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ASYMPTOTIC PROPERTIES OF INTEGRAL FUNCTIONALS OF FRACTIONAL BROWNIAN FIELDS

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Abstract. Two theorems describing the asymptotic behavior of integral functionals of multidimensional self-similar random fields are proved. For a d-dimensional fractional Brownian field depending on N parameters, a theorem on the convergence of the integral mean-type functional is established. The weak convergence of an integral functional of a d-dimensional anisotropic self-similar random field with N parameters to the local time is proved under the assumption that the continuous local time exists for this field.

1. Introduction

Let $\{\Omega, \mathcal{F}, \mathsf{P}\}$ be a probability space where all random objects considered below are defined. Denote $\mathbb{R}_+^k = [0, +\infty)^k$, $k \geq 1$, and let $\mathbf{x} \cdot \mathbf{y}$ be the vector constituted from the products of the corresponding coordinates, that is,

$$\mathbf{x} \cdot \mathbf{y} = (x_1 y_1, \ldots, x_d y_d),$$

where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

Definition 1.1. A real valued random field $\{X^{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\}$ is called *self-similar* with index $\mathbf{H} = (H_1, \ldots, H_N) \in (0, +\infty)^N$ if

$$\left\{X(\mathbf{a} \cdot \mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\right\} \stackrel{d}{=} \left\{a_1^{H_1} \ldots a_N^{H_N} X(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\right\}$$

for an arbitrary $\mathbf{a} = (a_1, \ldots, a_N) \in (0, +\infty)^N$, where the symbol $\stackrel{d}{=}$ means the equality of finite dimensional distributions.

Definition 1.2. The random field $\{X^{\mathbf{H},d}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\}$ with values in $\mathbb{R}^d$ is defined by

$$X^{\mathbf{H},d} = \left(X_1^{\mathbf{H}}, \ldots, X_d^{\mathbf{H}}\right),$$

where $X_1^{\mathbf{H}}, \ldots, X_d^{\mathbf{H}}$ are independent copies of a real valued random field

$$\left\{X_0^{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\right\}.$$

If $\{X_0^{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\}$ is a self-similar field with index $\mathbf{H} = (H_1, \ldots, H_N) \in (0, +\infty)^N$, then we say that $\{X^{\mathbf{H},d}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\}$ is a *d-dimensional self-similar field* with index $\mathbf{H}$.

Remark 1.1. In the case of $N = 1$, we write H instead of $\mathbf{H}$. In this case, real valued and d-dimensional self-similar random fields are called *processes* and denoted by X^H and $X^{H,d}$, respectively.

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Studies of self-similar random fields (in other words, multiparameter random processes) are explained by the evidence of the property of self-similarity in the data from climatology, nature science and other fields (see [8, 10]). In particular, the so-called anisotropic random fields are used to model phenomena in the environmental statistics, statistical hydrology and image processing (see [2, 3, 4]). Strong limit theorems for self-similar random fields are proved in [9] and more references are given therein.

Definition 1.3. A real valued centered Gaussian random field $B^{\mathbf{H}} = \{B^{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\}$ with covariance function

$$\mathbf{E}\left(B^{\mathbf{H}}(\mathbf{t})B^{\mathbf{H}}(\mathbf{s})\right) = 2^{-N}\prod_{i=1}^{N}\left(|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}\right), \qquad \mathbf{t},\mathbf{s} \in \mathbb{R}_+^N,$$

is called an *anisotropic fractional Brownian field* with Hurst index $\mathbf{H} = (H_1, \dots, H_N) \in (0,1)^N$.

Definition 1.4. Let $\mathbf{H} = (H_1, \dots, H_N) \in (0,1)^N$. A random field

$$B^{\mathbf{H},d} = \left\{B^{\mathbf{H},\mathbf{d}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\right\}$$

defined by

$$B^{\mathbf{H},d} = \left(B_1^{\mathbf{H}}, \dots, B_d^{\mathbf{H}}\right),$$

where $B_1^{\mathbf{H}}, \dots, B_d^{\mathbf{H}}$ are independent copies of a real valued anisotropic fractional Brownian field $\{B_0^{\mathbf{H}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\}$ with Hurst index $\mathbf{H}$, is called a *d-dimensional anisotropic fractional Brownian field* with Hurst index $\mathbf{H}$.

Remark 1.2. A d-dimensional anisotropic fractional Brownian field is self-similar and its Hurst index is the index of the self-similarity. A real valued fractional Brownian motion $\{B^H(t), t \in \mathbb{R}_+\}$, $H \in (0,1)$, and a d-dimensional fractional Brownian motion $\{B^{H,d}(t), t \in \mathbb{R}_+\}$, $H \in (0,1)$, are, in fact, random fields $B^{\mathbf{H}}$ and $B^{\mathbf{H},d}$ if $N = 1$. If $H = 1/2$, then $B^{1/2,d}$ is a Brownian motion.

The main aim of this paper is to study the asymptotic behavior of integral functionals of a fractional Brownian field. The paper is motivated by corresponding results for a multidimensional Brownian motion and fractional Brownian motion. In particular, the Kallianpur–Robbins law [6] describes the long term asymptotic evolution of the distribution of the occupation measure for a two dimensional Brownian motion. Below is their corresponding result.

Theorem 1.1 ([6])**.** *Let $\{W^{(2)}(t), t \in \mathbb{R}_+\}$ be a two dimensional Brownian motion. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded integrable function such that $\overline{f} := \int_{\mathbb{R}^2} f(\mathbf{x})\,d\mathbf{x} \neq 0$. Then*

$$\lim_{T\to+\infty} \mathsf{P}\left[\frac{2\pi}{\overline{f}\log T}\int_0^T f\left(W^{(2)}(t)\right)\,dt \geq x\right] = e^{-x}, \qquad x > 0.$$

Kôno [7] generalizes this result and proves the following theorem for a d-dimensional fractional Brownian motion $\{B^{H,d}(t), t \in \mathbb{R}_+\}$.

Theorem 1.2 ([7])**.** *Let $g\colon \mathbb{R}^d \to \mathbb{R}_+$ be a bounded integrable function such that*

$$\overline{g} := \int_{\mathbb{R}^d} g(\mathbf{x})\,d\mathbf{x} \neq 0$$

and let $Hd = 1$, $d \geq 2$. Then

$$\lim_{T\to+\infty} \mathsf{P}\left[\frac{(2\pi)^{d/2}}{\overline{g}\log T}\int_0^T g\left(B^{H,d}(t)\right)\,dt \geq x\right] = e^{-x}, \qquad x > 0.$$

A partial generalization of the Kallianpur–Robbins law is obtained in this paper. More precisely, we find a normalization for the convergence of a mean-type integral functional for a d-dimensional fractional Brownian field that depends on N parameters.

The weak convergence of a normalized integral functional of a d-dimensional fractional Brownian field to the local time is proved in [11]. In this paper, we establish the convergence of a normalized integral functional of a d-dimensional anisotropic self-similar field that depends on N parameters to the local time under the assumption that its continuous local time exists. Note that the paper [12] proves that the local time for a d-dimensional fractional Brownian field exists and is continuous with respect to all its arguments for $d < 1/H_i$, $i = 1, \dots, N$.

The paper is organized as follows. A theorem on the convergence of a mean-type integral functional of a d-dimensional fractional Brownian field that depends on N parameters is proved in Section 2. The proof of the main result uses an auxiliary result describing the asymptotic behavior of integrals depending on parameters. In Section 3, the notion of the local time for random fields is discussed and a limit theorem for integral functionals of self-similar fields is proved. Concluding remarks are given in Section 4.

2. Theorem on the convergence of a mean type integral functional of a d-dimensional fractional Brownian field that depends on N parameters

In what follows we use the notation $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^N$ and

$$[\mathbf{a}, \mathbf{b}] := [a_1, b_1] \times \cdots \times [a_N, b_N],$$

where $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$ and $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{R}^N$. The following auxiliary result is needed for the proof of the main theorem.

Lemma 2.1. *Let $H \in (0, 1)$, $C > 0$, and $\mathbf{T} = (T_1, \dots, T_N)$. Put*

$$I_{H,d}(C, \mathbf{T}) = \int_{[\mathbf{0},\mathbf{T}]} \exp\left(-\frac{C}{(s_1 \dots s_N)^{2H}}\right) \frac{d\mathbf{s}}{(s_1 \dots s_N)^{Hd}}, \qquad C > 0. \tag{1}$$

Then the asymptotic behavior of $I_{H,d}(C, \mathbf{T})$ as $\min\{T, i = 1, \dots, N\} \to \infty$ is as follows.

(i) If $d - \frac{1}{H} > 0$, then

$$\begin{aligned} I_{H,d}(C, \mathbf{T}) &= \frac{1}{2H} \frac{1}{(N-1)!} C^{\frac{1}{2H} - \frac{d}{2}} \left(\log(T_1 \dots T_N)\right)^{N-1} \Gamma\left(\frac{d}{2} - \frac{1}{2H}\right) \\ &\quad + O\left(\left(\log(T_1 \dots T_N)\right)^{N-2}\right). \end{aligned} \tag{2}$$

(ii) If $d = \frac{1}{H}$, then

$$I_{H,d}(C, \mathbf{T}) = \frac{1}{N!} \left(\log(T_1 \dots T_N)\right)^N + O\left(\left(\log(T_1 \dots T_N)\right)^{N-1}\right). \tag{3}$$

(iii) If $d - \frac{1}{H} < 0$, then

$$I_{H,d}(C, \mathbf{T}) = (T_1 \dots T_N)^{1-Hd} \left(\frac{1}{1 - Hd}\right)^N + o\left((T_1 \dots T_N)^{1-Hd}\right). \tag{4}$$

Proof. We change the variables $u_1 = \frac{s_1}{T_1} \dots \frac{s_N}{T_N}, u_2 = \frac{s_2}{T_2} \dots \frac{s_N}{T_N}, \dots, u_N = \frac{s_N}{T_N}$ in integral (1). Then

$$\begin{aligned} I_{H,d}(C, \mathbf{T}) &= \int_{0 \le u_1 \le \dots \le u_N \le 1} \exp\left(-\frac{C}{(T_1 \dots T_N)^{2H} u_1^{2H}}\right) u_1^{-Hd} \frac{(T_1 \dots T_N)^{1-Hd}}{u_2 \dots u_N} \, d\mathbf{u} \\ &= \frac{1}{(N-1)!} \int_0^1 \left(\log \frac{1}{u_1}\right)^{N-1} \frac{(T_1 \dots T_N)^{1-Hd}}{u_1^{Hd}} \exp\left(-\frac{C}{(T_1 \dots T_N)^{2H} u_1^{2H}}\right) du_1. \end{aligned}$$

Changing the variable $C(T_1 \dots T_N)^{-2H} u_1^{-2H} = t$ we obtain

$$I_{H,d}(C,\mathbf{T}) = \frac{1}{2H}\frac{1}{(N-1)!}C^{\frac{1}{2H}-\frac{d}{2}} \times \int_{C(T_1\dots T_N)^{-2H}}^{+\infty} \left(\log\frac{T_1\dots T_N}{C^{1/(2H)}} + \frac{\log t}{2H}\right)^{N-1} e^{-t} t^{\frac{d}{2}-\frac{1}{2H}-1}\,dt$$
$$= \frac{1}{2H}\frac{1}{(N-1)!}C^{\frac{1}{2H}-\frac{d}{2}} \times \sum_{k=0}^{N-1} C_{N-1}^k \left(\log\frac{T_1\dots T_N}{C^{1/(2H)}}\right)^{N-k-1} \int_{C(T_1\dots T_N)^{-2H}}^{+\infty} \left(\frac{\log t}{2H}\right)^k e^{-t} t^{\frac{d}{2}-\frac{1}{2H}-1}\,dt.$$

The asymptotic behavior of the integral $I_{H,d}(C,\mathbf{T})$ as

$$\min\{T_i, i=1,\dots,N\} \to \infty$$

is considered separately for the following three cases.

Case 1. Let $d - \frac{1}{H} > 0$. Then the following integral:

$$\int_0^{+\infty} \left(\frac{\log t}{2H}\right)^k e^{-t} t^{\frac{d}{2}-\frac{1}{2H}-1}\,dt < \infty$$

converges for an arbitrary $k \in \mathbb{N}$. Thus

$$I_{H,d}(C,\mathbf{T}) = \frac{1}{2H}\frac{1}{(N-1)!}C^{\frac{H}{2}-\frac{d}{2}} \left(\log\frac{T_1\dots T_N}{C^{1/(2H)}}\right)^{N-1} \int_{C(T_1\dots T_N)^{-2H}}^{+\infty} e^{-t} t^{\frac{d}{2}-\frac{1}{2H}-1}\,dt + O\left(\left(\log\frac{T_1\dots T_N}{C^{1/(2H)}}\right)^{N-2}\right)$$
$$\sim \frac{1}{2H}\frac{1}{(N-1)!}C^{\frac{H}{2}-\frac{d}{2}} \left(\log(T_1\dots T_N)\right)^{N-1} \Gamma\left(\frac{d}{2}-\frac{1}{2H}\right) + O\left(\left(\log(T_1\dots T_N)\right)^{N-2}\right)$$

as $\min\{T_i, i=1,\dots,N\} \to \infty$.

Case 2. Let $d = \frac{1}{H}$. Then

$$I_{H,d}(C,\mathbf{T}) = \left(\frac{1}{2H}\right)^N \frac{1}{(N-1)!} \times \int_{C(T_1\dots T_N)^{-2H}}^{+\infty} e^{-t} t^{-1} \left(\log\frac{(T_1\dots T_N)^{2H} t}{C}\right)^{N-1} dt.$$

Integrating by parts, we get

$$I_{H,d}(C,\mathbf{T}) = \left(\frac{1}{2H}\right)^N \frac{1}{N!} \int_{C(T_1\dots T_N)^{-2H}}^{+\infty} e^{-t} \left(\log\frac{(T_1\dots T_N)^{2H} t}{C}\right)^{N} dt$$
$$= \frac{1}{N!}\exp\left(-\frac{C}{(T_1\dots T_N)^{2H}}\right)\left(\log\frac{T_1\dots T_N}{C^{1/(2H)}}\right)^N + O\left(\left(\log\frac{T_1\dots T_N}{C^{1/(2H)}}\right)^{N-1}\right)$$
$$\sim \frac{1}{N!}\left(\log(T_1\dots T_N)\right)^N + O\left(\left(\log(T_1\dots T_N)\right)^{N-1}\right).$$

Case 3. Let $d < \frac{1}{H}$. Turning to the integral in (1) we change the variables $u_i = \frac{s_i}{T_i}$, $i = 1, \dots, N$. Then

$$I_{H,d}(C, \mathbf{T}) = \int_{[\mathbf{0},\mathbf{1}]} \exp\left(-\frac{C(T_1 \dots T_N)^{-2H}}{(u_1 \dots u_N)^{2H}}\right) \frac{(T_1 \dots T_N)^{1-Hd}}{(u_1 \dots u_N)^{Hd}}\, d\mathbf{u}.$$

In this case, the integral $\int_{[\mathbf{0},\mathbf{1}]} (u_1 \dots u_N)^{-Hd}\, d\mathbf{u}$ converges and thus

$$\begin{aligned} \frac{I_{H,d}(C, \mathbf{T})}{(T_1 \dots T_N)^{1-Hd}} &= \int_{[\mathbf{0},\mathbf{1}]} \exp\left(-\frac{C(T_1 \dots T_N)^{-2H}}{(u_1 \dots u_N)^{2H}}\right) \frac{d\mathbf{u}}{(u_1 \dots u_N)^{Hd}} \\ &\sim \int_{[\mathbf{0},\mathbf{1}]} \frac{d\mathbf{u}}{(u_1 \dots u_N)^{Hd}} + o(1) \\ &= \left(\frac{1}{1-Hd}\right)^N + o(1) \end{aligned}$$

as $\min\{T_i, i = 1, \dots, N\} \to \infty$. Lemma 2.1 is proved. □

Theorem 2.1. *Let $\{B^{\mathbf{H},d}(\mathbf{s}), \mathbf{s} \in \mathbb{R}_+^N\}$ be a d-dimensional anisotropic fractional Brownian field that depends on N parameters and has Hurst index $\mathbf{H} = (H, \dots, H)$, $H \in (0, 1)$. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a bounded Borel function. Put*

$$J_{H,d}(V, \mathbf{T}) := \mathbf{E} \int_{[\mathbf{0},\mathbf{T}]} V\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s}, \quad \text{where } \mathbf{T} = (T_1, \dots, T_N) \in \mathbb{R}_+^N. \tag{5}$$

(*i*) *If $d - \frac{1}{H} > 0$ and*

$$\widetilde{V} := \int_{\mathbb{R}^d} V(\mathbf{x}) \|\mathbf{x}\|^{\frac{1}{H}-d}\, d\mathbf{x} < +\infty,$$

then

$$\frac{J_{H,d}(V, \mathbf{T})}{(\log(T_1 \dots T_N))^{N-1}} \to \frac{\widetilde{V}}{2^{\frac{1}{2H}} \pi^{d/2}} \frac{1}{2H} \frac{1}{(N-1)!} \Gamma\left(\frac{d}{2} - \frac{1}{2H}\right)$$

as $\min\{T_i, i = 1, \dots, N\} \to \infty$.

(*ii*) *If $d = \frac{1}{H}$ and*

$$\overline{V} := \int_{\mathbb{R}^d} V(\mathbf{x})\, d\mathbf{x} < +\infty,$$

then

$$\frac{J_{H,d}(V, \mathbf{T})}{(\log(T_1 \dots T_N))^{N}} \to \frac{\overline{V}}{(2\pi)^{d/2} N!}$$

as $\min\{T_i, i = 1, \dots, N\} \to \infty$.

(*iii*) *If $d - \frac{1}{H} < 0$ and $\overline{V} < +\infty$, then*

$$\frac{J_{H,d}(V, \mathbf{T})}{(T_1 \dots T_N)^{1-Hd}} \to \overline{V} \left(\frac{1}{1-Hd}\right)^N$$

as $\min\{T_i, i = 1, \dots, N\} \to \infty$.

Proof. The property of the self-similarity of a fractional Brownian field implies that

$$B^{\mathbf{H},d}(\mathbf{t}) \overset{d}{=} t_1^{H_1} \dots t_N^{H_N} B^{\mathbf{H},d}(\mathbf{1}), \qquad t_i > 0.$$

The coordinates of a d-dimensional random vector $B^{\mathbf{H},d}(\mathbf{1})$ are independent random variables with the standard normal distribution $N(0, 1)$. By assumption of the theorem,

$H_1 = \cdots = H_N = H$, whence we conclude that the density $g_{\mathbf{t}}(\mathbf{x})$ of the random vector $B^{\mathbf{H},d}(\mathbf{t})$ is given by

$$g_{\mathbf{t}}(\mathbf{x}) = \prod_{i=1}^{d} \frac{\exp\left(-\frac{x_i^2}{2(t_1\dots t_N)^{2H}}\right)}{\sqrt{2\pi(t_1\dots t_N)^{2H}}} = \frac{\exp\left(-\frac{\|\mathbf{x}\|^2}{2(t_1\dots t_N)^{2H}}\right)}{(2\pi)^{d/2}(t_1\dots t_N)^{Hd}}.$$

Next we consider the integral in (5):

$$\begin{aligned}\mathbf{E}\int_{[\mathbf{0},\mathbf{T}]} V\left(B^{\mathbf{H},d}(\mathbf{s})\right)\,d\mathbf{s} &= \int_{[\mathbf{0},\mathbf{T}]} \mathbf{E}\left(V\left(B^{\mathbf{H},d}(\mathbf{s})\right)\right)\,d\mathbf{s}\\ &= \int_{[\mathbf{0},\mathbf{T}]}\int_{\mathbb{R}^d} V(\mathbf{x})\frac{\exp\left(-\frac{\|\mathbf{x}\|^2}{2(s_1\dots s_N)^{2H}}\right)}{(2\pi)^{d/2}(s_1\dots s_N)^{Hd}}\,d\mathbf{x}\,d\mathbf{s}\\ &= \int_{\mathbb{R}^d}\frac{V(\mathbf{x})}{(2\pi)^{d/2}}\int_{[\mathbf{0},\mathbf{T}]}\frac{\exp\left(-\frac{\|\mathbf{x}\|^2}{2(s_1\dots s_N)^{2H}}\right)}{(s_1\dots s_N)^{Hd}}\,d\mathbf{s}\,d\mathbf{x}.\end{aligned}$$

Hence

$$J_{H,d}(V,\mathbf{T}) = \int_{\mathbb{R}^d}\frac{V(\mathbf{x})}{(2\pi)^{d/2}} I_{H,d}\left(\frac{\|\mathbf{x}\|^2}{2},\mathbf{T}\right)d\mathbf{x}, \tag{6}$$

where $I_{H,d}\big(\|\mathbf{x}\|^2/2,\mathbf{T}\big)$ is defined by (1). Let the function V have a compact support. We assume that the variable $\mathbf{x}$ in the integral $I_{H,d}(\|\mathbf{x}\|^2/2,\mathbf{T})$ belongs to the set

$$\{\|\mathbf{x}\| \le M\}$$

for some $M > 0$. Lemma 2.1 implies the asymptotic behavior of $I_{H,d}(\|\mathbf{x}\|^2/2,\mathbf{T})$ as

$$\min\{T_i, i = 1,\dots,N\} \to \infty,$$

which together with (6) yields the asymptotic behavior of $J_{H,d}(V,\mathbf{T})$. The rest of the proof is divided into several steps.

If $d - \frac{1}{H} > 0$ and $\widetilde{V} < +\infty$, then (2) implies that

$$\begin{aligned}J_{H,d}(V,\mathbf{T}) &\sim \int_{\mathbb{R}^d}\frac{V(\mathbf{x})}{(2\pi)^{d/2}}\frac{1}{2H}\frac{1}{(N-1)!}\frac{\|\mathbf{x}\|^{\frac{1}{H}-d}}{2^{\frac{1}{2H}-\frac{d}{2}}}\left(\log(T_1\dots T_N)\right)^{N-1}\Gamma\left(\frac{d}{2}-\frac{1}{2H}\right)d\mathbf{x}\\ &= \frac{\widetilde{V}}{2^{\frac{1}{2H}}\pi^{d/2}}\frac{1}{2H}\frac{1}{(N-1)!}\left(\log(T_1\dots T_N)\right)^{N-1}\Gamma\left(\frac{d}{2}-\frac{1}{2H}\right).\end{aligned}$$

If $d = \frac{1}{H}$ and $\overline{V} < +\infty$, then relation (3) implies that

$$J_{H,d}(V,\mathbf{T}) \sim \left(\int_{\mathbb{R}^d} V(\mathbf{x})\,d\mathbf{x}\right)\frac{\left(\log(T_1\dots T_N)\right)^N}{(2\pi)^{d/2}N!}$$

as $\min\{T_i, i = 1,\dots,N\} \to \infty$.

If $d - \frac{1}{H} < 0$ and $\overline{V} < +\infty$, then relation (4) implies that

$$J_{H,d}(V,\mathbf{T}) \sim \left(\int_{\mathbb{R}^d} V(\mathbf{x})\,d\mathbf{x}\right)(T_1\dots T_N)^{1-Hd}\left(\frac{1}{1-Hd}\right)^N$$

as $\min\{T_i, i = 1,\dots,N\} \to \infty$. Therefore the theorem is proved for the case where V has a compact support. In the general case, we introduce the function

$$V^M(\mathbf{x}) := V(\mathbf{x})\mathbf{1}_{\{\|\mathbf{x}\|\le M\}}, \qquad \mathbf{x}\in\mathbb{R}^d.$$

Now we consider the case of $d > \frac{1}{H}$. Note that the function $I_{H,d}(y, \mathbf{T})$, $y > 0$, is non-increasing with respect to y. Then

$$\begin{aligned}
&\frac{1}{(\log(T_1 \dots T_N))^{N-1}} \mathbf{E} \left| \int_{[\mathbf{0},\mathbf{T}]} V\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} - \int_{[\mathbf{0},\mathbf{T}]} V^M\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} \right| \\
&\quad \le \frac{1}{(\log(T_1 \dots T_N))^{N-1}} \int_{[\mathbf{0},\mathbf{T}]} \mathbf{E}\left| V\left(B^{\mathbf{H},d}(\mathbf{s})\right) - V^M\left(B^{\mathbf{H},d}(\mathbf{s})\right)\right| d\mathbf{s} \\
&\quad \le \frac{1}{(\log(T_1 \dots T_N))^{N-1}} \int_{\mathbb{R}^d} \left| V(\mathbf{x}) - V^M(\mathbf{x})\right| I_{H,d}\left(\frac{\|x\|^2}{2}, \mathbf{T}\right) d\mathbf{x} \\
&\quad \le \frac{1}{(\log(T_1 \dots T_N))^{N-1}} \int_{\mathbb{R}^d} \left| V(\mathbf{x}) - V^M(\mathbf{x})\right| d\mathbf{x} I_{H,d}\left(\frac{M^2}{2}, \mathbf{T}\right) \\
&\quad \overset{(2)}{\le} C_1 \frac{\|V - V^M\|_1}{M^{d-\frac{1}{H}}},
\end{aligned}$$

where $\|\cdot\|_1$ is a norm in the space $L^1(\mathbb{R}^d)$.

The same method as that shown above proves similar inequalities for the cases $d = \frac{1}{H}$ and $d - \frac{1}{H} < 0$. Indeed,

$$\begin{aligned}
&\frac{1}{(\log(T_1 \dots T_N))^{N}} \mathbf{E} \left| \int_{[\mathbf{0},\mathbf{T}]} V\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} - \int_{[\mathbf{0},\mathbf{T}]} V^M\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} \right| \\
&\quad \le C_2 \|V - V^M\|_1.
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{(T_1 \dots T_N)^{1-Hd}} \mathbf{E} \left| \int_{[\mathbf{0},\mathbf{T}]} V\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} - \int_{[\mathbf{0},\mathbf{T}]} V^M\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} \right| \\
&\quad \le C_3 \|V - V^M\|_1.
\end{aligned}$$

Thus the theorem is proved. □

3. Weak convergence of a normalized integral functional of a d-dimensional self-similar field that depends on N parameters to a local time

Let λ_k be the Lebesgue measure in $\mathbb{R}^k$ and let $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^N\}$ be a d-dimensional random field.

Definition 3.1. Given an arbitrary Borel set $T \subseteq \mathbb{R}^N$, the occupation measure of the field X in the set T is defined by

$$\mu_T(B) = \lambda_N \{\mathbf{t} \in T; X(\mathbf{t}) \in B\}$$

for an arbitrary Borel set $B \subseteq \mathbb{R}^d$. If μ_T is absolutely continuous with respect to λ_d, then the Radon–Nikodym derivative of the measure μ_T with respect to the measure λ_d is called the local time $L(\cdot, T)$ of the random field X in T, that is,

$$L(\mathbf{x}, T) = \frac{d\mu_T}{d\lambda_d}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

We write $L(\mathbf{x}, \mathbf{t}) = L(\mathbf{x}, [\mathbf{0}, \mathbf{t}])$ if $\mathbf{t} \in \mathbb{R}^N$.

If X has a local time in T, then X has a local time in an arbitrary Borel set $I \subseteq T$.

The local time satisfies the following occupation density formula (see Theorems 6.3 and 6.4 in [5]):

$$\int_T f(X(\mathbf{t}))\,d\mathbf{t} = \int_{\mathbb{R}^d} f(\mathbf{x})L(\mathbf{x},T)\,d\mathbf{x}, \tag{7}$$

where $T \subseteq \mathbb{R}^N$ is an arbitrary Borel set and $f\colon \mathbb{R}^d \to \mathbb{R}$ is an arbitrary measurable function.

Put $T = \prod_{i=1}^N [a_i, a_i + h_i]$. If some version of the local time

$$L\left(\mathbf{x}, \prod_{i=1}^N [a_i, a_i + h_i]\right)$$

is a continuous function with respect to $(\mathbf{x}, t_1, \dots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i]$, then we say that X has a local time in T being continuous with respect to all arguments.

As proved in the paper [12], the local time for a d-dimensional fractional Brownian field exists and is continuous with respect to all arguments if $d < 1/H_i$, $i = 1, \dots, N$. The existence of a continuous local time for such a field is proved in the paper [1] for the case of $d < \sum_{i=1}^N 1/H_i$.

Below we denote the weak convergence of finite dimensional distributions by $\xrightarrow{W}$. Using the above occupation density formula for self-similar fields, we prove the following result.

Theorem 3.1. *Let $\{X^{\mathbf{H},d}(\mathbf{s}), \mathbf{s} \in \mathbb{R}_+^N\}$ be a d-dimensional self-similar random field with Hurst index $\mathbf{H} = (H_1, \dots, H_N) \in (0, +\infty)^N$. Assume that a continuous local time $\{L(\mathbf{x}, \mathbf{t}), \mathbf{x} \in \mathbb{R}^d, \mathbf{t} \in \mathbb{R}_+^N\}$ exists. Then*

$$\lambda^{(H_1+\dots+H_N)d-N} \int_{[0,\lambda\mathbf{t}]} f\left(X^{\mathbf{H},d}(\mathbf{s})\right)\,d\mathbf{s} \xrightarrow{W} \overline{f} L(\mathbf{0}, \mathbf{t}), \qquad \lambda \to \infty, \tag{8}$$

for all $\mathbf{t} \in \mathbb{R}_+^N$ and for an arbitrary bounded function $f\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\overline{f} := \int_{\mathbb{R}^d} f(\mathbf{x})\,d\mathbf{x} < \infty.$$

Proof. Consider the integral $\int_{[\mathbf{0},\lambda\mathbf{t}]} f\left(X^{\mathbf{H},d}(\mathbf{s})\right)d\mathbf{s}$:

$$\begin{aligned}
\int_0^{\lambda t_1} \dots \int_0^{\lambda t_N} f\left(X^{\mathbf{H},d}(\mathbf{s})\right)d\mathbf{s} &= \lambda^N \int_0^{t_1} \dots \int_0^{t_N} f(X^{\mathbf{H},d}(\lambda\mathbf{v}))\,d\mathbf{v} \\
&\overset{d}{=} \lambda^N \int_0^{t_1} \dots \int_0^{t_N} f\left(\lambda^{H_1+\dots+H_N} X^{\mathbf{H},d}(\mathbf{v})\right)d\mathbf{v} \overset{(7)}{=} \lambda^N \int_{\mathbb{R}^d} f\left(\lambda^{H_1+\dots+H_N}\mathbf{x}\right)L(\mathbf{x},\mathbf{t})\,d\mathbf{x} \\
&= \lambda^N \lambda^{-(H_1+\dots+H_N)d} \int_{\mathbb{R}^d} f(\mathbf{y}) L\left(\lambda^{-(H_1+\dots+H_N)}\mathbf{y}, \mathbf{t}\right)d\mathbf{y}.
\end{aligned}$$

We are going to prove the following convergence:

$$\int_{\mathbb{R}^d} f(\mathbf{y})L(\lambda^{-(H_1+\dots+H_N)}\mathbf{y}, \mathbf{t})\,d\mathbf{y} \to \int_{\mathbb{R}^d} f(\mathbf{y})L(\mathbf{0},\mathbf{t})\,d\mathbf{y}, \qquad \lambda \to +\infty, \tag{9}$$

almost surely. Since f is an integrable and bounded function, one can uniformly approximate it by simple functions $f_n \to f$, $n \to \infty$. Thus

$$\int_{\mathbb{R}^d} f_n(\mathbf{y}) L\left(\lambda^{-(H_1+\dots+H_N)}\mathbf{y}, \mathbf{t}\right)\,d\mathbf{y} \to \int_{\mathbb{R}^d} f(\mathbf{y}) L\left(\lambda^{-(H_1+\dots+H_N)}\mathbf{y}, \mathbf{t}\right)\,d\mathbf{y}, \qquad n \to \infty.$$

Now we prove convergence (9) for an indicator function $\chi(\mathbf{x}) = \mathbf{1}_{[\mathbf{a},\mathbf{b}]}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$. Indeed,

$$\int_{\mathbb{R}^d} \chi(\mathbf{y}) L\left(\lambda^{-(H_1+\cdots+H_N)}\mathbf{y}, \mathbf{t}\right) d\mathbf{y} = \int_{[\mathbf{a},\mathbf{b}]} L\left(\lambda^{-(H_1+\cdots+H_N)}\mathbf{y}, \mathbf{t}\right) d\mathbf{y}$$
$$= \lambda^{(H_1+\cdots+H_N)d} \int_{[\lambda^{-(H_1+\cdots+H_N)}\mathbf{a}, \lambda^{-(H_1+\cdots+H_N)}\mathbf{b}]} L(\mathbf{x}, \mathbf{t})\, d\mathbf{x}.$$

Therefore

$$\left| \int_{\mathbb{R}^d} \chi(\mathbf{y}) L\left(\lambda^{-(H_1+\cdots+H_N)}\mathbf{y}, \mathbf{t}\right) d\mathbf{y} - \overline{\chi} L(\mathbf{0}, \mathbf{t}) \right|$$
$$= \lambda^{(H_1+\cdots+H_N)d} \left| \int_{[\lambda^{-(H_1+\cdots+H_N)}\mathbf{a}, \lambda^{-(H_1+\cdots+H_N)}\mathbf{b}]} (L(\mathbf{x}, \mathbf{t}) - L(\mathbf{0}, \mathbf{t}))\, d\mathbf{x} \right|$$
$$\leq \lambda^{(H_1+\cdots+H_N)d} \int_{[\lambda^{-(H_1+\cdots+H_N)}\mathbf{a}, \lambda^{-(H_1+\cdots+H_N)}\mathbf{b}]} |L(\mathbf{x}, \mathbf{t}) - L(\mathbf{0}, \mathbf{t})|\, d\mathbf{x}.$$

Since $L(\mathbf{y}, \mathbf{t})$ is continuous, we conclude that, for any given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|L(\mathbf{y}, \mathbf{t}) - L(\mathbf{0}, \mathbf{t})| < \varepsilon$$

almost surely for all $\mathbf{y} \in [-\delta, \delta]^d$.

Hence $[\lambda^{-(H_1+\cdots+H_N)}\mathbf{a}, \lambda^{-(H_1+\cdots+H_N)}\mathbf{b}] \subset [-\delta, \delta]^d$ for all

$$\lambda^{(H_1+\cdots+H_N)} > \frac{1}{\delta} \max_{i=1,\ldots,d} \{|a_i|, |b_i|\},$$

whence

$$\lambda^{(H_1+\cdots+H_N)d} \int_{[\lambda^{-(H_1+\cdots+H_N)}\mathbf{a}, \lambda^{-(H_1+\cdots+H_N)}\mathbf{b}]} |L(\mathbf{x}, \mathbf{t}) - L(\mathbf{0}, \mathbf{t})|\, d\mathbf{x} \leq \varepsilon \prod_{i=1}^{d} (b_i - a_i).$$

Thus

$$\lambda^{(H_1+\cdots+H_N)d-N} \int_{[\mathbf{0}, \lambda\mathbf{t}]} f\left(X^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} \overset{d}{=} \int_{\mathbb{R}^d} f(\mathbf{y}) L\left(\lambda^{-(H_1+\cdots+H_N)}\mathbf{y}, \mathbf{t}\right) d\mathbf{y}$$
$$\overset{\mathsf{P1}}{\to} \overline{f} L(\mathbf{0}, \mathbf{t}), \qquad \lambda \to \infty,$$

which proves the weak convergence. The theorem is proved. □

If $d < \sum_{i=1}^{N} 1/H_i$, then the local time exists and is continuous with respect to all arguments in the case of an anisotropic fractional Brownian field $B^{\mathbf{H},d}$. Thus Theorem 3.1 holds for the field $B^{\mathbf{H},d}$. In particular, if $\mathbf{t} = \mathbf{1}$ in (8), then

$$\lambda^{(H_1+\cdots+H_N)d-N} \int_{[\mathbf{0}, \lambda\mathbf{1}]} f\left(B^{\mathbf{H},d}(\mathbf{s})\right) d\mathbf{s} \overset{W}{\to} \overline{f} L(\mathbf{0}, \mathbf{1}), \qquad \lambda \to \infty.$$

4. Concluding remarks

The convergence of a mean-type normalized integral functional of a d-dimensional anisotropic fractional Brownian field $B^{\mathbf{H},d}$ that depends on N parameters is established in Theorem 2.1 for the case where

$$\mathbf{H} = (H, \ldots, H).$$

The normalization and conditions imposed on the integrand depend on the parameters H and d. Namely the normalizing factor is $\log(T_1 \ldots T_N)^{N-1}$ if $Hd > 1$, or $\log(T_1 \ldots T_N)^N$ if $Hd = 1$, or $(T_1 \ldots T_N)^{1-Hd}$ if $Hd < 1$. Note that this result holds for the case of $d = \frac{1}{H} + \cdots + \frac{1}{H} = \frac{N}{d}$ as well, that is, for the case where a local time does not exist for an anisotropic fractional Brownian field.

The weak convergence of a normalized integral functional of a d-dimensional anisotropic self-similar field that depends on N parameters to its local time is proved in Section 3. Theorem 3.1 holds under the assumption that a continuous local time exists for such a field. The normalizing factor in this theorem equals $\lambda^{N-(H_1+\cdots+H_N)H}$. In particular, Theorem 3.1 holds for a d-dimensional anisotropic fractional Brownian field $B^{\mathbf{H},d}$ that depends on N parameters if $d < \frac{1}{H_1} + \cdots + \frac{1}{H_N}$.

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