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THE WEAK CONVERGENCE OF GREEK SYMBOLS FOR PRICES OF EUROPEAN OPTIONS: FROM DISCRETE TIME TO CONTINUOUS

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Abstract. The behavior of the so-called "Greeks" that characterize the financial market and assets on it for the Black–Scholes model is investigated in this paper. Discrete analogues of these quantities are introduced for the binomial model. The weak convergence of these analogues to the Greeks in the Black–Scholes model is established under the condition that the number of periods tends to infinity.

1. Introduction

The weak convergence of the binomial model to the Black–Scholes model for every fixed time moment is proved even in the first papers devoted to the binomial model of financial markets in the scheme of series (in particular, see [3]). Since then there appeared a number of various generalizations and improvements of this convergence (see, for example, [6]). Conditions for the weak convergence of the measures generated by the binomial model as well as by more general prelimit models to the measure generated by the limit diffusion process are obtained in [7]. Several papers deal with bounds of the rate of convergence of option prices if the prices of risky assets converge ([1, 5]). However the binomial analogues of "Greeks" remain insufficiently investigated till today.

The necessity for finding discrete models being close in the asymptotic sense to the Black–Scholes model is dictated by the fact that real financial markets are discrete, while the Black–Scholes model is a convenient and well studied approximation of more realistic discrete models. Therefore the study of the asymptotical behavior of models with discrete time as the number of periods tend to infinity is of a practical significance.

In the current paper, some analogues of the Greek functionals of the European options are studied for the case of discrete time if the model of a financial market is symmetric and binomial.

The paper is organized as follows: the limit Black–Scholes model is discussed in Section 2; the main object of this paper, the Greek delta, is described therein. A prelimit model, the Cox–Ross–Rubinstein symmetric binomial model, is described in Section 3; the main properties of this model are also discussed and the notion of a delta-hedge is introduced in Section 3. The weak convergence of the measures generated by the symmetric binomial model to the measure generated by the limit Black–Scholes process is proved in Section 4. Section 5 contains some auxiliary results needed in the proof of the weak convergence of Greek functionals. The main result of the paper, the weak convergence of a discrete analogue of the Greek delta in the binomial model to the Greek

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delta itself as the number of periods tends to infinity, is stated and proved in Section 6. Concluding remarks are given in Section 7.

2. The limit Black–Scholes model and its components related to strategies and capital

Let $\{\Omega, \mathfrak{F}, \mathbb{P}\}$ be a complete probability space equipped with a filtration $\{\mathfrak{F}_t, t \geq 0\}$ satisfying the standard conditions (see, for example, [2]).

Consider a financial market with two assets, one being risky and the other one being non-risky. Let the evolution of the non-risky asset be given by

$$B(t) = e^{rt},$$

while the price of the risky asset is described by the stochastic process

$$S(t) = S_0 \exp\{\mu t + \sigma W_t\}, \qquad S_0 > 0,$$

where $\{W_t, \mathfrak{F}_t^{\mathbb{P}}, t \geq 0\}$ is a Wiener process with respect to the measure $\mathbb{P}$. Let

$$X(t) = S(t) e^{-rt}$$

be the evolution of the discounted price of the risky asset. It is proved for such a model of a financial market (see, for example, [4]) that the risk-neutral (martingale) measure $\mathbb{P}^*$ exists and is unique. Recall that process $X = \{X(t), t \geq 0\}$ is a martingale with respect to this measure and admits the representation

$$X(t) = S_0 \exp\left\{\sigma W_t^{\mathbb{P}^*} - \frac{1}{2}\sigma^2 t\right\}.$$

Without loss of generality we assume in what follows that $S_0 = 1$ and drop the superscript $\mathbb{P}^*$ in the notation for the Wiener process; we also assume that W is a Wiener process with respect to the measure $\mathbb{P}$.

Let

$$V(S,t) = e^{-r(T-t)} \mathsf{E}\{(S(T) - K)^+ \mid \mathfrak{F}_t\}$$

be the fair price of a European call option with strike price K and exercise time T given the current price S of the risky asset. The Black–Scholes partial differential equation is written as follows ([4]):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad t \in [0, T].$$

The so-called Greek functionals, or just Greeks, correspond to different partial derivatives of the function V. In particular, let $\Delta = \frac{\partial V}{\partial S}$. We consider this representative of the Greek functionals throughout the paper and call it "delta". The delta characterizes the change of the option price or a portfolio of options with the price of the asset S. The delta describes the correlation between the changes of option prices and changes of the prices of the corresponding asset.

The Black–Scholes formula for the option price ([4])

$$V(S(t), t) = S(t)\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

implies the known expression for Δ in terms of a fixed $S(t) = x$, namely

$$\Delta(x, T-t) = \frac{\partial V}{\partial x} = \Phi(d_+(x, T-t)),$$

where $\Phi(x)$ is the standard normal distribution function,

$$d_\pm(x, T-t) = \frac{\ln \frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

The problem is to introduce an analogue of the delta for the Cox–Ross–Rubinstein model of a financial market described below and to study its convergence to $\Delta(x, T-t)$.

3. The symmetric Cox–Ross–Rubinstein model

To describe the prelimit model of a market in the n^{th} series, introduce the probability space

$$\Omega^{(n)} := \{-1,+1\}^n = \left\{\omega^{(n)} = \left(y_1^{(n)}, \dots, y_n^{(n)}\right) \,\Big|\, y_i^{(n)} \in \{-1,+1\}\right\}$$

and two sequences a_n and b_n, $n \geq 1$. For $\omega^{(n)} = \left(y_1^{(n)}, \dots, y_n^{(n)}\right)$, denote by

$$Y_k\left(\omega^{(n)}\right) := y_k^{(n)}$$

the projection of the k^{th} coordinate and let

$$R_k^{(n)}\left(\omega^{(n)}\right) := a_n \frac{1 - Y_k\left(\omega^{(n)}\right)}{2} + b_n \frac{1 + Y_k\left(\omega^{(n)}\right)}{2}.$$

We assume that the prelimit model in the n^{th} series is a Cox–Ross–Rubinstein model with

(1) n periods;
(2) the interest rate
$$r_n = \frac{rT}{n}$$
determining the evolution of the non-risky asset $B_k^{(n)} = (1 + r_n)^k$, $k = 0, \dots, n$;
(3) a single risky asset $S_k^{(n)}$, $k = 0, \dots, n$, and
(4) gains defined by $R_k^{(n)} = \left(S_k^{(n)} - S_{k-1}^{(n)}\right)/S_{k-1}^{(n)}$, $k = 1, \dots, n$. Moreover, the gains are independent and may assume only two values, either a_n or b_n, $k = 1, \dots, n$.

When passing to the next period of this model, the price of the asset jumps from $S_{k-1}^{(n)}$ to a larger value $S_k^{(n)} = S_{k-1}^{(n)}(1+b_n)$ (if $b_n > 0$) or to a smaller value $S_k^{(n)} = S_{k-1}^{(n)}(1+a_n)$ (if $a_n < 0$). This case is considered throughout the paper below.

As far as the limit model is concerned, we assume without loss of generality that the price of the asset at the initial moment is such that $S_0^{(n)} = 1$. Then we introduce the discounted value process for the risky asset

$$X_k^{(n)} := \frac{S_k^{(n)}}{(1 + r_n)^k} = \prod_{i=1}^{k} \frac{1 + R_i^{(n)}}{1 + r_n}.$$

The filtration in the space under consideration is given by

$$\mathfrak{F}_k^{(n)} := \sigma\left(S_1^{(n)}, \dots, S_k^{(n)}\right) = \sigma\left(X_1^{(n)}, \dots, X_k^{(n)}\right), \qquad k = 1, \dots, n.$$

Let $\mathfrak{F}^{(n)} = \mathfrak{F}_n^{(n)}$. The objective probability measure in the space $\left(\Omega^{(n)}, \mathfrak{F}^{(n)}\right)$ is denoted by $\mathbb{P}^{(n)}$. We also assume that the model is symmetric, that is,

$$\hat{a}_n = 1 + a_n = \exp\left\{-\sigma\sqrt{T/n}\right\}, \qquad \hat{b}_n = 1 + b_n = \exp\left\{\sigma\sqrt{T/n}\right\} \tag{1}$$

for some $\sigma > 0$. Since

$$\sqrt{n} r_n \to 0, \qquad \sqrt{n} a_n \to -\sigma\sqrt{T}, \qquad \sqrt{n} b_n \to \sigma\sqrt{T} \quad \text{as } n \to \infty, \tag{2}$$

we conclude that $a_n < r_n < b_n$ starting with some number n_0. Then, for $n \geq n_0$, such a model is arbitrage free, the market is complete, and the unique equivalent martingale measure $\mathbb{P}_n^*$ is characterized by

$$\mathbb{P}_n^*\left\{R_k^{(n)} = b_n\right\} =: p_n^* = \frac{r_n - a_n}{b_n - a_n}$$

(see, for example, [4]). It is easy to conclude from (2) that $\lim_{n\to\infty} p_n^* = 1/2$. We refer to such a model as a symmetric binomial model.

Let C be some contingent claim being measurable with respect to the σ-algebra $\mathfrak{F}_n^{(n)}$. Then there exists a Borel function h such that the discounted contingent claim

$$H = Ce^{-rT}$$

admits the following representation:

$$H = h\left(S_1^{(n)}, \dots, S_n^{(n)}\right).$$

Since the market is complete, the contingent claim C is attainable.

Definition 3.1. A trading strategy $\bar{\xi}^{(n)} = \left(\xi_B^{(n)}, \xi_S^{(n)}\right)$ is called *self-financing* if

$$\bar{\xi}_k^{(n)} \bar{S}_k^{(n)} = \bar{\xi}_{k+1}^{(n)} \bar{S}_k^{(n)}, \qquad k = 1, \dots, n-1.$$

Definition 3.2. A self-financing trading strategy $\bar{\xi}^{(n)} = \left(\xi_B^{(n)}, \xi_S^{(n)}\right)$ is called a *replicating strategy* for an attainable contingent claim C if $C = \bar{\xi}_n^{(n)} \bar{S}_n^{(n)}$ almost surely.

Definition 3.3. The *value process* $V = \left(V_k^{(n)}\right)_{k=0,\dots,n}$ associated to a trading strategy $\bar{\xi}^{(n)}$ is defined by

$$V_0^{(n)} := \bar{\xi}_1^{(n)} \bar{X}_0^{(n)}, \qquad V_k^{(n)} := \bar{\xi}_k^{(n)} \bar{X}_k^{(n)}, \quad k = 1, \dots, n,$$

where $\bar{X}_k^{(n)} = \left(1, X_k^{(n)}\right)$, $k = 0, \dots, n$, are the discounted price processes for the assets.

The following result can be found in [4, p. 250].

Proposition 3.1. *The value process of a replicating strategy for a contingent claim H is equal to*

$$V_i^{(n)} = \mathbb{E}^{\mathbb{P}_n^*}\left[H \,\middle|\, \mathfrak{F}_i^{(n)}\right], \qquad i = 0, \dots, n,$$

and admits the following representation:

$$V_i^{(n)}(\omega) = v_i^{(n)}\left(S_1^{(n)}(\omega), \dots, S_i^{(n)}(\omega)\right),$$

where the function $v_i^{(n)}$ is given by

$$v_i^{(n)}(x_0, \dots, x_i) = \mathbb{E}^{\mathbb{P}_n^*}\left[h\left(x_0, \dots, x_i, x_i S_1^{(n)}, \dots, x_i S_{n-i}^{(n)}\right)\right]. \tag{3}$$

Then $v_i^{(n)}$ is called the *capital function.* Note that the capital function can be defined by using a backward recursion

$$v_n^{(n)}(x_0, \dots, x_n) = h(x_0, \dots, x_n),$$
$$v_i^{(n)}(x_0, \dots, x_i) = p_n^* v_{i+1}^{(n)}(x_0, \dots, x_i, x_i \hat{b}_n) + (1 - p_n^*) v_{i+1}^{(n)}(x_0, \dots, x_i, x_i \hat{a}_n).$$

A replicating strategy for a contingent claim H is called a *delta-hedge* and is denoted by $\Delta_i^{(n)} := \Delta_i^{(n)}\left(S_1^{(n)}(\omega), \dots, S_{i-1}^{(n)}(\omega)\right)$.

Below we obtain the expression for a delta-hedge

$$C^{(n)} = f\left(S_n^{(n)}\right) = \left(S_n^{(n)} - K\right)^+$$

in the case of a European call option in a market with discrete time described above.

Lemma 3.1. *A replicating strategy for a European call option is given by*

$$\Delta_k^{(n)}\left(S_{k-1}^{(n)}\right) = (1+r_n)^k \, \frac{v_k^{(n)}\left(S_{k-1}^{(n)}\hat{b}_n\right) - v_k^{(n)}\left(S_{k-1}^{(n)}\hat{a}_n\right)}{S_{k-1}^{(n)}\big(\hat{b}_n - \hat{a}_n\big)}, \tag{4}$$

where

$$v_k^{(n)}(y) = \sum_{i=0}^{n-k} \frac{\left(\hat{b}_n^i \hat{a}_n^{n-i-k} y - K\right)^+}{(1+r_n)^n} C_{n-k}^i (p_n^*)^i (1-p_n^*)^{n-i-1}. \tag{5}$$

Proof. Equality (4) follows from [4, Proposition 5.45, p. 252] where a general form of a replicating strategy $\Delta_k^{(n)}$ is obtained.

In the case under consideration, $H = h\big(S_n^{(n)}\big)$, $h(x) = (x-K)^+/(1+r_n)^n$. Thus H depends only on the price of an asset at the exercise time. In this case, $V_k^{(n)}$ depends only on the current price of an asset,

$$V_k^{(n)}(\omega) = v_k^{(n)}\left(S_k^{(n)}(\omega)\right).$$

Moreover, the right hand side of relation (3) for the capital function equals the expectation of a function determined by a binomial random variable with parameter p_n^*:

$$v_k^{(n)}(x_k) = \sum_{i=0}^{n-k} h\big(x_k \hat{b}_n^i \hat{a}_n^{n-i-k}\big) C_{n-k}^i (p_n^*)^i (1-p_n^*)^{n-i-k},$$

whence (5) follows. □

In view of the definition given above one can expect that the delta-hedge can be viewed as the desired analogue of the delta. The fraction on the right hand side of (4) is a certain discrete analogue of the capital function and thus there is a reason to expect that one obtains the delta by passing to the limit provided the capital function converges to a limit value.

4. The weak convergence of the symmetric Cox–Ross–Rubinstein model to the Black–Scholes model

Since the delta-hedge and delta of an option are defined for markets with different structures of times, denote by k_t^n the integer number such that $\frac{k_t^n T}{n} \le t < \frac{(k_t^n+1)T}{n}$ and $0 \le k_t^n \le n$. Hence $k_t^n = [\frac{nt}{T}]$. Consider the following sequence of stochastic processes:

$$S_n(t) = S_{k_t^n}^{(n)}.$$

Denote by $\xrightarrow{\mathrm{d}}$ the weak convergence in distribution, by $\xrightarrow{\mathsf{P}}$ the convergence in probability, and by $\xrightarrow{\mathrm{W}}$ the weak convergence of the measures generated by stochastic processes. Note that Theorem 5.53 and Corollary 5.55 of [4] imply the weak convergence of one dimensional distributions, that is, $S_{k_t^n}^{(n)} \xrightarrow{\mathrm{d}} S(t)$ for every $t \in [0,T]$. We apply Theorem 3.3 of [7] to prove the weak convergence of the measures $\mathbb{Q}_n$ generated by the stochastic processes S_n to the measure $\mathbb{Q}$ generated by the stochastic process S.

Below we recall the statement of Theorem 3.3 in [7].

Theorem 4.1. *Let* $B_t^{(n)} = (1+rT/n)^{k_t^n}$ *and* $S_t^{(n)} = \prod_{k=1}^{k_t^n}\big(1+R_k^{(n)}\big)$. *Assume that conditions* (i)–(iv) *hold*:

(i) $\sup_{1\le k\le n}\big|R_k^{(n)}\big| \xrightarrow{\mathsf{P}} 0$ *as* $n\to\infty$;

(ii) *for all* $a \in (0,1]$,

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}_n^* \left(\sum_{1\le k\le n} \left| \mathbb{E}_n^* \left(R_k^{(n)} 1_{|R_k^{(n)}|\le a} \mid \mathcal{F}_{k-1}^{(n)} \right) \right| \ge C \right)$$
$$= \lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}_n^* \left(\sum_{1\le k\le n} \mathbb{E}_n^* \left(\left(R_k^{(n)} \right)^2 1_{\left|R_k^{(n)}\right|\le a} \mid \mathcal{F}_{k-1}^{(n)} \right) \ge C \right) = 0;$$

(iii) *for all* $\varepsilon > 0$ *and* $a \in (0,1]$,

$$\lim_n \mathbb{P}_n^* \left(\sup_{t\in[0,T]} \left| \sum_{1\le k\le [\frac{nt}{T}]} \mathbb{E}_n^* \left(R_k^{(n)} 1_{\left|R_k^{(n)}\right|\le a} \mid \mathcal{F}_{k-1}^{(n)} \right) - rt \right| \ge \varepsilon \right) = 0;$$

(iv) *for all* $\varepsilon > 0$ *and* $a \in (0,1]$,

$$\lim_n \mathbb{P}_n^* \left(\sup_{t\in[0,T]} \left| \sum_{1\le k\le [\frac{nt}{T}]} \mathbb{E}_n^* \left((R_k^{(n)})^2 1_{\left|R_k^{(n)}\right|\le a} \mid \mathcal{F}_{k-1}^{(n)} \right) - \sigma^2 t \right| \ge \varepsilon \right) = 0.$$

Then

$$\mathbb{Q}_n \overset{\mathrm{W}}{\to} \mathbb{Q}.$$

Put $R := r - \frac{\sigma^2}{2}$. The following result is proved below with the help of Theorem 4.1.

Theorem 4.2. *The symmetric binomial model satisfies conditions* (i)–(iv) *of Theorem* 4.1. *Thus*

$$\mathbb{Q}_n \overset{\mathrm{W}}{\to} \mathbb{Q}.$$

Proof. Recall that $R_k^{(n)} = b_n = \exp\{\sigma\sqrt{T/n}\} - 1 = \sigma\sqrt{T/n} + \sigma^2 T/(2n) + o(1/n)$ or $R_k^{(n)} = a_n = \exp\{-\sigma\sqrt{T/n}\} - 1 = -\sigma\sqrt{T/n} + \sigma^2 T/(2n) + o(1/n)$ with probabilities

$$\mathbb{P}_n^* \left\{ R_k^{(n)} = b_n \right\} = p_n^* = \frac{r_n - a_n}{b_n - a_n} = \frac{\sigma\sqrt{T/n} + \frac{RT}{n} + o\left(\frac{1}{n}\right)}{2\sigma\sqrt{T/n} + o\left(\frac{1}{\sqrt{n}}\right)}$$
$$(6) \qquad = \frac{1}{2} + \frac{R\sqrt{T}}{2\sigma\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

or

$$1 - p_n^* = \frac{\sigma\sqrt{T/n} - \frac{RT}{n} + o\left(\frac{1}{n}\right)}{2\sigma\sqrt{T/n} + o\left(\frac{1}{\sqrt{n}}\right)} = \frac{1}{2} - \frac{R\sqrt{T}}{2\sigma\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

respectively.

Condition (i) is obvious.

Now, for an arbitrary $a \in (0,1]$ and starting with a certain number n, we have

$$\left| \mathbb{E}_n^* \left(R_k^{(n)} 1_{\left|R_k^{(n)}\right|\le a} \mid \mathcal{F}_{k-1}^{(n)} \right) \right| = \left| \mathbb{E}_n^* \left(R_k^{(n)} \mid \mathcal{F}_{k-1}^{(n)} \right) \right| = \left| \mathbb{E}_n^* \left(R_k^{(n)} \right) \right| = \frac{rT}{n} + o\left(\frac{1}{n}\right),$$
$$\mathbb{E}_n^* \left(\left(R_k^{(n)} \right)^2 1_{\left|R_k^{(n)}\right|\le a} \,\middle|\, \mathcal{F}_{k-1}^{(n)} \right) = \mathbb{E}_n^* \left(\left(R_k^{(n)} \right)^2 \,\middle|\, \mathcal{F}_{k-1}^{(n)} \right)$$
$$= \mathbb{E}_n^* \left(R_k^{(n)} \right)^2 = \frac{\sigma^2 T}{n} + o\left(\frac{1}{n}\right),$$

whence condition (ii) follows. The same reasoning shows that conditions (iii) and (iv) hold and this completes the proof of the theorem. □

Remark 4.1. Now we are in position to apply the Skorokhod representation method of constructing stochastic processes on a common probability space and define all stochastic processes S and S_n in a common probability space in such a way that all finite dimensional distributions remain the same. At the same time, the almost sure convergence of the prelimit processes to the limit process is valid at every point.

5. Auxiliary results

We start with a result that allows one to find an alternative representation for $\Delta_k^{(n)}$ being more convenient than (4) when establishing the convergence. According to Remark 4.1, all price processes are defined on a common probability space. This allows one to drop the random variable $S_{k-1}^{(n)}$ in the definition of $\Delta_k^{(n)}\big(S_{k-1}^{(n)}\big)$ and to study the convergence of $\Delta_k^{(n)}(x_n)$ under the assumption that the sequence x_n converges to some fixed $x > 0$. First we consider $\Delta_k^{(n)}(x)$ at every fixed $x > 0$.

Theorem 5.1. *For every $x > 0$, there exists a number $n = n(x)$ such that one can find integer numbers $m_a = m_a(x) \geq n$ and $m_b = m_b(x) \geq n$ with $0 \leq m_a(x) \leq m_b(x) \leq n - k_t^n$ and $0 < p_n^1 < 1$ for which the delta-hedge for a European call option admits the following representation:*

$$\Delta_{k_t^n}^{(n)}(x) = P\left(n - k_t^n, m_a(x), p_n^1\right) + M_{k_t^n}^{(n)}(x, p_n^*)1_{\{m_a(x)\neq m_b(x)\}}, \tag{7}$$

where $p_n^1 = \frac{\hat b_n}{(1+r_n)}p_n^*$, $P(L, l, p)$ is the probability that at least l successes occur in L Bernoulli trials with parameter p, and

$$M_k^{(n)}(x, p) = \frac{\left(x\hat b_n^{m_b+1}\hat a_n^{n-m_b-k} - K\right)}{(1 + r_n)^{n-k}} C_{n-k}^{m_b} p^{m_b}(1 - p)^{n-m_b-k}. \tag{8}$$

Proof. It is easy to see that (5) implies that the non-zero terms denoted by $v_k^{(n)}(x\hat a_n)$ and $v_k^{(n)}(x\hat b_n)$ in the sums involved in definition (4) are written successfully and occur starting with some integer numbers $m_a(x)$ and $m_b(x)$, respectively. We are going to determine these numbers. First, note that the greatest term corresponding to the index $i = n - k_t^n$ in the sum is of the order

$$x \exp\{\sigma\sqrt{nT}(1 - t/T)\} \to \infty \quad \text{as } n \to \infty.$$

Thus there exists a number $n(x)$ such that the set of non-zero terms in the sums is non-empty if $n \geq n(x)$. Further, a term is involved in the first sum if

$$\left(x\hat b_n^{m_b+1}\hat a_n^{n-k-m_b} - K\right)^+ > 0.$$

Rewrite this condition in the following (equivalent) form:

$$x\hat b_n^{m_b+1}\hat a_n^{n-k-m_b} > K$$

and transform it for $k = k_t^n$:

$$(m_b + 1)\ln \hat b_n + (n - k_t^n - m_b)\ln \hat a_n + \ln x > \ln K$$

or

$$m_b \ln\big(\hat b_n/\hat a_n\big) > \ln K - (n - k_t^n)\ln \hat a_n - \ln x - \ln \hat b_n.$$

Note that $\ln \hat b_n = \sigma\sqrt{T}/\sqrt{n}$, $\ln \hat a_n = -\sigma\sqrt{T}/\sqrt{n}$, and $\ln \hat b_n/\hat a_n = 2\sigma\sqrt{T}/\sqrt{n}$, whence

$$m_b > \frac{\ln(K/x)}{2\sigma\sqrt{\frac{T}{n}}} + \frac{1}{2}(n - k_t^n - 1).$$

Therefore one can put

$$m_b = 0 \vee \left(\left[\frac{\ln(K/x)}{2\sigma\sqrt{\frac{T}{n}}} + \frac{1}{2}(n - k_t^n - 1) \right] + 1 \right) \tag{9}$$

for the first sum and similarly

$$m_a = 0 \vee \left(\left[\frac{\ln(K/x)}{2\sigma\sqrt{\frac{T}{n}}} + \frac{1}{2}(n - k_t^n + 1) \right] + 1 \right)$$

for the second sum.

It is obvious that, for $n \geq n(x)$, only one of the following two cases may occur, either $m_a > m_b$ or $m_a = m_b = 0$. In any case, $\Delta_{k_t^n}^{(n)}(x)$ contains non-zero terms of both sums for i ranging from m_a to $n - k_t^n$.

Consider the case of $m_a > m_b$. The terms of the first sum are non-zero for i ranging from m_b to m_a. It turns out that there exists only one value of i with this property and hence it equals m_b. Indeed,

$$m_a - m_b = \left[\frac{\ln(K/x)}{2\sigma\sqrt{\frac{T}{n}}} + \frac{1}{2}(n - k_t^n + 1) \right] - \left[\frac{\ln(K/x)}{2\sigma\sqrt{\frac{T}{n}}} + \frac{1}{2}(n - k_t^n - 1) \right] = 1.$$

Thus

$$\begin{aligned}
\Delta_{k_t^n}^{(n)}(x) &= \frac{\sum_{i=m_a}^{n-k_t^n} C_{n-k_t^n}^i (p_n^*)^i (1-p_n^*)^{n-i-k_t^n} x \hat{a}_n^{n-k_t^n-i} \hat{b}_n^i \left(\hat{b}_n - \hat{a}_n\right)}{(1+r_n)^n x \left(\hat{b}_n - \hat{a}_n\right)} (1+r_n)^{k_t^n} \\
&\quad + M_{k_t^n}^{(n)} 1_{\{m_a \neq m_b\}} \\
&= \sum_{i=m_a}^{n-k_t^n} \frac{C_{n-k_t^n}^i (p_n^*)^i (1-p_n^*)^{n-i-k_t^n} \hat{a}_n^{n-i-k_t^n} \hat{b}_n^i}{(1+r_n)^{n-k_t^n}} + M_{k_t^n}^{(n)} 1_{\{m_a \neq m_b\}} \\
&= \sum_{i=m_a}^{n-k_t^n} C_{n-k_t^n}^i \left(\frac{\hat{b}_n}{(1+r_n)} p_n^* \right)^i \left(\frac{\hat{a}_n}{(1+r_n)} (1-p_n^*) \right)^{n-k_t^n-i} \\
&\quad + M_{k_t^n}^{(n)} 1_{\{m_a \neq m_b\}} \\
&= \sum_{i=m_a}^{n-k_t^n} C_{n-k_t^n}^i \left(p_n^1\right)^i \left(1-p_n^1\right)^{n-k_t^n-i} + M_{k_t^n}^{(n)} 1_{\{m_a \neq m_b\}} \\
&= P\left(n-k_t^n, m_a(x), p_n^1\right) + M_{k_t^n}^{(n)} 1_{\{m_a \neq m_b\}},
\end{aligned} \tag{10}$$

where $p_n^1 = \frac{\hat{b}_n}{(1+r_n)} p_n^*$ and $1 - p_n^1 = \frac{\hat{a}_n}{(1+r_n)} (1-p_n^*)$. This is true, since

$$p_n^* = \frac{r_n - a_n}{b_n - a_n} = \frac{r_n - \hat{a}_n + 1}{\hat{b}_n - \hat{a}_n}, \quad \text{whence} \quad \frac{\hat{b}_n}{(1+r_n)} p_n^* + \frac{\hat{a}_n}{(1+r_n)} (1-p_n^*) = 1. \qquad \square$$

The following analogue of the Esseen inequality is needed for the proof of the main result of this paper (see [8, p. 111]).

Theorem 5.2. *Let Y_j^n, $j = 1, \ldots, n$, be independent random variables. Assume that* $\mathsf{E}\, Y_j^n = 0$, $\mathsf{E}\left(Y_j^n\right)^2 = \sigma_j{}^2 > 0$, *and* $\mathsf{E}\left|Y_j^n\right|^3 < \infty$, $j = 1, \ldots, n$. *Put* $B_n = \sum_{j=1}^n \sigma_j^2$, $F_n(x) = \mathsf{P}\left(B_n^{-1/2} \sum_{j=1}^n Y_j^n < x\right)$, *and* $L_n = B_n^{-3/2} \sum_{j=1}^n \mathsf{E}\left|Y_j^n\right|^3$.

Then

$$\sup_x |F_n(x) - \Phi(x)| \leq AL_n, \tag{11}$$

where A is a positive constant and $\Phi(x)$ is the standard normal distribution function.

6. The convergence of $\Delta_k^{(n)}$ to $\Delta(x, T-t)$

Theorem 5.1 allows us to consider separately the convergence of components of the discrete Greek functional $\Delta_k^{(n)}$. Throughout below we assume that $0 < t < T$, since the results for $t = 0$ and $t = T$ are analogous and proofs are simpler.

First we prove the following result.

Theorem 6.1. *The convergence $M_{k_t^n}^{(n)}(x, p_n^*) \to 0$ as $n \to \infty$ is uniform with respect to $x > 0$ belonging to an arbitrary bounded set.*

Proof. Note that the inequality $m_a > m_b$ implies that $m_a > 0$ and thus the maximum in the definition of m_a can be dropped by the reasoning used in the proof of Theorem 5.1.

Next,

$$m_a > m_b \geq \frac{\ln(K/x)}{2\sigma\sqrt{\frac{T}{n}}} \to \infty$$

and $m_b = m_a - 1 \to \infty$ as $n \to \infty$. Moreover,

$$n - k_t^n - m_b \approx \frac{\ln(K/x)}{2\sigma\sqrt{\frac{T}{n}}} + \frac{1}{2}n\left(1 - \frac{t}{T}\right) \to \infty$$

as $n \to \infty$.

To apply the local de Moivre–Laplace theorem (see [9, p. 67]) we only need to prove that the sequence

$$\alpha(m_b, n) := \frac{m_b - (n - k_t^n)p_n^*}{\sqrt{(n - k_t^n)p_n^*(1 - p_n^*)}}$$

is uniformly bounded with respect to n. We use relations (6) and (9) and rewrite the latter sequence as follows:

$$\alpha(m_b, n) = \frac{\frac{\sqrt{n}\ln(K/x)}{2\sigma\sqrt{T}} - (n - k_t^n)\frac{R\sqrt{T}}{2\sigma\sqrt{n}} + O(1)}{\sqrt{n\left(1 - \frac{t}{T}\right)\left(\frac{1}{4} - \frac{R^2T}{4\sigma^2 n}\right)} + O(1)} \to \frac{\ln(K/x) - (T-t)R}{\sqrt{T-t}}. \tag{12}$$

Thus the sequence $\alpha(m_b, n)$, $n \geq 1$, is bounded for every $x > 0$. Therefore all the conditions for the local de Moivre–Laplace theorem are valid and hence, for sufficiently large n,

$$M_{k_t^n}^{(n)}(x, p_n^*) = \frac{\left(x\hat{b}_n^{m_b+1}\hat{a}_n^{n-m_b-k_t^n} - K\right)}{(1 + r_n)^{n-k_t^n}} \frac{e^{-\varphi(n)}}{\sqrt{n - k_t^n}} + o\left(\frac{1}{\sqrt{n}}\right), \tag{13}$$

where $\varphi(n) \geq 0$, $n \geq 1$, is some function (the precise expression for this function does not matter for our purposes; the procedure for obtaining this function is described in [9]).

Let $x \leq C$. We rewrite the first fraction in representation (13) as follows:

$$0 \leq \frac{\left(x\hat{b}_n^{m_b+1}\hat{a}_n^{n-m_b-k_t^n} - K\right)}{(1 + r_n)^{n-k_t^n}} \leq C \exp\left\{\sigma\sqrt{\frac{T}{n}}(2m_b + 1 - (n - k_t^n))\right\} \leq K + o(1).$$

Thus $M_{k_t^n}^{(n)}(x, p_n^*) \to 0$ as $n \to \infty$ uniformly with respect to any bounded set of x. □

Now we are ready to state and prove the main result of this paper.

Theorem 6.2. *Consider a prelimit symmetric binomial model. Let the limit model be the Black–Scholes model. Then the delta-hedge for a European call option in the discrete time weakly converges to the delta functional of a European call option in the continuous time as the number of periods in the discrete time model tends to infinity, that is,*

$$\Delta_k^{(n)}\left(S_{k_t^n}^{(n)}\right) \xrightarrow{\mathrm{d}} \Delta(S(t), T-t), \qquad n \to \infty.$$

Proof. It is clear from what we discussed above that one only needs to check that

$$\Delta_k^{(n)}(x_n) \to \Delta(x, T-t), \qquad n \to \infty,$$

under the condition that $x_n \to x > 0$.

We apply representation (7) of Theorem 5.1. By Theorem 6.1, $M_{k_t^n}^{(n)}(x_n, p_n^*) \to 0$ as $n \to \infty$ in this representation. Now the convergence of $\Delta_k^{(n)}(x_n)$ to $\Delta(x, T-t)$ follows from the convergence of $P(n-k_t^n, m_a(x_n), p_n^1)$ to $\Delta(x, T-t)$. Here $P(n-k_t^n, m_a(x_n), p_n^1)$ is defined in (10), where x_n substitutes for x. For the sake of a simpler notation we write m_a instead of $m_a(x_n)$.

Consider the probability that at least m_a successes occur in a sequence of n Bernoulli trials with parameters $n-k$ and p_n^1, that is,

$$P(n-k, m_a, p_n^1) = \sum_{i=m_a}^{n-k} C_{n-k}^i \left(p_n^1\right)^i \left(1-p_n^1\right)^{n-i-k}.$$

In other words,

$$\sum_{i=m_a}^{n-k} C_{n-k}^i \left(p_n^1\right)^i \left(1-p_n^1\right)^{n-i-k} = 1 - \mathsf{P}\left(\nu_n < m_a\right),$$

where

$$\nu_n = \sum_{i=1}^{n-k} X_i^n, \qquad X_i^n = \begin{cases} 0 & \text{with probability } q_n^1, \\ 1 & \text{with probability } p_n^1, \end{cases} \qquad q_n^1 := 1 - p_n^1.$$

Thus it remains to note that

$$\mathsf{P}(\nu_n < m_a) \to 1 - \Phi(d_+(x, T-t)) = \Phi\left(\frac{\ln(K/x) - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right), \qquad n \to \infty. \tag{14}$$

The next step of the proof is to apply Theorem 5.2 to the sum under consideration. Since $\mathsf{E}\, X_j^n = p_n^1$, we pass to the centered terms, that is, put $Y_j^n = X_j^n - p_n^1$, $j = 1, \dots, n$. Then

$$Y_j^n = \begin{cases} -p_n^1 & \text{with probability } q_n^1, \\ 1 - p_n^1 & \text{with probability } p_n^1, \end{cases} \qquad \mathsf{E}\, Y_j^n = 0, \ i = 1, \dots, n.$$

Now we evaluate the second and third moments:

$$\mathsf{E}\left(Y_j^n\right)^2 = p_n^1\left(1-p_n^1\right)^2 + \left(p_n^1\right)^2 q_n^1 = \left(1-p_n^1\right)\left(p_n^1\left(1-p_n^1\right) + \left(p_n^1\right)^2\right) = q_n^1 p_n^1,$$

$$\begin{aligned} \mathsf{E}\left|Y_j^n\right|^3 &= \left(1-p_n^1\right)^3 p_n^1 + \left(p_n^1\right)^3\left(1-p_n^1\right) \\ &= p_n^1 - 3\left(p_n^1\right)^2 + 3\left(p_n^1\right)^3 - \left(p_n^1\right)^4 + \left(p_n^1\right)^3 - \left(p_n^1\right)^4 \\ &= p_n^1\left(2\left(p_n^1\right)^3 - 4\left(p_n^1\right)^2 + 3p_n^1 - 1\right). \end{aligned}$$

It is clear that $\mathsf{E}\left|Y_j^n\right|^3 < \infty$ and

$$B_n = \sum_{j=1}^{n-k} p_n^1 q_n^1 = (n-k) p_n^1 q_n^1.$$

Therefore all the assumptions of Theorem 5.2 hold and inequality (11) can be applied to the sum under consideration. Consider L_n and take into account that $k = k_t^n$:

$$\begin{aligned} L_n &= B_n^{-1/2} \sum_{j=1}^{n-k} \mathsf{E}\left|Y_j^n\right|^3 = \frac{\sum_{j=1}^{n-k} p_n^1 \left(2\left(p_n^1\right)^3 - 4\left(p_n^1\right)^2 + 3p_n^1 - 1\right)}{\sqrt{\left((n-k)p_n^1 q_n^1\right)^3}} \\ &= -\frac{(n-k)p_n^1 \left(2\left(p_n^1\right)^3 - 4\left(p_n^1\right)^2 + 3p_n^1 - 1\right)}{\sqrt{\left((n-k)p_n^1 q_n^1\right)^3}} = -\frac{2\left(p_n^1\right)^3 - 4\left(p_n^1\right)^2 + 3p_n^1 - 1}{\sqrt{(n-k)p_n^1 \left(q_n^1\right)^3}} \\ &= \frac{\left(1 - p_n^1\right)\left(2\left(p_n^1\right)^2 - 2p_n^1 + 1\right)}{\sqrt{(n-k)p_n^1 \left(q_n^1\right)^3}} = \frac{2\left(p_n^1\right)^2 - 2p_n^1 + 1}{\sqrt{(n-k)p_n^1 q_n^1}} \to 0, \end{aligned}$$

that is, $L_n \to 0$ as $n \to \infty$, whence we conclude that

$$\sup_{y \in \mathbb{R}} \left|F_n(y) - \Phi(y)\right| \to 0, \qquad n \to \infty. \tag{15}$$

Recall that

$$F_n(y) = \mathsf{P}\left(B_n^{-1/2} \sum_{j=1}^{n-k} Y_j^n < y\right) = \mathsf{P}\left(\frac{\sum_{j=1}^{n-k} X_j^n - (n-k)p_n^1}{\sqrt{(n-k)p_n^1 q_n^1}} < y\right)$$

by the definition of Y_j^n. It is obvious that $F_n(\alpha_{m_a}) = \mathsf{P}(\nu_n < m_a)$, where

$$\alpha_{m_a} := \frac{m_a - (n - k_t^n)p_n^1}{\sqrt{(n - k_t^n)p_n^1 q_n^1}}.$$

Finally, it remains to prove that

$$F_n(\alpha_{m_a}) \to \Phi\left(\frac{\ln(K/x) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

In order to prove the latter relation, we transform α_{m_a} to a more convenient form. First we use representation (6) and transform the probabilities as follows:

$$p_n^1 = \frac{e^{\sigma\sqrt{\frac{T}{n}}}}{1 + \frac{rT}{n}} p_n^* = \frac{1}{2}\left(1 + \left(\sigma + \frac{R}{\sigma}\right)\sqrt{\frac{T}{n}}\right) + o\left(\frac{1}{\sqrt{n}}\right),$$

whence a similar asymptotic expansion for q_n^1 follows. As in the case of (12), we get

$$\alpha_{m_a} = \frac{m_a(x_n) - (n - k_t^n)p_n^1}{\sqrt{(n - k_t^n)p_n^1 q_n^1}} = \frac{\ln(K/x_n) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} + o\left(\frac{1}{\sqrt{n}}\right). \tag{16}$$

Now we put

$$d(x) = \frac{\ln(K/x) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and estimate

$$\left|F_n\left(d(x_n)+o\left(\frac{1}{\sqrt{n}}\right)\right)-\Phi(d(x))\right|$$
$$\leq \sup_{y\in\mathbb{R}}\left|F_n\left(y\right)-\Phi\left(y\right)\right|+\left|\Phi(d(x))-\Phi\left(d(x_n)+o\left(\frac{1}{\sqrt{n}}\right)\right)\right|\to 0$$

as $n\to\infty$ which completes the proof of the theorem. □

7. Concluding remarks

An analogue of the Greek functional delta for a price of a European call option considered in the Black–Scholes model is introduced for the Cox–Ross-Rubinstein symmetric binomial model. It is proved that the so-called delta-hedge in the binomial model is an analogue of the delta functional for the Black–Scholes model of a financial market. By using the Esseen inequality, it is proved that the sequence of delta-hedges converges to the delta functional in the limit model as the number of periods tends to infinity.

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