{23,T}^2=\frac{T^3}{3},\qquad D_{33,T}^2=\frac{\big(\overline{A}+\overline{B}\big)^2}{5}T^5.$$

Hence

$$\frac{D^2_{j3,T}}{d_{jT}(\theta)d_{3T}(\theta)} = O\left(T^{-1/2}\right), \qquad j=1,2,3;$$

that is, condition **B2** holds.

It is quite easy to see (see, for example, [6, 10]) that the functions g_1, g_2, and g_3 possess the same atomic spectral measure $\mu(d\lambda)$ such that $\mu(\{\pm\varphi\}) = 1/2$. This implies condition **B3**.

Note that if $f(\varphi) \neq 0$, then

$$\mathbf{B}_j(z) = 2\pi \int_{-\infty}^{\infty} \cos\lambda z f(\lambda)\,\mu(d\lambda) = 2\pi f(\varphi)\cos\varphi z, \qquad j=1,2,3.$$

This means that the limit processes $\xi_j(z)$, $z\in[0,H]$, are random harmonic oscillations; namely $\xi_j(z) = \eta_j\cos\varphi z + \zeta_j \sin\varphi z$, where η_j and ζ_j are independent $N(0, 2\pi f(\varphi))$ random variables.

Next we check condition **B4**. This condition is clear if $j = 1,2$. Now let $z_2 > z_1$. Then

$$|g_3(t+z_1,\theta) - g_3(t+z_2,\theta)| = |g_3'(t^*,\theta)|\cdot|z_1-z_2|,$$

where $t^* = t + z_1 + \nu(z_2 - z_1) \le t + H$ and $\nu \in (0,1)$ is a certain number. We get

$$g_3(t,\theta) = -At\sin\varphi t + Bt\cos\varphi t,$$

$$\begin{aligned}|g_3'(t^*,\theta)|^2 &= |-A\sin\varphi t^* - A\varphi t^*\cos\varphi t^* + B\cos\varphi t^* - B\varphi t^*\sin\varphi t^*|^2 \\ &\le (A+B)^2(1+\varphi H+\varphi t)^2,\end{aligned}$$

whence

$$d_{3T}^{-2}(\theta)\Psi_{3T}(z_1,z_2;\theta) \le \left(1+O\left(T^{-1}\right)\right)^{-1}\left(\frac{2\varphi^2(A+B)^2}{A^2+B^2} + O\left(T^{-1}\right)\right)|z_1-z_2|^2.$$

As far as condition **AB** is concerned, the asymptotic normality of the least squares estimator $d_T(\theta)(\widehat{\theta}_T - \theta)$ for the trigonometric regression function is obtained, for example, in [10, 7].

4. Concluding remarks

We considered a nonlinear regression model with continuous time and mean square continuous as well as almost surely continuous Gaussian stationary random noise with zero mean and absolutely integrable covariance function. A functional theorem on the asymptotic normality of the estimator of the covariance function of the noise is proved in the space of continuous functions. This estimator allows one to construct asymptotic confidence intervals for an unknown covariance function of the random noise.

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