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THE CONCAVITY OF THE PAYOFF FUNCTION OF A SWING OPTION IN A BINOMIAL MODEL

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Abstract. We use the lattice method to price a swing option. We show that the payoff function at each node of the lattice is concave and piecewise linear. A corollary of this result is that there exists a bang-bang control such that if the loan at a certain moment is integer, then the optimal purchased quantity at this moment is equal to either 0 or 1. If the loan at a certain moment is not integer, then the fair price is a convex combination of the nearest pay-off values with integer loans.

1. Introduction

The first tree model for pricing an option, known nowadays as the Cox–Ross–Rubinstein binomial model, is attributed to Cox et al. [1]. The tree method still is an important tool for finding fair prices of options; a survey of the modern models can be found in Seydel [2]. As a rule, the tree method requires an amount of computation and is used to make decisions on whether to exercise an option depending on a trajectory of the stochastic process describing the evolution of the price of the underlying asset. A swing option being popular in the energy derivatives market is an example of such a kind of contingent claim.

The definition and main properties of swing options can be found in Breslin et al. [3]. A typical gas swing contract is an agreement between a supplier and a purchaser for the delivery of variable daily quantities of gas over a certain number of years at a specified set of contract prices. The contract assumes that the quantities lie between specified minimum and maximum daily limits. The main features of swing contracts that make them difficult to value and manage risks are the constraints on the quantity of gas that can be taken. The main constraint, called take-or-pay, is that in each year there is a minimum volume of gas (termed “take-or-pay” or “minimum bill”) for which the purchaser will be charged at the end of the year (or at a “penalty date”).

While swing contracts have been used for many years to manage the inherent uncertainty of gas supply and demand, it is only in recent years with the deregulation of the energy markets that there has been an interest in understanding and valuing the optionality contained in these contracts. The volatility in the model described in Breslin et al. [3] is a deterministic function with respect to both variables, namely with respect to the time variable as well as with respect to the variable measuring the time until maturity.

There nevertheless is much evidence that the volatility in gas markets is random. It is claimed in Chiarella et al. [4] that a model with varying volatility better describes the stochastic nature of the volatility function in gas markets. Their investigation is based

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on a model described in Wahab, Lee [5] and Wahab et al. [6]. This model relies on the pentanomial lattice approach with changing regimes considered in Bollen [7]. The model in Wahab, Lee [5] and Wahab et al. [6] considers the geometrical Brownian motion as an evolution process for the price of the underlying asset and does not involve the "take-or-pay" principle when pricing an option. In contrast, the model in Chiarella et al. [4] involves the "take-or-pay" principle when pricing an option and considers the mean reverting process to describe dynamics of the asset.

Wahab, Lee [5] and Wahab et al. [6] use the dynamic programming principle and take the maximum of two values when pricing an option at every time instance, namely they take the price of an option in the event of purchasing a maximal amount of the underlying asset or the price of an option if it is not exercised at a current moment. It is also assumed that the loan is expressed as an integer number at each time instance. Such a strategy is called the "bang-bang" control. It is worth mentioning that Wahab, Lee [5] and Wahab et al. [6] do not show that such a strategy maximizes the profit of a holder of a swing option.

We prove for the binomial model that, under certain assumptions imposed on the global limits for purchasing the underlying asset written in the contract, the optimal strategy corresponds to the "bang-bang" regime indeed if the loan is expressed as an integer number. Moreover we show that the pay-off function at every node of the tree is concave and step wise linear.

General results concerning the existence of the "bang-bang" regime can be found in Bardou et al. [8]. Our aim is to obtain some properties of the pay-off function of a swing option for a more practical setting. Discussing the property of concavity of the pay-off function of a swing option we provide a clear financial explanation of this property.

2. The optimal strategy of a holder of a swing option

Consider the Cox–Ross–Rubinstein binomial model in the time interval from 0 to N. Let $S_0(0)$ and $S_j(n)$ be the spot price of the underlying asset at the moments $t = 0$ and $t = n$, respectively. The number of moments in the interval $[0, n]$ when the spot price increases is denoted by j. For the Cox–Ross–Rubinstein model, the spot price of the underlying asset changes as follows:

$$S_j(n) = \frac{1}{1+r}(\tilde{p}S_{j+1}(n+1) + \tilde{q}S_j(n+1)),$$

where r is a fixed non-risky interest rate, and $\tilde{p}$ and $\tilde{q}$ are risk-neutral probabilities of increase and decrease of the spot price, respectively, over one period. The probabilities $\tilde{p}$ and $\tilde{q}$ are such that

$$\tilde{p}u + \tilde{q}d = 1 + r,$$

where u and d are the coefficients of growth and decrease, respectively. The numbers $\tilde{p}$ and $\tilde{q}$ are positive and their sum equals one if and only if the arbitrage free condition holds, namely

$$0 < d < 1 + r < u.$$

Let V_N denote the pay-off related to an option at moment N. We assume that V_N is an $\mathcal{F}_N$-measurable random variable. Here $\mathcal{F}_N$ is the σ-algebra of random events observable up to the moment N. It is clear that $\mathcal{F}_n \subset \mathcal{F}_m \subset \mathcal{F}_N$ for all $0 \le n \le m \le N$. Then the price of a derivative at a moment n is given by the following *risk-neutral pricing formula*:

$$V_n = \widetilde{\mathbb{E}}\left[\frac{V_N}{(1+r)^{N-n}} \,\middle|\, \mathcal{F}_n\right]$$

for all n such that $0 \le n \le N$, where $\widetilde{\mathbb{E}}[\cdot \mid \mathcal{F}_n]$ is the conditional expectation with respect to the risk-neutral probabilities.

Moreover the discounted price of the derivative is a martingale with respect to the risk-neutral measure,

$$\frac{V_n}{(1+r)^n} = \widetilde{\mathbb{E}}\left[\frac{V_{n+1}}{(1+r)^{n+1}} \,\middle|\, \mathcal{F}_n\right].$$

Now we are going to introduce the notion of a swing option for the Cox–Ross–Rubinstein model. We consider the so-called *normalized swing option* which means that one can purchase no more than one unit of the underlying asset at each moment. More precisely, the amount of purchase of the underlying asset varies from zero to one at every instance of time.

Thus *the global restriction* are as follows: if a contract is conducted at the initial (zero) time and expires at a moment N, then the holder of the contract is not allowed to purchase more than $N+1$ units of the asset during the period from 0 to N. Let m be the amount of the underlying asset allowed to be purchased after the moment $N-l$. Then $0 \le m \le N+1$ at the initial time, and $0 \le m \le l+1$ at the moment $N-l$. If $m = l+1$ at the moment $N-l$, then the strategy of the contract holder is unique, namely he/she should purchase one unit of the underlying asset at every moment starting with $N-l$. Let K be the option price of one unit of the underlying asset (in other words, K is the strike price).

Let $V(j,m,l)$ be the price of a swing option at moment $N-l$ when the spot price increased j times preceding the moment $N-l$ and let m be the amount of the underlying asset free for purchasing at moment $N-l$. Then the boundary conditions are written as follows:

$$V(j,0,0) = (S_j(N)-K)^+, \qquad V(j,1,0) = S_j(N)-K.$$

If $\alpha \in (0,1)$, then

$$V(j,\alpha,0) = \alpha(S_j(N)-K) + (1-\alpha)(S_j(N)-K)^+ = \alpha V(j,1,0) + (1-\alpha)V(j,0,0).$$

Thus $V(j,0,0) = (S_j(N)-K)^+$ means that the option holder purchased all amounts of gas until time N and thus can proceed as follows: if $S_j(N)-K>0$, then the option holder may purchase one unit of gas for the strike price K according to the option and then immediately sell it for the price $S_j(N)$ in the spot market making therefore the profit $S_j(N)-K$; otherwise, that is, if $S_j(N) \le K$, the option holder is not obliged to purchase anything and apply a strategy similar to that in the case of $S_j(N) > K$ because it makes him/her out the money, since $m=0$. Both cases are described by the following formula:

$$(S_j(N)-K)^+ = \max(S_j(N)-K, 0)$$

for $m=0$.

If $m=1$, then the pay-off at moment N equals $V(j,1,0) = S_j(N)-K$. The option holder is obliged to purchase one unit of gas at the moment N and the pay-off equals $S_j(N)-K$ according to the option (an analogous reasoning is shown above).

To price this derivative we make a key assumption that a decision on whether or not an investor purchases gas is made according to the evolution of the spot price of the underlying asset irrespective of a real need in gas. Then, in view of the principle that investor behaviors are rational, the price of the derivative at every time instant $N-l$ is a solution of the following maximization problem:

$$V(j,m,l) = \sup_{\alpha\in[0,1]}\Big[\alpha(S_j(N-l)-K) + \frac{1}{1+r}(\tilde{p}V(j+1,(m-\alpha)^+,l-1) + \tilde{q}V(j,(m-\alpha)^+,l-1)\Big].$$

Below we prove a number of properties of the function $V(j,m,l)$ needed to justify the optimal strategy of a holder of a swing option in the Cox–Ross–Rubinstein model under the assumption that his/her aim is to maximize the profit.

For the sake of simplicity we put $r=0$.

Theorem 2.1. *The price function $V(j,m,l)$ of a swing option is not increasing with m for all l and j.*

Proof. We prove this result by induction with respect to l for all j.

I. *Base of induction $l=0$.* It is clear that $V(j,\alpha,0)\le V(j,\beta,0)$ if $0\le\beta\le\alpha\le 1$, since $V(j,\beta,0)$ means more opportunity (the amount of the underlying asset to be purchased at the moment N is less in this case).

II. *Step of induction.* Assume that the statement is true for $l_0 = l-1$. Let

$$V(j,m_1,l) = \alpha(S_j(N-l)-K) + \tilde{p}V(j+1,(m_1-\alpha)^+,l-1) + \tilde{q}V(j,(m_1-\alpha)^+,l-1).$$

Choose m_2 such that $0\le m_2\le m_1\le l+1$. By the definition of the function V,

$$V(j,m_2,l) \ge \alpha(S_j(N-l)-K) + \tilde{p}V(j+1,(m_2-\alpha)^+,l-1) + \tilde{q}V(j,(m_2-\alpha)^+,l-1)$$

for all $\alpha\in[0,1]$.

Since $m_2\le m_1$, the assumption of induction implies that

$$V(j+1,(m_2-\alpha)^+,l-1) \ge V(j+1,(m_1-\alpha)^+,l-1)$$

and

$$V(j,(m_2-\alpha)^+,l-1) \ge V(j,(m_1-\alpha)^+,l-1).$$

Then

$$\begin{aligned} V(j,m_2,l) &\ge \alpha(S_j(N-l)-K) + \tilde{p}V(j+1,(m_2-\alpha)^+,l-1) + \tilde{q}V(j,(m_2-\alpha)^+,l-1) \\ &\ge \alpha(S_j(N-l)-K) + \tilde{p}V(j+1,(m_1-\alpha)^+,l-1) + \tilde{q}V(j,(m_1-\alpha)^+,l-1) \\ &= V(j,m_1,l). \end{aligned}$$

This completes the proof, since α is arbitrary. □

Theorem 2.2. *The following properties of the price function of a swing option $V(j,m,l)$ hold for all l and j:*

(1) *given an arbitrary integer number m such that $0\le m\le l$, the function $V(j,m,l)$ can be written as follows:*

$$V(j,m,l) = \max\Big[S_j(N-l)-K+\tilde{p}V\left(j+1,(m-1)^+,l-1\right)+\tilde{q}V\left(j,(m-1)^+,l-1\right); \tilde{p}V(j+1,m,l-1)+\tilde{q}V(j,m,l-1)\Big];$$

(2) *$V(j,m,l)$ is concave with respect to m, where $m\in\mathbb{Z}_+$;*

(3) *the function $V(j,m,l)$ is step wise linear, that is,*

$$V(j,m+\alpha,l) = \alpha V(j,m+1,l) + (1-\alpha)V(j,m,l)$$

for all $\alpha\in[0,1]$ and $m\in\mathbb{Z}_+$ such that $0\le m\le l$.

Proof. We use the induction with respect to l for all j. We prove all statements (1)–(3) simultaneously, since each step of induction needs all of them to be true for the preceding step.

I. *Base of induction* obviously follows from the boundary conditions. Indeed, if $l=0$, then consider all possible prices of a swing option at the moment N. Since $0\le m\le l+1$ at each time moment, the integer number m is equal to either 0 or to 1 if $l=0$:

$$V(j,0,0)=(S_j(N)-K)^+,\qquad V(j,1,0)=S_j(N)-K.$$

If $\alpha\in[0,1]$, then

$$V(j,\alpha,0)=\alpha(S_j(N)-K)+(1-\alpha)(S_j(N)-K)^+=\alpha V(j,1,0)+(1-\alpha)V(j,0,0).$$

We see that $V(j,m,l)$ for $l=0$ is a linear function in the interval $[0,1]$.

II. *Step of induction.* Let all the three properties (1)–(3) of the function $V(j,m,l)$ hold for all $l<l_0$. First we consider the case of $l=l_0$.

Step 1. For all $m\in\mathbb{Z}_+$ and all $\alpha\in[0,1]$,

$$V(j,m,l)=\sup_{\alpha\in[0,1]}\bigl\{\alpha(S_j(N-l)-K)+\tilde pV(j+1,m-\alpha,l-1)+\tilde qV(j,m-\alpha,l-1)\bigr\}.$$

By the inductive assumption,

$$V(j,m-\alpha,l-1)=(1-\alpha)V(j,m,l-1)+\alpha V(j,m-1,l-1).$$

Then

$$\begin{aligned}V(j,m,l)=\sup_{\alpha\in[0,1]}\Bigl\{&\alpha(S_j(N-l)-K)\\&+(1-\alpha)\bigl[\tilde pV(j+1,m,l-1)+\tilde qV(j,m,l-1)\bigr]\\&\qquad+\alpha\bigl[\tilde pV(j+1,m-1,l-1)+\tilde qV(j,m-1,l-1)\bigr]\Bigr\}.\end{aligned}$$

At the same time, $\sup_{\alpha\in[0,1]}\{\alpha\, a+(1-\alpha)b\}=\max\{a,b\}$, whence

$$\begin{aligned}V(j,m,l)=\max\Bigl[&S_j(N-l)-K+\tilde pV(j+1,m-1,l-1)+\tilde qV(j,m-1,l-1);\\&\tilde pV(j+1,m,l-1)+\tilde qV(j,m,l-1)\Bigr].\end{aligned}$$

Step 2. We prove the concavity with respect to $m\in\mathbb{Z}_+$. Step 1 implies that the following functions are well defined for $0\le m\le l-1$:

$$\begin{aligned}V(j,m,l)&=\max\bigl[S_j(N-l)-K+\tilde pV(j+1,m-1,l-1)+\tilde qV(j,m-1,l-1);\\&\qquad\tilde pV(j+1,m,l-1)+\tilde qV(j,m,l-1)\bigr],\\V(j,m+1,l)&=\max\bigl[S_j(N-l)-K+\tilde pV(j+1,m,l-1)+\tilde qV(j,m,l-1);\\&\qquad\tilde pV(j+1,m+1,l-1)+\tilde qV(j,m+1,l-1)\bigr],\\V(j,m-1,l)&=\max\bigl[S_j(N-l)-K+\tilde pV(j+1,m-2,l-1)+\tilde qV(j,m-2,l-1);\\&\qquad\tilde pV(j+1,m-1,l-1)+\tilde qV(j,m-1,l-1)\bigr].\end{aligned}$$

If $m=l$, then

$$V(j,m+1,l)=S_j(N-l)-K+\tilde pV(j+1,m,l-1)+\tilde qV(j,m,l-1).$$

Now we consider four cases for $0\le m\le l-1$:

Case a. Let

$$\begin{aligned}S_j(N-l)-K+\tilde pV(j+1,m,l-1)+\tilde qV(j,m,l-1)\\\ge\tilde pV(j+1,m+1,l-1)+\tilde qV(j,m+1,l-1)\end{aligned}$$

and

$$S_j(N-l)-K+\tilde{p}V(j+1,m-2,l-1)+\tilde{q}V(j,m-2,l-1) \\ \geq \tilde{p}V(j+1,m-1,l-1)+\tilde{q}V(j,m-1,l-1).$$

By the inductive assumption, $V(j,m,l-1)$ is concave, whence

$$\begin{aligned} V(j,m,l) &\geq S_j(N-l)-K+\tilde{p}V(j+1,m-1,l-1)+\tilde{q}V(j,m-1,l-1) \\ &\geq \frac{1}{2}\Big(S_j(N-l)-K+\tilde{p}V(j+1,m,l-1)+\tilde{q}V(j,m,l-1) \\ &\qquad + S_j(N-l)-K+\tilde{p}V(j+1,m-2,l-1)+\tilde{q}V(j,m-2,l-1)\Big) \\ &= \frac{V(j,m+1,l)+V(j,m-1,l)}{2}. \end{aligned}$$

Case b. Now let

$$S_j(N-l)-K+\tilde{p}V(j+1,m,l-1)+\tilde{q}V(j,m,l-1) \\ < \tilde{p}V(j+1,m+1,l-1)+\tilde{q}V(j,m+1,l-1)$$

and

$$S_j(N-l)-K+\tilde{p}V(j+1,m-2,l-1)+\tilde{q}V(j,m-2,l-1) \\ < \tilde{p}V(j+1,m-1,l-1)+\tilde{q}V(j,m-1,l-1).$$

By the inductive assumption,

$$\begin{aligned} V(j,m,l) &\geq \tilde{p}V(j+1,m,l-1)+\tilde{q}V(j,m,l-1) \\ &\geq \frac{1}{2}\Big(\tilde{p}V(j+1,m+1,l-1)+\tilde{q}V(j,m+1,l-1) \\ &\qquad + \tilde{p}V(j+1,m-1,l-1)+\tilde{q}V(j,m-1,l-1)\Big) \\ &= \frac{V(j,m+1,l)+V(j,m-1,l)}{2}. \end{aligned}$$

Case c. Consider the case of

$$S_j(N-l)-K+\tilde{p}V(j+1,m,l-1)+\tilde{q}V(j,m,l-1) \\ \geq \tilde{p}V(j+1,m+1,l-1)+\tilde{q}V(j,m+1,l-1)$$

and

$$S_j(N-l)-K+\tilde{p}V(j+1,m-2,l-1)+\tilde{q}V(j,m-2,l-1) \\ < \tilde{p}V(j+1,m-1,l-1)+\tilde{q}V(j,m-1,l-1).$$

Then we get

$$\begin{aligned} 2V(j,m,l) &\geq S_j(N-l)-K \\ &\quad + \big(\tilde{p}V(j+1,m-1,l)+\tilde{q}V(j,m-1,l)+\tilde{p}V(j+1,m,l)+\tilde{q}V(j,m,l)\big) \\ &= V(j,m-1,l)+V(j,m+1,l). \end{aligned}$$

If $m=l$, then Cases **a** and **c** are exhaustive, since

$$V(j,m+1,l)=S_j(N-l)-K+\tilde{p}V(j+1,m,l-1)+\tilde{q}V(j,m,l-1).$$

Case d. Let $A_1=\max\{a,b\}$ and $A_2=\max\{c,d\}$. Assume that $a-b\geq c-d$. If $A_2=c$, then $c-d>0$ and $a>b$, whence $A_1=a$. Thus the case where $A_1=b$ and $A_2=c$ is excluded if $a-b\geq c-d$.

Now we specify $A_1 = V(j, m+1, l)$ and $A_2 = V(j, m-1, l)$. Then

$$\begin{aligned} a &= S_j(N-l) - K + \tilde{p}V(j+1, m, l-1) + \tilde{q}V(j, m, l-1), \\ b &= \tilde{p}V(j+1, m+1, l-1) + \tilde{q}V(j, m+1, l-1), \\ c &= S_j(N-l) - K + \tilde{p}V(j+1, m-2, l-1) + \tilde{q}V(j, m-2, l-1), \\ d &= \tilde{p}V(j+1, m-1, l-1) + \tilde{q}V(j, m-1, l-1). \end{aligned}$$

Our aim is to prove that $a - b \geq c - d$. Indeed,

$$\begin{aligned} a - b = {} & S_j(N-l) - K + \tilde{p}(V(j+1, m, l-1) - V(j+1, m+1, l-1)) \\ & + \tilde{q}(V(j, m, l-1) - V(j+1, m+1, l-1)), \\ c - d = {} & S_j(N-l) - K + \tilde{p}(V(j+1, m-2, l-1) - V(j, m-1, l-1)) \\ & + \tilde{q}(V(j, m-2, l-1) - V(j, m-1, l-1)). \end{aligned}$$

By the inductive assumption,

$$V(j, m-1, l-1) \geq \frac{V(j, m-2, l-1) + V(j, m, l-1)}{2}$$

for all m such that $0 \leq m \leq l-1$, where l is fixed. In other words,

$$V(j, m-2, l-1) - V(j, m-1, l-1) \leq V(j, m-1, l-1) - V(j, m, l-1).$$

A similar reasoning shows that

$$\begin{aligned} V(j, m-2, l-1) - V(j, m-1, l-1) &\leq V(j, m-1, l-1) - V(j, m, l-1) \\ &\leq V(j, m, l-1) - V(j, m+1, l-1). \end{aligned}$$

This easily implies that $a - b \geq c - d$. Thus if $A_2 = c$, then $A_1 = a$ and the case $A_2 = c$ and $A_1 = b$ does not occur.

Therefore, we proved the concavity of the function $V(j, m, l)$ with respect to $m \in \mathbb{Z}_+$ for all l and j and hence we conclude that $0 \leq m \leq l+1$.

Step 3. We show that if

$$V(j, m+\alpha, l) = \alpha V(j, m+1, l) + (1-\alpha)V(j, m, l)$$

for $0 \leq m \leq l$ with $m \in \mathbb{Z}_+$ and real $0 \leq \alpha \leq 1$, then $V(j, m+\alpha, l)$ is the fair price of a swing option. In other words,

$$\begin{aligned} V(j, m+\alpha, l) = \sup_{\beta \in [0,1]} \big[& \beta(S_j(N-l) - K) + \tilde{p}V(j+1, (m+\alpha-\beta)^+, l-1) \\ & + \tilde{q}V(j, (m+\alpha-\beta)^+, l-1) \big]. \end{aligned}$$

Recall that we are interested in the case of $S_j(N-l) - K < 0$, since otherwise, that is if

$$S_j(N-l) - K \geq 0,$$

the supremum is attained at $\beta = 1$. Therefore we may assume without loss of generality that

$$S_j(N-l) - K < 0.$$

We further assume that $1 \leq m \leq l-1$ at the moment $N-l$. Two other cases $m = 0$ and $m = l$ are similar.

By the inductive assumption, if m is an integer number, then the price of a swing option is given by

$$V(j,m,l) = \max\Big[S_j(N-l) - K + \tilde{p}V(j+1,m-1,l-1) + \tilde{q}V(j,m-1,l-1);\\ \tilde{p}V(j+1,m,l-1) + \tilde{q}V(j,m,l-1)\Big],$$

$$V(j,m+1,l) = \max\Big[S_j(N-l) - K + \tilde{p}V(j+1,m,l-1) + \tilde{q}V(j,m,l-1);\\ \tilde{p}V(j+1,m+1,l-1) + \tilde{q}V(j,m+1,l-1)\Big].$$

The function $V(j,m,l-1)$ is concave with respect to m and this implies that

$$V(j,m,l-1) - V(j,m+1,l-1) \geq V(j,m-1,l-1) - V(j,m,l-1),$$
$$V(j,m,l-1) - V(j,m-1,l-1) \geq V(j,m+1,l-1) - V(j,m,l-1).$$

For the sake of convenience we introduce the following notation:

$$x = S_j(N-l) - K,$$
$$V(j,m,l) = \max\{a;b\},$$
$$V(j,m+1,l) = \max\{c;d\}.$$

If $\max\{a;b\} = a$, then the concavity mentioned above implies that $\max\{c;d\} = c$. This means that if an investor gets a profit when purchasing an amount of the underlying asset at the current moment when the loan is m units, then the investor gets an even higher profit when purchasing a unit of the underlying asset if the loan is larger, that is, if it equals $m+1$ units. Then the assumption of induction yields

$$V(j,m+\alpha,l) = \alpha\, c + (1-\alpha)\, a = \alpha\left(x + \tilde{p}V(j+1,m,l-1) + \tilde{q}V(j,m,l-1)\right)\\ + (1-\alpha)\big(x + \tilde{p}V(j+1,m-1,l-1) + \tilde{q}V(j,m-1,l-1)\big)\\ = x + \tilde{p}V(j+1,m+\alpha-1,l-1) + \tilde{q}V(j,m+\alpha-1,l-1).$$

Put

$$f = x + \tilde{p}V(j+1,m+\alpha-1,l-1) + \tilde{q}V(j,m+\alpha-1,l-1).$$

Next we consider the case of $\max\{c;d\} = d$. The concavity implies that $\max\{a;b\} = b$. This means that if, at the current moment when the loan equals $m+1$ units, an investor will not get a profit when purchasing an amount of the underlying asset, then the same is true if the loan is smaller, that is, if the loan equals m units. By the inductive assumption,

$$V(j,m+\alpha,l) = \alpha d + (1-\alpha)\, b = \alpha\left(\tilde{p}V(j+1,m+1,l-1) + \tilde{q}V(j,m+1,l-1)\right)\\ + (1-\alpha)\big(\tilde{p}V(j+1,m,l-1) + \tilde{q}V(j,m,l-1)\big)\\ = \tilde{p}V(j+1,m+\alpha,l-1) + \tilde{q}V(j,m+\alpha,l-1).$$

Put

$$g = \tilde{p}V(j+1,m+\alpha,l-1) + \tilde{q}V(j,m+\alpha,l-1).$$

The case of $a > b$, $c < d$ is not possible if $c - d \geq a - b$. This can be shown similarly to the proof of the concavity. The financial meaning of this situation can be explained as follows. If the loan equals m units and an investor gets profit from purchasing an amount of the underlying asset, then it is not possible that an investor gets profit if he/she avoids purchasing the underlying asset in case the loan is $m+1$ units at the same moment.

It remains to consider the case where it is profitable for an investor to purchase an amount of the underlying asset at the moment $N-l$ when the loan is equal to $m+1$ but this is not the case if the loan equals m units, that is, if $a < b$, $c > d$. By the assumption of induction,

$$\begin{aligned} V(j, m+\alpha, l) &= \alpha c + (1-\alpha) b \\ &= \alpha\big(x + \tilde{p}V(j+1, m, l-1) + \tilde{q}V(j, m, l-1)\big) \\ &\quad + (1-\alpha)\big(\tilde{p}V(j+1, m, l-1) + \tilde{q}V(j, m, l-1)\big) \\ &= \alpha\, x + \tilde{p}V(j+1, m, l-1) + \tilde{q}V(j, m, l-1). \end{aligned}$$

Put

$$h = \alpha\, x + \tilde{p}V(j+1, m, l-1) + \tilde{q}V(j, m, l-1).$$

Summarizing, if

$$V(j, m+\alpha, l) = \alpha\, V(j, m+1, l) + (1-\alpha)V(j, m, l),$$

then

$$V(j, m+\alpha, l) \in \big\{f, g, h\big\}.$$

Next we show that

$$\begin{aligned} V(j, m+\alpha, l) &= \max\big\{f, g, h\big\} \\ &= \sup_{\beta\in[0,1]} \big[\beta(S_j(N-l) - K) + \tilde{p}V(j+1, (m+\alpha-\beta)^+, l-1) \\ &\qquad\qquad + \tilde{q}V(j, (m+\alpha-\beta)^+, l-1)\big] \end{aligned}$$

for an arbitrary $\alpha \in [0, 1]$. This follows if

$$V(j, m+\alpha, l) \ge \beta(S_j(N-l) - K) + \tilde{p}V(j+1, m+\alpha-\beta, l-1) + \tilde{q}V(j, m+\alpha-\beta, l-1)$$

for all $\beta \in [0, 1]$ and all $\alpha \in [0, 1]$. Considering $\max\big\{f, g, h\big\}$ we treat two cases, namely $f \ge g$ and $g \ge f$.

Case I. Let $f - g \ge 0$, where

$$\begin{aligned} f - g = x &+ \alpha\Big(\tilde{p}\big(V(j+1, m, l-1) - V(j+1, m+1, l-1)\big) \\ &\qquad + \tilde{q}\big(V(j, m, l-1) - V(j, m+1, l-1)\big)\Big) \\ &+ (1-\alpha)\Big(\tilde{p}\big(V(j+1, m-1, l-1) - V(j+1, m, l-1)\big) \\ &\qquad + \tilde{q}\big(V(j, m-1, l-1) - V(j, m, l-1)\big)\Big). \end{aligned}$$

By concavity,

$$x + \tilde{p}\big(V(j+1, m, l-1) - V(j+1, m+1, l-1)\big) + \tilde{q}\big(\ldots\big) \ge f - g \ge 0.$$

It is also clear that

$$h - g = \alpha x + \alpha\big(\tilde{p}(V(j+1, m, l-1) - V(j+1, m+1, l-1)) + \tilde{q}(\ldots)\big) \ge 0.$$

As a result, $\max\big\{f, g, h\big\} = \max\{f, h\}$.

Further we treat the following two cases.

Case I.1. Let $\max\{f, h\} = f$, whence $f \ge h \ge g$. We are going to prove that

$$\begin{aligned} \max\{f, g, h\} &= f \ge s \\ &:= \beta(S_j(N-l) - K) + \tilde{p}V(j+1, m+\alpha-\beta, l-1) \\ &\quad + \tilde{q}V(j, m+\alpha-\beta, l-1) \end{aligned}$$

for all $\beta \in [0, 1]$ and all $\alpha \in [0, 1]$.

Case I.1.1. First we treat the case of $0 \le \alpha \le \beta \le 1$. Then it is easy to see that $m-1 \le m-(\beta-\alpha) \le m$. By the assumption of induction,

$$\begin{aligned} s &= \beta(S_j(N-l)-K) + \tilde{p}V(j+1, m+\alpha-\beta, l-1) + \tilde{q}V(j, m+\alpha-\beta, l-1) \\ &= \beta x + (\beta-\alpha)\tilde{p}V(j+1, m-1, l-1) + \big(1-(\beta-\alpha)\big)\tilde{p}V(j+1, m, l-1) \\ &\quad + (\beta-\alpha)\tilde{q}V(j, m-1, l-1) + \big(1-(\beta-\alpha)\big)\tilde{q}V(j, m, l-1). \end{aligned}$$

Then

$$\begin{aligned} f-s &= (1-\beta)x + (1-\beta)\tilde{p}\big[V(j+1, m-1, l-1) - V(j+1, m, l-1)\big] \\ &\quad + (1-\beta)\tilde{q}\big[V(j, m-1, l-1) - V(j, m, l-1)\big]. \end{aligned}$$

Since

$$\begin{aligned} f-h &= (1-\alpha)x + (1-\alpha)\tilde{p}\big[V(j+1, m-1, l-1) - V(j+1, m, l-1)\big] \\ &\quad + (1-\alpha)\tilde{q}\big[V(j, m-1, l-1) - V(j, m, l-1)\big] \ge 0, \end{aligned}$$

we get $f - s \ge 0$.

Case I.1.2. Now we consider the case of $0 \le \beta \le \alpha \le 1$. Then it is easy to see that $m \le m + (\alpha-\beta) \le m+1$. By the assumption of induction,

$$\begin{aligned} s &= \beta(S_j(N-l)-K) + \tilde{p}V(j+1, m+\alpha-\beta, l-1) + \tilde{q}V(j, m+\alpha-\beta, l-1) \\ &= \beta x + (\alpha-\beta)\tilde{p}V(j+1, m+1, l-1) + (1-\alpha+\beta)\tilde{p}V(j+1, m, l-1) \\ &\quad + (\alpha-\beta)\tilde{q}V(j, m+1, l-1) + (1-\alpha+\beta)\tilde{q}V(j, m, l-1). \end{aligned}$$

Thus

$$\begin{aligned} h-s &= (\alpha-\beta)x + (\alpha-\beta)\tilde{p}\big[V(j+1, m, l-1) - V(j+1, m+1, l-1)\big] \\ &\quad + (\alpha-\beta)\tilde{q}\big[V(j, m, l-1) - V(j, m+1, l-1)\big]. \end{aligned}$$

On the other hand,

$$\begin{aligned} h-g &= \alpha x + \alpha\tilde{p}\big[V(j+1, m, l-1) - V(j+1, m+1, l-1)\big] \\ &\quad + \alpha\tilde{q}\big[V(j, m, l-1) - V(j, m+1, l-1)\big] \\ &\ge 0, \end{aligned}$$

whence $h \ge s$ and $f \ge h \ge s$.

Therefore

$$\begin{aligned} \max\{f, g, h\} = f &\ge s \\ &:= \beta(S_j(N-l)-K) + \tilde{p}V(j+1, m+\alpha-\beta, l-1) \\ &\quad + \tilde{q}V(j, m+\alpha-\beta, l-1) \end{aligned}$$

for all $\beta \in [0,1]$ and all $\alpha \in [0,1]$.

Case I.2. Now let $\max\{f, h\} = h$. Then $h \ge f \ge g$. We are going to prove that

$$\begin{aligned} \max\{f, g, h\} = h &\ge s \\ &:= \beta(S_j(N-l)-K) + \tilde{p}V(j+1, m+\alpha-\beta, l-1) \\ &\quad + \tilde{q}V(j, m+\alpha-\beta, l-1) \end{aligned}$$

for all $\beta \in [0,1]$ and all $\alpha \in [0,1]$.

Case I.2.1. First we check the inequality $h \ge s$ for the case of $0 \le \alpha \le \beta \le 1$. In this case,

$$\begin{aligned} s &= \beta x + (\beta-\alpha)\tilde{p}V(j+1, m-1, l-1) + \big(1-(\beta-\alpha)\big)\tilde{p}V(j+1, m, l-1) \\ &\quad + (\beta-\alpha)\tilde{q}V(j, m-1, l-1) + \big(1-(\beta-\alpha)\big)\tilde{q}V(j, m, l-1). \end{aligned}$$

Since $h \geq f$, we get

$$\begin{aligned} h - f &= -(1-\alpha)x + (1-\alpha)\tilde{p}\big[V(j+1,m,l-1) - V(j+1,m-1,l-1)\big] \\ &\quad + (1-\alpha)\tilde{q}[V(j,m,l-1) - V(j,m-1,l-1)] \\ &\geq 0, \end{aligned}$$

whence

$$\begin{aligned} h - s &= -(\beta-\alpha)x + (\beta-\alpha)\tilde{p}\big[V(j+1,m,l-1) - V(j+1,m-1,l-1)\big] \\ &\quad + (\beta-\alpha)\tilde{q}\big[V(j,m,l-1) - V(j,m-1,l-1)\big] \\ &\geq 0. \end{aligned}$$

Case I.2.2. We have already proved in Case I.1.2 for $0 \leq \beta \leq \alpha \leq 1$ that if $h \geq g$, then $h \geq s$. Therefore

$$\begin{aligned} \max\{f,g,h\} = h \geq s & \\ := \beta(S_j(N-l) - K) &+ \tilde{p}V(j+1, m+\alpha-\beta, l-1) \\ &+ \tilde{q}V(j, m+\alpha-\beta, l-1) \end{aligned}$$

for all $\beta \in [0,1]$ and all $\alpha \in [0,1]$.

Case II. A similar reasoning shows that if $g \geq f$, then $h \geq f$, whence

$$\max\{f,g,h\} = \max\{g,h\}.$$

Then one can prove that

$$\max\{g,h\} \geq \beta(S_j(N-l) - K) + \tilde{p}V(j+1, m+\alpha-\beta, l-1) + \tilde{q}(\ldots)$$

for all $\beta \in [0,1]$ and all $\alpha \in [0,1]$. This can be done by considering separately the cases $g \geq h \geq f$ and $h \geq g \geq f$.

Therefore we have shown that

$$\begin{aligned} V(j, m+\alpha, l) = \sup_{\beta\in[0,1]} & \big[\beta(S_j(N-l) - K) + \tilde{p}V(j+1, (m+\alpha-\beta)^+, l-1) \\ & \qquad + \tilde{q}V(j, (m+\alpha-\beta)^+, l-1)\big] \\ = \max\big\{f,g,h\big\} & \end{aligned}$$

for all $\alpha \in [0,1]$. Moreover we proved that

$$V(j, m+\alpha, l) \in \{f,g,h\}$$

if

$$V(j, m+\alpha, l) = \alpha\, V(j, m+1, l) + (1-\alpha)V(j,m,l)$$

for $0 \leq m \leq l$, $0 \leq \alpha \leq 1$. This means that $\max\big\{f,g,h\big\}$ is represented as follows:

$$V(j, m+\alpha, l) = \max\big\{f,g,h\big\} = \alpha V(j, m+1, l) + (1-\alpha)V(j,m,l). \qquad \square$$

3. Concluding remarks

When pricing a swing option with the help of the lattice method we obtain an optimal bang-bang strategy if the loan is an integer number, namely, an investor should purchase, at every moment, a maximal amount of the underlying asset or keep the loan at the same level if this is possible. If the loan is not integer, the pay-off function for a swing option at each node of the tree has a linear structure depending on the loan.

In further research, one may try to adopt the "bang-bang" regime for the case where a penalty function is present for every purchase of the underlying asset as proposed by Fusai and Roncoroni [9].

Bibliography

1. J. C. Cox, S. Ross, and M. Rubinstein, *Option pricing: A simplified approach*, J. Financial Economics **7** (1979), no. 3, 229–264.
2. M. R. Seydel, *Lattice Approach and Implied Trees*, Handbook of Computational Finance, Second Edition (J.-C. Duan et al., eds.), Springer, New York, 2012, pp. 551–577. MR2908485
3. J. Breslin, L. Clewlow, C. Strickland, and D. van der Zee, *Swing contracts: take it or leave it?*, Energy Risk (2008), no. 2, 64–68.
4. C. Chiarella, L. Clewlow, and B. Kang, *The Evaluation of Gas Swing Contracts with Regime Switching*, Topics in Numerical Methods for Finance (M. Cummins et al., eds), Springer, New York, 2012, pp. 155–176.
5. M. I. M. Wahab and C.-G. Lee, *Pricing swing options with regime switching*, Ann. Oper. Res. **185** (2011), no. 1, 139–160. MR2788791
6. M. I. M. Wahab, Z. Yin, and N. C. Edirsinghe, *Pricing swing options in the electricity markets under regime-switching uncertainty*, Quantitative Finance **10** (2010), no. 9, 975–994. MR2738822 (2012a:91211)
7. N. Bollen, *Valuing options in regime-switching models*, J. Derivatives **6** (1998), no. 1, 38–49.
8. O. Bardou, S. Bouthemy, and G. Pages, *When are swing options bang-bang?*, Internat. J. Theoret. Appl. Finance **13** (2010), no. 7, 867–899. MR2718983 (2012d:91194)
9. G. Fusai and A. Roncoroni, *Implementing Models in Quantitative Finance: Methods and Cases*, Springer-Verlag, Berlin, 2008. MR2386793 (2009g:65003)

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