ASYMPTOTIC PROPERTIES OF INTEGRAL FUNCTIONALS
OF FRACTIONAL BROWNIAN FIELDS

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V. I. MAKOGIN

Abstract. Two theorems describing the asymptotic behavior of integral functionals of multidimensional self-similar random fields are proved. For a d-dimensional fractional Brownian field depending on N parameters, a theorem on the convergence of the integral mean-type functional is established. The weak convergence of an integral functional of a d-dimensional anisotropic self-similar random field with N parameters to the local time is proved under the assumption that the continuous local time exists for this field.

1. Introduction

Let \{\Omega, \mathcal{F}, P\} be a probability space where all random objects considered below are defined. Denote \(\mathbb{R}^k_+ = [0, +\infty)^k, k \geq 1\), and let \(x \cdot y\) be the vector constituted from the products of the corresponding coordinates, that is, \(x \cdot y = (x_1 y_1, \ldots, x_d y_d)\), where \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\) and \(y = (y_1, \ldots, y_d) \in \mathbb{R}^d\).

Definition 1.1. A real valued random field \(\{X^H(t), t \in \mathbb{R}^N_+\}\) is called self-similar with index \(H = (H_1, \ldots, H_N) \in (0, +\infty)^N\) if

\[\{X(a \cdot t), t \in \mathbb{R}^N_+\} \overset{d}{=} \{a_1^{H_1} \cdots a_N^{H_N} X(t), t \in \mathbb{R}^N_+\}\]

for an arbitrary \(a = (a_1, \ldots, a_N) \in (0, +\infty)^N\), where the symbol \(\overset{d}{=}\) means the equality of finite dimensional distributions.

Definition 1.2. The random field \(\{X^{H,d}(t), t \in \mathbb{R}^N_+\}\) with values in \(\mathbb{R}^d\) is defined by

\[X^{H,d} = (X_1^H, \ldots, X_d^H),\]

where \(X_1^H, \ldots, X_d^H\) are independent copies of a real valued random field

\[\{X_0^H(t), t \in \mathbb{R}^N_+\}.\]

If \(\{X_0^H(t), t \in \mathbb{R}^N_+\}\) is a self-similar field with index \(H = (H_1, \ldots, H_N) \in (0, +\infty)^N\), then we say that \(\{X^{H,d}(t), t \in \mathbb{R}^N_+\}\) is a d-dimensional self-similar field with index \(H\).

Remark 1.1. In the case of \(N = 1\), we write \(H\) instead of \(H\). In this case, real valued and d-dimensional self-similar random fields are called processes and denoted by \(X^H\) and \(X^{H,d}\), respectively.

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Studies of self-similar random fields (in other words, multiparameter random processes) are explained by the evidence of the property of self-similarity in the data from climatology, nature science and other fields (see [3, 11]). In particular, the so-called anisotropic random fields are used to model phenomena in the environmental statistics, statistical hydrology and image processing (see [2, 3, 4]). Strong limit theorems for self-similar random fields are proved in [9] and more references are given therein.

**Definition 1.3.** A real valued centered Gaussian random field $B^H = \{B^H(t), t \in \mathbb{R}^N_+\}$ with covariance function

$$E(B^H(t)B^H(s)) = 2^{-N} \prod_{i=1}^{N} (|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}), \quad t, s \in \mathbb{R}^N_+,$$

is called an anisotropic fractional Brownian field with Hurst index $H = (H_1, \ldots, H_N) \in (0, 1)^N$.

**Definition 1.4.** Let $H = (H_1, \ldots, H_N) \in (0, 1)^N$. A random field

$$B^{H,d} = \{B^{H,d}(t), t \in \mathbb{R}^N_+\}$$

defined by

$$B^{H,d} = (B^H_1, \ldots, B^H_d),$$

where $B^H_1, \ldots, B^H_d$ are independent copies of a real valued anisotropic fractional Brownian field $\{B^H_0(t), t \in \mathbb{R}^N_+\}$ with Hurst index $H$, is called a $d$-dimensional anisotropic fractional Brownian field with Hurst index $H$.

**Remark 1.2.** A $d$-dimensional anisotropic fractional Brownian field is self-similar and its Hurst index is the index of the self-similarity. A real valued fractional Brownian motion $\{B^H(t), t \in \mathbb{R}_+\}$, $H \in (0, 1)$, and a $d$-dimensional fractional Brownian motion $\{B^{H,d}(t), t \in \mathbb{R}_+\}$, $H \in (0, 1)$, are, in fact, random fields $B^H$ and $B^{H,d}$ if $N = 1$. If $H = 1/2$, then $B^{1/2,d}$ is a Brownian motion.

The main aim of this paper is to study the asymptotic behavior of integral functionals of a fractional Brownian field. The paper is motivated by corresponding results for a multidimensional Brownian motion and fractional Brownian motion. In particular, the Kallianpur–Robbins law [6] describes the long term asymptotic evolution of the distribution of the occupation measure for a two dimensional Brownian motion. Below is their corresponding result.

**Theorem 1.1** ([6]). Let $\{W^{(2)}(t), t \in \mathbb{R}_+\}$ be a two dimensional Brownian motion. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a bounded integrable function such that $\overline{f} := \int_{\mathbb{R}^2} f(x) \, dx \neq 0$. Then

$$\lim_{T \to +\infty} P \left[ \frac{2\pi}{f \log T} \int_0^T f(W^{(2)}(t)) \, dt \geq x \right] = e^{-x}, \quad x > 0.$$

Kôno [7] generalizes this result and proves the following theorem for a $d$-dimensional fractional Brownian motion $\{B^{H,d}(t), t \in \mathbb{R}_+\}$.

**Theorem 1.2** ([7]). Let $g: \mathbb{R}^d \to \mathbb{R}_+$ be a bounded integrable function such that

$$\overline{g} := \int_{\mathbb{R}^d} g(x) \, dx \neq 0$$

and let $Hd = 1$, $d \geq 2$. Then

$$\lim_{T \to +\infty} P \left[ \frac{(2\pi)^{d/2}}{g \log T} \int_0^T g(B^{H,d}(t)) \, dt \geq x \right] = e^{-x}, \quad x > 0.$$
A partial generalization of the Kallianpur–Robbins law is obtained in this paper. More precisely, we find a normalization for the convergence of a mean-type integral functional for a $d$-dimensional fractional Brownian field that depends on $N$ parameters.

The weak convergence of a normalized integral functional of a $d$-dimensional fractional Brownian field to the local time is proved in [11]. In this paper, we establish the convergence of a normalized integral functional of a $d$-dimensional anisotropic self-similar field that depends on $N$ parameters to the local time under the assumption that its continuous local time exists. Note that the paper [12] proves that the local time for a $d$-dimensional fractional Brownian field exists and is continuous with respect to all its arguments for $d < 1/H_i$, $i = 1, \ldots, N$.

The paper is organized as follows. A theorem on the convergence of a mean-type integral functional of a $d$-dimensional fractional Brownian field that depends on $N$ parameters is proved in Section 2. The proof of the main result uses an auxiliary result describing the asymptotic behavior of integrals depending on parameters. In Section 3 the notion of the local time for random fields is discussed and a limit theorem for integral functionals of self-similar fields is proved. Concluding remarks are given in Section 4.

2. THEOREM ON THE CONVERGENCE OF A MEAN TYPE INTEGRAL FUNCTIONAL OF A $d$-DIMENSIONAL FRACTIONAL BROWNIAN FIELD THAT DEPENDS ON $N$ PARAMETERS

In what follows we use the notation $0 = (0, \ldots, 0) \in \mathbb{R}^N$ and

$$[a, b] := [a_1, b_1] \times \cdots \times [a_N, b_N],$$

where $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ and $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$. The following auxiliary result is needed for the proof of the main theorem.

**Lemma 2.1.** Let $H \in (0, 1)$, $C > 0$, and $T = (T_1, \ldots, T_N)$. Put

$$(1) \quad I_{H,d}(C, T) = \int_{[0, \mathbf{T}]} \exp \left( -\frac{C}{(s_1 \ldots s_N)^{2H}} \right) \frac{ds}{s_1 \ldots s_N^{Hd}}, \quad C > 0.$$  

Then the asymptotic behavior of $I_{H,d}(C, T)$ as $\min\{T, i = 1, \ldots, N\} \to \infty$ is as follows.

(i) If $d - \frac{1}{H} > 0$, then

$$(2) \quad I_{H,d}(C, T) = \frac{1}{2H} \frac{1}{(N-1)!} C^{\frac{1}{H} - \frac{d}{2}} \left( \log(T_1 \ldots T_N) \right)^{N-1} \Gamma \left( \frac{d}{2} - \frac{1}{2H} \right) + O \left( \left( \log(T_1 \ldots T_N) \right)^{N-2} \right).$$

(ii) If $d = \frac{1}{H}$, then

$$(3) \quad I_{H,d}(C, T) = \frac{1}{N!} \left( \log(T_1 \ldots T_N) \right)^N + O \left( \left( \log(T_1 \ldots T_N) \right)^{N-1} \right).$$

(iii) If $d - \frac{1}{H} < 0$, then

$$(4) \quad I_{H,d}(C, T) = (T_1 \ldots T_N)^{1-Hd} \left( \frac{1}{1-Hd} \right)^N + o \left( (T_1 \ldots T_N)^{1-Hd} \right).$$

**Proof.** We change the variables $u_1 = \frac{T_1}{T_1^H}, \ldots, \frac{T_N}{T_N^H}, u_2 = \frac{T_2}{T_1^H}, \ldots, \frac{T_N}{T_1^H}, \ldots, u_N = \frac{T_N}{T_1^H}$ in integral (1). Then

$$I_{H,d}(C, T) = \int_{0 \leq u_1 \leq \ldots \leq u_N \leq \frac{1}{C}} \exp \left( -\frac{C}{(T_1 \ldots T_N)^{2H} u_1^{2H}} \right) u_1^{-Hd} \frac{(T_1 \ldots T_N)^{1-Hd}}{u_2 \ldots u_N} \, du.$$  

$$= \frac{1}{(N-1)!} \int_0^1 \left( \log \frac{1}{u_1} \right)^{N-1} \frac{(T_1 \ldots T_N)^{1-Hd}}{u_1^{Hd}} \exp \left( -\frac{C}{(T_1 \ldots T_N)^{2H} u_1^{2H}} \right) \, du_1.$$
Changing the variable $C(T_1 \ldots T_N)^{-2H} u_1^{-2H} = t$ we obtain

$$I_{H,d}(C, T) = \frac{1}{2H} \frac{1}{(N-1)!} C^{\frac{1}{2H}-\frac{d}{2}}$$

$$\times \int_{C(T_1 \ldots T_N)^{-2H}}^{+\infty} \left( \log \frac{T_1 \ldots T_N}{C^{1/(2H)}} + \log t \right)^{N-1} e^{-t \frac{d}{2H} - \frac{1}{2H} dt}$$

$$= \frac{1}{2H} \frac{1}{(N-1)!} C^{\frac{1}{2H}-\frac{d}{2}}$$

$$\times \sum_{k=0}^{N-1} C_{N-1}^k \left( \log \frac{T_1 \ldots T_N}{C^{1/(2H)}} \right)^{N-k-1} \int_{C(T_1 \ldots T_N)^{-2H}}^{+\infty} \left( \log t \right)^k e^{-t \frac{d}{2H} - \frac{1}{2H} dt}.$$

The asymptotic behavior of the integral $I_{H,d}(C, T)$ as

$$\min \{T_i, i = 1, \ldots, N \} \to \infty$$

is considered separately for the following three cases.

Case 1. Let $d - \frac{1}{H} > 0$. Then the following integral:

$$\int_0^{+\infty} \left( \log \frac{t}{2H} \right)^k e^{-t \frac{d}{2H} - \frac{1}{2H} dt} < \infty$$

converges for an arbitrary $k \in \mathbb{N}$. Thus

$$I_{H,d}(C, T) = \frac{1}{2H} \frac{1}{(N-1)!} C^{\frac{1}{2H}-\frac{d}{2}} \left( \log \frac{T_1 \ldots T_N}{C^{1/(2H)}} \right)^{N-1} \int_{C(T_1 \ldots T_N)^{-2H}}^{+\infty} e^{-t \frac{d}{2H} - \frac{1}{2H} dt}$$

$$+ O \left( \left( \log \frac{T_1 \ldots T_N}{C^{1/(2H)}} \right)^{N-2} \right)$$

$$\sim \frac{1}{2H} \frac{1}{(N-1)!} C^{\frac{1}{2H}-\frac{d}{2}} \left( \log(T_1 \ldots T_N) \right)^{N-1} \Gamma \left( \frac{d}{2} - \frac{1}{2H} \right)$$

$$+ O \left( \left( \log(T_1 \ldots T_N) \right)^{N-2} \right)$$

as $\min \{T_i, i = 1, \ldots, N \} \to \infty$.

Case 2. Let $d = \frac{1}{H}$. Then

$$I_{H,d}(C, T) = \left( \frac{1}{2H} \right)^N \frac{1}{(N-1)!}$$

$$\times \int_{C(T_1 \ldots T_N)^{-2H}}^{+\infty} e^{-t} \left( \log \frac{T_1 \ldots T_N}{C} \right)^{N-1} dt.$$

Integrating by parts, we get

$$I_{H,d}(C, T) = \left( \frac{1}{2H} \right)^N \frac{1}{N!} \int_{C(T_1 \ldots T_N)^{-2H}}^{+\infty} e^{-t} \left( \log \frac{T_1 \ldots T_N}{C^{1/(2H)}} \right)^N dt$$

$$= \frac{1}{N!} \exp \left( -\frac{C}{(T_1 \ldots T_N)^{2H}} \right) \left( \log \frac{T_1 \ldots T_N}{C^{1/(2H)}} \right)^N$$

$$+ O \left( \left( \log \frac{T_1 \ldots T_N}{C^{1/(2H)}} \right)^{N-1} \right)$$

$$\sim \frac{1}{N!} \left( \log(T_1 \ldots T_N) \right)^N + O \left( \left( \log(T_1 \ldots T_N) \right)^{N-1} \right).$$
Let $J(\mathbf{s})$ be a fractional Brownian field that depends on $\mathbf{s}$. The property of the self-similarity of a fractional Brownian field implies that

$$J(\mathbf{s}) \equiv J(t_1 \ldots t_n) = J(t_1^H \ldots t_n^H),$$

for $t_i > 0$. Then

$$J(\mathbf{s}) \neq J(\mathbf{s})$$

Proof. The property of the self-similarity of a fractional Brownian field implies that

$$J(\mathbf{s}) \equiv J(t_1 \ldots t_n) = J(t_1^H \ldots t_n^H),$$

for $t_i > 0$. Then

$$J(\mathbf{s}) \neq J(\mathbf{s})$$

The coordinates of a $d$-dimensional random vector $B^{H,d}(1)$ are independent random variables with the standard normal distribution $N(0, 1)$. By assumption of the theorem,
\[ H_1 = \ldots = H_N = H, \] whence we conclude that the density \( g_t(x) \) of the random vector \( B^{H,d}(t) \) is given by
\[
g_t(x) = \prod_{i=1}^d \exp \left( -\frac{x_i^2}{2(t_1 \ldots t_N)^{2H}} \right) \sqrt{2\pi (t_1 \ldots t_N)^{2H}} \cdot \exp \left( -\frac{\|x\|^2}{2(t_1 \ldots t_N)^{2H}} \right) \left( 2\pi \right)^{d/2 (t_1 \ldots t_N)^{Hd}}.
\]

Next we consider the integral in (5):
\[
E \int_{[0,T]} V(B^{H,d}(s)) \, ds = \int_{[0,T]} E(V(B^{H,d}(s))) \, ds
= \int_{[0,T]} \int_{\mathbb{R}^d} V(x) \exp \left( -\frac{\|x\|^2}{2(s_1 \ldots s_N)^{2H}} \right) \frac{d^n}{(2\pi)^{d/2}} d^n x \, ds
= \int_{\mathbb{R}^d} V(x) \exp \left( -\frac{\|x\|^2}{2(s_1 \ldots s_N)^{2H}} \right) \frac{d^n}{(s_1 \ldots s_N)^{Hd}} \, ds \, dx.
\]

Hence
\[
J_{H,d}(V, T) = \int_{\mathbb{R}^d} \frac{V(x)}{(2\pi)^{d/2}} I_{H,d} \left( \frac{x}{2} T \right) d^n x,
\]
where \( I_{H,d}(\|x\|^2/2, T) \) is defined by (11). Let the function \( V \) have a compact support. We assume that the variable \( x \) in the integral \( I_{H,d}(\|x\|^2/2, T) \) belongs to the set
\[
\{\|x\| \leq M\}
\]
for some \( M > 0 \). Lemma 2.1 implies the asymptotic behavior of \( I_{H,d}(\|x\|^2/2, T) \) as
\[
\min\{T_i, i = 1, \ldots, N\} \to \infty,
\]
which together with (3) yields the asymptotic behavior of \( J_{H,d}(V, T) \). The rest of the proof is divided into several steps.

If \( d - \frac{1}{H} > 0 \) and \( \bar{V} < +\infty \), then (2) implies that
\[
J_{H,d}(V, T) \sim \int_{\mathbb{R}^d} \frac{V(x)}{(2\pi)^{d/2}} \frac{1}{2^{H/2} (N-1)!} \frac{\|x\|^{1-d}}{2^{\pi/2} - \frac{d}{2}} \left( \log(T_1 \ldots T_N) \right)^{N-1} \Gamma \left( \frac{d}{2} - \frac{1}{2H} \right) \, dx
= \frac{\bar{V}}{2^\pi \pi^{d/2} 2 H (N-1)!} \left( \log(T_1 \ldots T_N) \right)^{N-1} \Gamma \left( \frac{d}{2} - \frac{1}{2H} \right).
\]
If \( d = \frac{1}{H} \) and \( \bar{V} < +\infty \), then relation (3) implies that
\[
J_{H,d}(V, T) \sim \left( \int_{\mathbb{R}^d} V(x) \, dx \right) \left( \log(T_1 \ldots T_N) \right)^N \left( 2\pi \right)^{d/2 N!}
\]
as \( \min\{T_i, i = 1, \ldots, N\} \to \infty \).

If \( d - \frac{1}{H} < 0 \) and \( \bar{V} < +\infty \), then relation (4) implies that
\[
J_{H,d}(V, T) \sim \left( \int_{\mathbb{R}^d} V(x) \, dx \right) (T_1 \ldots T_N)^{1-Hd} \left( \frac{1}{1-Hd} \right)^N
\]
as \( \min\{T_i, i = 1, \ldots, N\} \to \infty \). Therefore the theorem is proved for the case where \( V \) has a compact support. In the general case, we introduce the function
\[
V^M(x) := V(x) 1_{\{\|x\| \leq M\}}, \quad x \in \mathbb{R}^d.
\]
Now we consider the case of \( d > \frac{1}{H} \). Note that the function \( I_{H,d}(y,T) \), \( y > 0 \), is non-increasing with respect to \( y \). Then

\[
\frac{1}{(\log(T_1 \ldots T_N))^{N-1}} E \left| \int_{[0,T]} V(B^{H,d}(s)) \ ds - \int_{[0,T]} V^M(B^{H,d}(s)) \ ds \right| \\
\leq \frac{1}{(\log(T_1 \ldots T_N))^{N-1}} \int_{[0,T]} E \left| V(B^{H,d}(s)) - V^M(B^{H,d}(s)) \right| \ ds \\
\leq \frac{1}{(\log(T_1 \ldots T_N))^{N-1}} \int_{\mathbb{R}^d} |V(x) - V^M(x)| I_{H,d} \left( \frac{\|x\|^2}{2}, T \right) \ dx \\
\leq \frac{1}{(\log(T_1 \ldots T_N))^{N-1}} \int_{\mathbb{R}^d} |V(x) - V^M(x)| \ dx I_{H,d} \left( \frac{M^2}{2}, T \right) \\
\leq C_1 \|V - V^M\|_1 \left( \frac{1}{M^{d-H}} \right),
\]

where \( \| \cdot \|_1 \) is a norm in the space \( L^1(\mathbb{R}^d) \).

The same method as that shown above proves similar inequalities for the cases \( d = \frac{1}{H} \) and \( d - \frac{1}{H} < 0 \). Indeed,

\[
\frac{1}{(\log(T_1 \ldots T_N))^N} E \left| \int_{[0,T]} V(B^{H,d}(s)) \ ds - \int_{[0,T]} V^M(B^{H,d}(s)) \ ds \right| \\
\leq C_2 \|V - V^M\|_1,
\]

and

\[
\frac{1}{(T_1 \ldots T_N)^{1-Hd}} E \left| \int_{[0,T]} V(B^{H,d}(s)) \ ds - \int_{[0,T]} V^M(B^{H,d}(s)) \ ds \right| \\
\leq C_3 \|V - V^M\|_1.
\]

Thus the theorem is proved. \( \square \)

3. **Weak convergence of a normalized integral functional of a \( d \)-dimensional self-similar field that depends on \( N \) parameters to a local time**

Let \( \lambda_k \) be the Lebesgue measure in \( \mathbb{R}^k \) and let \( \{X(t), t \in \mathbb{R}^N\} \) be a \( d \)-dimensional random field.

**Definition 3.1.** Given an arbitrary Borel set \( T \subseteq \mathbb{R}^N \), the occupation measure of the field \( X \) in the set \( T \) is defined by

\[
\mu_T(B) = \lambda_N \{t \in T; X(t) \in B\}
\]

for an arbitrary Borel set \( B \subseteq \mathbb{R}^d \). If \( \mu_T \) is absolutely continuous with respect to \( \lambda_d \), then the Radon–Nikodym derivative of the measure \( \mu_T \) with respect to the measure \( \lambda_d \) is called the local time \( L(\cdot, T) \) of the random field \( X \) in \( T \), that is,

\[
L(x,T) = \frac{d\mu_T}{d\lambda_d}(x), \quad \forall x \in \mathbb{R}^d.
\]

We write \( L(x,t) = L(x, [0,t]) \) if \( t \in \mathbb{R}^N \).

If \( X \) has a local time in \( T \), then \( X \) has a local time in an arbitrary Borel set \( I \subseteq T \).
The local time satisfies the following occupation density formula (see Theorems 6.3 and 6.4 in [5]):

\[ \int_T f(X(t)) \, dt = \int_{\mathbb{R}^d} f(x)L(x, T) \, dx, \]

where \( T \subseteq \mathbb{R}^N \) is an arbitrary Borel set and \( f: \mathbb{R}^d \to \mathbb{R} \) is an arbitrary measurable function.

Put \( T = \prod_{i=1}^N [a_i, a_i + h_i] \). If some version of the local time

\[ L \left( x, \prod_{i=1}^N [a_i, a_i + h_i] \right) \]

is a continuous function with respect to \((x, t_1, \ldots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i], \) then we say that \( X \) has a local time in \( T \) being continuous with respect to all arguments.

As proved in the paper [12], the local time for a \( d \)-dimensional fractional Brownian field exists and is continuous with respect to all arguments if \( d < 1/H_i, \ i = 1, \ldots, N. \) The existence of a continuous local time for such a field is proved in the paper [1] for the case of \( d < \sum_{i=1}^N 1/H_i. \)

Below we denote the weak convergence of finite dimensional distributions by \( \overset{W}{\to} \). Using the above occupation density formula for self-similar fields, we prove the following result.

**Theorem 3.1.** Let \( \{X^{H,d}(s), s \in \mathbb{R}^N_+\} \) be a \( d \)-dimensional self-similar random field with Hurst index \( H = (H_1, \ldots, H_N) \in (0, +\infty)^N \). Assume that a continuous local time \( \{L(x, t), x \in \mathbb{R}^d, t \in \mathbb{R}^N_+\} \) exists. Then

\[ \lambda^{(H_1 + \cdots + H_N)d - N} \int_{[0, \lambda t]} f \left( X^{H,d}(s) \right) \, ds \overset{W}{\to} \int f(L(0, t)), \quad \lambda \to \infty, \]

for all \( t \in \mathbb{R}^N_+ \) and for an arbitrary bounded function \( f: \mathbb{R}^d \to \mathbb{R} \) such that

\[ \int \int_{\mathbb{R}^d} f(x) \, dx < \infty. \]

**Proof.** Consider the integral \( \int_{[0, \lambda t]} f \left( X^{H,d}(s) \right) \, ds: \)

\[
\begin{align*}
\int_0^{\lambda t_1} \cdots \int_0^{\lambda t_N} f \left( X^{H,d}(s) \right) \, ds &= \lambda^N \int_0^{t_1} \cdots \int_0^{t_N} f \left( X^{H,d}(v) \right) \, dv \\
&\overset{d}{=} \lambda^N \int_0^{t_1} \cdots \int_0^{t_N} f \left( \lambda^{H_1+\cdots+H_N} X^{H,d}(v) \right) \, dv \\
&\overset{1}{=} \lambda^N \lambda^{-(H_1+\cdots+H_N)d} \int_{\mathbb{R}^d} f(y) L \left( \lambda^{-(H_1+\cdots+H_N)} y, t \right) \, dy.
\end{align*}
\]

We are going to prove the following convergence:

\[ \int_{\mathbb{R}^d} f(y) L \left( \lambda^{-(H_1+\cdots+H_N)} y, t \right) \, dy \to \int_{\mathbb{R}^d} f(y) L(0, t) \, dy, \quad \lambda \to +\infty, \]

almost surely. Since \( f \) is an integrable and bounded function, one can uniformly approximate it by simple functions \( f_n \to f, n \to \infty \). Thus

\[ \int_{\mathbb{R}^d} f_n(y) L \left( \lambda^{-(H_1+\cdots+H_N)} y, t \right) \, dy \to \int_{\mathbb{R}^d} f(y) L \left( \lambda^{-(H_1+\cdots+H_N)} y, t \right) \, dy, \quad n \to \infty. \]
Now we prove convergence (9) for an indicator function \( \chi(x) = 1_{[a,b]}(x), \ x \in \mathbb{R}^d \). Indeed,
\[
\int_{\mathbb{R}^d} \chi(y) L \left( \lambda^{-(H_1 + \cdots + H_N)} y, t \right) \, dy = \int_{[a,b]} L \left( \lambda^{-(H_1 + \cdots + H_N)} y, t \right) \, dy
\]
\[
= \lambda^{(H_1 + \cdots + H_N)d} \int_{[\lambda^{-(H_1 + \cdots + H_N)} a, \lambda^{-(H_1 + \cdots + H_N)} b]} L(x, t) \, dx.
\]
Therefore
\[
\left| \int_{\mathbb{R}^d} \chi(y) L \left( \lambda^{-(H_1 + \cdots + H_N)} y, t \right) \, dy - \chi L(0, t) \right|
\]
\[
= \lambda^{(H_1 + \cdots + H_N)d} \left| \int_{[\lambda^{-(H_1 + \cdots + H_N)} a, \lambda^{-(H_1 + \cdots + H_N)} b]} (L(x, t) - L(0, t)) \, dx \right|
\]
\[
\leq \lambda^{(H_1 + \cdots + H_N)d} \int_{[\lambda^{-(H_1 + \cdots + H_N)} a, \lambda^{-(H_1 + \cdots + H_N)} b]} \left| L(x, t) - L(0, t) \right| \, dx.
\]
Since \( L(y, t) \) is continuous, we conclude that, for any given \( \varepsilon > 0 \), there exists a number \( \delta > 0 \) such that
\[
\left| L(y, t) - L(0, t) \right| < \varepsilon
\]
a almost surely for all \( y \in [-\delta, \delta]^d \).
Hence \( [\lambda^{-(H_1 + \cdots + H_N)} a, \lambda^{-(H_1 + \cdots + H_N)} b] \subset [-\delta, \delta]^d \) for all
\[
\lambda^{(H_1 + \cdots + H_N)} > \frac{1}{\delta} \max_i \{|a_i|, |b_i|\},
\]
whence
\[
\int_{\mathbb{R}^d} \lambda^{(H_1 + \cdots + H_N)d} \left| L(x, t) - L(0, t) \right| \, dx \leq \varepsilon \prod_{i=1}^d (b_i - a_i).
\]
Thus
\[
\lambda^{(H_1 + \cdots + H_N)d - N} \int_{[0, \lambda t]} f \left( X^{H.d}(s) \right) \, ds ^= \int_{\mathbb{R}^d} f(y) L \left( \lambda^{-(H_1 + \cdots + H_N)} y, t \right) \, dy
\]
\[
\overset{P_1}{\longrightarrow} f L(0, t), \quad \lambda \to \infty,
\]
which proves the weak convergence. The theorem is proved.

If \( d < \sum_{i=1}^N 1/H_i \), then the local time exists and is continuous with respect to all arguments in the case of an anisotropic fractional Brownian field \( B^{H.d} \). Thus Theorem 3.1 holds for the field \( B^{H.d} \). In particular, if \( t = 1 \) in \( 8 \), then
\[
\lambda^{(H_1 + \cdots + H_N)d - N} \int_{[0, \lambda]} f \left( B^{H.d}(s) \right) \, ds \overset{L^1}{\to} f L(0, 1), \quad \lambda \to \infty.
\]

4. CONCLUDING REMARKS

The convergence of a mean-type normalized integral functional of a \( d \)-dimensional anisotropic fractional Brownian field \( B^{H,d} \) that depends on \( N \) parameters is established in Theorem 2.1 for the case where
\[
H = (H, \ldots, H).
\]
The normalization and conditions imposed on the integrand depend on the parameters \( H \) and \( d \). Namely the normalizing factor is \( \log(T_1 \cdots T_N)^{N-1} \) if \( Hd > 1 \), or \( \log(T_1 \cdots T_N)^N \) if \( Hd = 1 \), or \( (T_1 \cdots T_N)^{1-Hd} \) if \( Hd < 1 \). Note that this result holds for the case of
\[
d = \frac{1}{H_1} + \cdots + \frac{1}{H_N} = \frac{N}{d} \text{ as well, that is, for the case where a local time does not exist for an anisotropic fractional Brownian field.}
The weak convergence of a normalized integral functional of a $d$-dimensional anisotropic self-similar field that depends on $N$ parameters to its local time is proved in Section 3. Theorem 3.1 holds under the assumption that a continuous local time exists for such a field. The normalizing factor in this theorem equals $\lambda^{N-(H_1+\cdots+H_N)H}$. In particular, Theorem 3.1 holds for a $d$-dimensional anisotropic fractional Brownian field $B^{H,d}$ that depends on $N$ parameters if $d < \frac{1}{H_1} + \cdots + \frac{1}{H_N}$.

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Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue, 6, Kyiv 03127, Ukraine
E-mail address: makoginv@ukr.net

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