Теорія Ймовір. та Матем. Статист.
Вип. 91, 2014

Theor. Probability and Math. Statist.
No. 91, 2015, Pages 115–129
http://dx.doi.org/10.1090/tpms/971
Article electronically published on February 4, 2016

A COUNTING PROCESS IN THE MAX-SCHEME

UDC 519.21

I. K. MATSAK

Abstract. The exact asymptotic behavior of a counting process is studied for the max-scheme in the case of independent random variables.

1. Introduction

Consider a sequence (ξ_n), $n \geq 1$, of independent identically distributed random variables with the distribution function $F(t) = \mathsf{P}(\xi_n < t)$. Let

$$z_n = \max_{1\leq i\leq n} \xi_i, \qquad N(t) = \min(n \geq 1 \colon z_n \geq t). \tag{1}$$

Then $N(t)$ is called a counting process constructed from the sequence (z_n).

It is clear that the process $N(t)$ can be defined in an equivalent way as follows:

$$N(t) = \min(n \geq 1 \colon \xi_n \geq t),$$

that is, $N(t)$ is the first moment when the sequence (ξ_n) hits the set $[t, \infty)$.

We are interested in studying the almost sure asymptotic behavior of the process $N(t)$ as $t \to \infty$. Definition (1) better fits our purpose, since the behavior of the process $N(t)$ is closely related to that of the sequence (z_n).

The following reasoning shows that extensions of results known for (z_n) to the case of $N(t)$ is not always trivial. It is known that $N(t)$ has a geometric distribution at each point t, that is,

$$\mathsf{P}(N(t) = k) = q(1-q)^{k-1}, \qquad k = 1, 2, \dots, \quad q = 1 - F(t).$$

This implies the following weak convergence:

$$(1 - F(t))N(t) \xrightarrow{D} \tau^e$$

as $t \to \infty$, where the random variable τ^e has the standard exponential distribution, $\mathsf{P}(\tau^e < x) = 1 - \exp(-x)$.

The latter relation means that there is no nonrandom function $a(t)$ such that the following strong law of large numbers

$$\frac{N(t)}{a(t)} \longrightarrow 1$$

holds almost surely (see [1, p. 42–43]). On the other hand, it is known that

$$\frac{z_n^e}{\ln n} \longrightarrow 1 \quad \text{a.s.}$$

2010 *Mathematics Subject Classification.* Primary 60G70.

Key words and phrases. Maximum of independent random variables, counting process, almost sure behavior.

©2016 American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

as $n \to \infty$ (see [2, Chapter 4, §3, Example 4.3.3]). Here and in what follows "a.s." abbreviates "almost surely".

Throughout the paper, we denote by $N^e(t)$ and (z_n^e) the counting process and sequence of maximums, respectively, both constructed from (τ_k^e) being independent copies of τ^e.

The asymptotic behavior of (z_n) has been studied by many authors ([3]–[8]). Clearly, the counting process $N(t)$ has also been studied (see [9]–[10] and the references therein to earlier papers).

The book [1] containing a rather thorough survey of general results for the so-called generalized renewal processes must also be mentioned here. In our language, the generalized renewal processes mean counting processes.

Below is one of the basic results concerning the asymptotic behavior of the counting process in the max-scheme [9].

Put

$$R(t) = -\ln(1 - F(t)) \quad \text{or} \quad F(t) = 1 - \exp(-R(t)),$$

Theorem A. *Let the distribution function $F(t)$ be continuous and strictly increasing. Furthermore assume that $F(t) < 1$ for all $t \in \mathbb{R}$. Then*

$$\limsup_{t\to\infty} \frac{\ln N(t) - R(t)}{\ln \ln R(t)} = 1 \quad a.s. \tag{2}$$

and

$$\liminf_{t\to\infty} \frac{\ln N(t) - R(t)}{\ln R(t)} = -1 \quad a.s. \tag{3}$$

It turns out that results like (2)–(3) hold in the discrete time only under additional assumptions on the rate of increase of the function $R(t)$. In particular, these results hold for the geometric and Poisson distributions; see [10].

If one considers the process $N(t)$ itself instead of $\ln N(t)$, then the problem becomes even more complicated and results like the law of the iterated logarithm of type (2)–(3) are not valid at all.

In this paper, we obtain some results on the upper and lower limits for appropriately normalized processes $N(t)$.

2. Asymptotic behavior of the counting process for continuous distributions

Theorem 1. *Let the distribution function $F(t)$ be continuous and strictly increasing. Furthermore assume that $F(t) < 1$ for all $t \in \mathbb{R}$. Then*

$$\limsup_{t\to\infty} \frac{N(t)}{\exp(R(t)) \ln R(t)} = 1 \quad a.s. \tag{4}$$

The proof of Theorem 1 is based on several auxiliary results.

Lemma 1. *Let ν be a geometric random variable with parameter q:*

$$\mathsf{P}(\nu = k) = p_k = q(1-q)^{k-1}, \qquad k \ge 1,\ 0 < q < 1.$$

Then

$$\mathsf{P}(q\nu > x) \le \frac{1}{1-q} \exp(-x)$$

for $x > 0$.

The proof of Lemma 1 follows easily from the following inequality:

$$\left(1 - \frac{1}{y}\right)^y \le \exp(-1) \quad \text{for } y \ge 1. \tag{5}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Indeed,

$$\mathsf{P}(q\nu > x) = (1-q)^{[x/q]} \le \frac{1}{1-q}(1-q)^{x/q} \le \frac{1}{1-q}\exp(-x).$$

Lemma 2. *Let (ξ_i) be a sequence of independent identically distributed random variables with the distribution function $F(x)$. Let (u_n) be a nondecreasing sequence of real numbers such that the sequence $n\,[1-F(u_n)]$ is nondecreasing, as well. Furthermore we assume that the function $F(x)$ is continuous. If $u_n < \omega(F)$ with $\omega(F) = \sup\,(x\colon F(x) < 1)$, then the probability*

$$\mathsf{P}\,(z_n \le u_n \ i.o.)$$

equals zero or one according to how the series

$$\sum_{n=1}^{\infty} [1-F(u_n)] \exp\{-n\,[1-F(u_n)]\} \tag{6}$$

converges or diverges where the abbreviation "i.o." means "infinitely often".

Lemma 2 is a known result (see [2, Theorem 4.3.1] or [6]).

Lemma 3. *We have*

$$\mathsf{P}(z_n^e \le \ln n - \ln\ln\ln n \ \ i.o.) = 1. \tag{7}$$

Proof of Lemma 3. Consider series (6) with $u_n = \ln n - \ln\ln\ln n$, where

$$F(t) = 1 - \exp(-t).$$

Then

$$\sum_{n=3}^{\infty} \exp\{-\ln n + \ln\ln\ln n\}\exp\{-n\exp\{-\ln n + \ln\ln\ln n\}\} = \sum_{n=3}^{\infty} \frac{(\ln\ln n)}{n\ln n} = \infty.$$

Since series (6) diverges, the latter result together with Lemma 2 implies (7). □

Proof of Theorem 1. It is well known (see, for example, [5]) that the assumptions of Theorem 1 imply that the random variables $\tau_i^e = R(\xi_i)$, $i \ge 1$, have the standard exponential distributions. If (z_n^e) and $N^e(t)$ are the corresponding sequence of maximums and counting process, then

$$\begin{aligned} N(t) &= \min(n \ge 1\colon z_n \ge t) = \min(n \ge 1\colon R(z_n) \ge R(t)) \\ &= \min(n \ge 1\colon z_n^e \ge R(t)) = N^e(R(t)). \end{aligned} \tag{8}$$

This makes clear that one may restrict the consideration to the case of the standard exponential distribution $F(t) = 1 - \exp(-t)$, $R(t) = t$ and prove that

$$\limsup_{t\to\infty} \frac{N^e(t)}{\exp(t)\ln t} = 1 \quad \text{a.s.} \tag{9}$$

First we show that

$$\limsup_{t\to\infty} \frac{N^e(t)}{\exp(t)\ln t} \le 1 + \varepsilon \quad \text{a.s.} \tag{10}$$

for all $\varepsilon > 0$.

Since $N^e(t)$ is geometrically distributed with parameter $q = 1 - F(t) = \exp(-t)$, we derive from Lemma 1 that

$$\mathsf{P}(\exp(-t)N^e(t) > x) \le \frac{\exp(-x)}{1-\exp(-t)}. \tag{11}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Further, we choose a sufficiently small $\delta > 0$ and put $t_k = k\delta$, $k = 1, 2, \dots$. Inequality (11) with $x = (1 + \varepsilon/2) \ln t_k$ and $t = t_k$ yields

$$\mathsf{P}\big(N^e(t_k) > (1 + \varepsilon/2)(\ln t_k) \exp(t_k)\big) \le \frac{(k\delta)^{-(1+\varepsilon/2)}}{1 - \exp(-k\delta)}.$$

Therefore the series

$$\sum_{k \ge 1} \mathsf{P}\big(N^e(t_k) > (1 + \varepsilon/2)(\ln t_k) \exp(t_k)\big)$$

converges and the Borel–Cantelli lemma implies that

$$\limsup_{k \to \infty} \frac{N^e(t_k)}{\exp(t_k) \ln t_k} \le 1 + \varepsilon/2 \quad \text{a.s.} \tag{12}$$

If $t_k < t \le t_{k+1}$, then

$$\frac{N^e(t)}{\exp(t) \ln t} \le \frac{N^e(t_{k+1})}{\exp(t_k) \ln t_k} = \frac{N^e(t_{k+1})}{\exp(t_{k+1}) \ln t_{k+1}} \frac{\exp(\delta) \ln \delta k}{\ln \delta(k+1)}.$$

Since

$$\frac{\ln \delta k}{\ln \delta(k+1)} \to 1$$

as $k \to \infty$, we deduce from (12) that

$$\limsup_{t \to \infty} \frac{N^e(t)}{\exp(t) \ln t} \le (1 + \varepsilon/2) \exp(\delta) \quad \text{a.s.}$$

Since δ is arbitrary, one can choose it so small that relation (10) holds.

It remains to show that

$$\limsup_{t \to \infty} \frac{N^e(t)}{\exp(t) \ln t} \ge (1 - \varepsilon) \quad \text{a.s.} \tag{13}$$

for all $\varepsilon > 0$.

Using Lemma 3 and equality (7) we prove that there exists a sequence of integer numbers $r_n = r_n(\omega)$ such that

$$z^e_{r_n} \le \ln r_n - \ln \ln \ln r_n \quad \text{a.s.} \tag{14}$$

for all $n \ge 1$.

Note that the set of values of the function $[(1 - \varepsilon) \exp(t) \ln t] + 1$ for t between 1 and ∞ coincides with the set of all positive integer numbers. Hence if ε is fixed, then there exists a sequence $t_n = t_n(\omega)$ such that

$$r_n = [(1 - \varepsilon) \exp(t_n) \ln t_n] + 1.$$

A simple algebra shows that

$$\ln r_n = t_n + \ln \ln t_n + \ln(1 - \varepsilon) + o(1)$$

and

$$\ln \ln \ln r_n = \ln \ln t_n + o(1).$$

The latter two equalities together with (14) show that

$$z^e_{r_n} \le t_n + \ln(1 - \varepsilon) + o(1) \quad \text{a.s.}$$

for all $n \ge 1$. In view of $\ln(1 - \varepsilon) < 0$, we conclude that there exists a positive integer number n_0 such that

$$z^e_{r_n} \le t_n \quad \text{a.s.}$$

for all $n > n_0$. This together with the equality

$$(N^e(t_n) \ge r_n) = (z^e_{r_n} \le t_n) \quad \text{a.s.} \tag{15}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

implies that there exists a positive integer number n_0 such that

$$N^e(t_n) \geq (1-\varepsilon)\exp(t_n)\ln t_n \quad \text{a.s.}$$

for all $n > n_0$. This proves relation (13).

Combining inequalities (10) and (13) we complete the proof of (9). □

Theorem 2. *Let the distribution function $F(t)$ be continuous and strictly increasing. Assume that $F(t) < 1$ for all $t \in \mathbb{R}$ and that $\psi(t)$ is a continuous strictly increasing function for $t > t_0$.*

(i) *If the series*

$$\sum_{n>1} \frac{1}{n\psi(n)} \tag{16}$$

converges, then

$$\lim_{t\to\infty} \frac{N(t)\psi(\exp(R(t)))}{\exp(R(t))} = \infty \quad a.s. \tag{17}$$

(ii) *If $\psi(t)$ is a slowly varying at infinity function such that series* (16) *diverges and*

$$\frac{\psi(t/\psi(t))}{\psi(t)} \to 1, \qquad t \to \infty, \tag{18}$$

then

$$\liminf_{t\to\infty} \frac{N(t)\psi(\exp(R(t)))}{\exp(R(t))} = 0 \quad a.s. \tag{19}$$

Remark 1. A typical function for which all the assumptions (ii) of Theorem 2 hold for sufficiently large t is given by

$$\psi(t) = \prod_{k=1}^{m} L_k(t),$$
$$L_1(t) = \ln t, \qquad L_2(t) = \ln L_1(t),$$
$$L_m(t) = \ln L_{m-1}(t).$$

Corollary 1. *Let all the assumptions of Theorem 2 hold. Then, for all $m \geq 1$ and $\varepsilon > 0$,*

$$\liminf_{t\to\infty} \frac{N(t)R(t)\prod_{k=1}^{m} L_k(R(t))}{\exp(R(t))} = 0 \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{N(t)R(t)\prod_{k=1}^{m-1} L_k(R(t))L_m^{1+\varepsilon}(R(t))}{\exp(R(t))} = \infty \quad a.s.$$

Proof of Theorem 2. (i) The argument in the proof of Theorem 1 makes it clear that one can restrict the consideration to the case of the standard exponential distribution. Relation (17) is rewritten in this case as follows:

$$\lim_{t\to\infty} \frac{N^e(t)\psi(\exp(t))}{\exp(t)} \quad \text{a.s.} = \infty. \tag{20}$$

We choose an arbitrary $C > 0$ and sufficiently small $\delta > 0$. Then we put

$$u_k = \frac{C\exp(t_k)}{\psi(\exp(t_k))}, \qquad t_k = k\delta, \qquad k = 1, 2, \dots .$$

Without loss of generality we assume that $\psi(t) = o(t)$ and $u_k \to \infty$ as $k \to \infty$ (bound (20) is obvious otherwise).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

The random variable $N^e(t)$ is geometrically distributed with parameter $q = \exp(-t)$. Thus

$$\mathsf{P}(N(t) \le u_k) = 1 - (1-q)^{[u_k]}. \tag{21}$$

Then we apply the following known inequalities:

$$1 - \exp(-x) = x + O\left(x^2\right) \quad \text{for small } x > 0$$

and

$$0 \le \exp(-x) - \left(1 - \frac{x}{n}\right)^n \le \frac{x^2 \exp(-x)}{2(n-1)}, \qquad n > 1,\ 0 < x \le n,$$

(see [12, Chapter 2, §4]).

Using equality (21) and the above inequalities we obtain

$$\mathsf{P}(N(t_k) \le u_k) = x + O\left(x^2\right) \sim \frac{C}{\psi\left(s^k\right)}, \tag{22}$$

where $s = \exp(\delta)$ and $x = C/\psi(\exp(t_k))$. As usual, the notation $g_1(t) \sim g_2(t)$ means that the functions g_1 and g_2 are asymptotically equivalent, that is, $g_1(t)/g_2(t) \to 1$ as $t \to \infty$.

The following Cauchy theorem is well known in calculus (see [11]): if $H(t)$ is a decreasing function and $s > 1$, then two series

$$\sum_{n \ge 1} H(n) \quad \text{and} \quad \sum_{k \ge 1} s^k H\left(s^k\right)$$

converge or diverge simultaneously. Choosing $H(t) = 1/(t\psi(t))$ and applying the above Cauchy theorem we conclude that

$$\sum_{k \ge 1} \frac{1}{\psi(s^k)} < \infty,$$

since series (16) converges. The latter estimate together with relation (22) and the Borel–Cantelli lemma yields

$$\mathsf{P}(\exists k_0, \forall k \ge k_0 \colon N(t_k) \ge u_k) = 1$$

or

$$\liminf_{k \to \infty} \frac{N(t_k)\psi(\exp(t_k))}{\exp(t_k)} \ge C \quad \text{a.s.},$$

where C is an arbitrary positive number, whence

$$\lim_{k \to \infty} \frac{N(t_k)\psi(\exp(t_k))}{\exp(t_k)} = \infty \quad \text{a.s.}$$

Let $t \in [t_k, t_{k+1})$ be an arbitrary number. Then

$$\frac{N(t)\psi(\exp(t))}{\exp(t)} \ge \frac{N(t_k)\psi(\exp(t_k))}{\exp(t_k)} \exp(-\delta) \to \infty \quad \text{a.s.}$$

as $k \to \infty$, that is, equality (20) is established.

(ii) First we recall a result for Karamata's regularly varying functions.

Let $g(t) = tL(t)$, where $L(t)$ is a slowly varying at infinity function such that $g(t)$ is nondecreasing in $[A, \infty)$ for some A. Then

$$g^{-1}(y) = \inf(t \colon g(t) \ge y, t \in [A, \infty)) = yL^*(y), \tag{23}$$

where $L^*(t)$ is a conjugate function to $L(t)$. It is well known that $L^*(t)$ is slowly varying at infinity, and

$$L(L^*(t)) \sim t, \qquad L^*(L(t)) \sim t$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

(see [13, Chapter 1, Lemmas 1.8 and 1.10]). Moreover, if the function $L(t)$ satisfies condition (18), then

$$L^*(t) \sim \frac{1}{L(t)}. \tag{24}$$

We consider a random variable $\widetilde{\xi}$ with the distribution function $\widetilde{F}(t) = 1 - 1/t$, $t \geq 1$. Then $\widetilde{R}(t) = \ln t$. Now our aim is to check equality (19).

Let $(\widetilde{z}_n)$ and $(\widetilde{N}(t))$ be the sequence of maximums and counting process, respectively, constructed from the sequence $(\widetilde{\xi}_k)$, where $\widetilde{\xi}_k$ are independent copies of the random variable $\widetilde{\xi}$. If series (16) diverges, then

$$\limsup_{n\to\infty} \frac{\widetilde{z}_n}{n\psi(n)} = \infty \quad \text{a.s.}$$

(see [2, Example 4.3.4]). Let $C > 0$ be an arbitrary number. Then

$$\mathsf{P}(\widetilde{z}_n \geq Cn\psi(n) \text{ i.o.}) = 1.$$

We obtain from the latter equality and (15) that

$$\mathsf{P}(\widetilde{N}\big(Cn\psi(n)\big) \leq n \text{ i.o.}) = 1. \tag{25}$$

Put $\varphi(t) = t\psi(t)$, $t_n = Cn\psi(n)$ and fix $\varepsilon > 0$. According to equalities (25) and (23) we get

$$\mathsf{P}\left(\widetilde{N}(t_n) \leq \varphi^{-1}\left(\frac{t_n}{C}\right) \text{ i.o.}\right) = \mathsf{P}\left(\widetilde{N}(t_n) \leq (1+\varepsilon)\frac{t_n}{C}\psi^*\left(\frac{t_n}{C}\right) \text{ i.o.}\right) = 1.$$

Since $\psi^*(t)$ is a regularly varying function, we obtain the following estimate:

$$\liminf_{n\to\infty} \frac{\widetilde{N}}{(t_n)\psi^*((t_n))} \leq \frac{1+2\varepsilon}{C} \quad \text{a.s.}$$

Since C is an arbitrary number, the latter result together with (24) yields

$$\liminf_{t\to\infty} \frac{\widetilde{N}(t)}{t\psi^*(t)} = \liminf_{t\to\infty} \frac{\widetilde{N}(t)\psi(t)}{t} = 0 \quad \text{a.s.} \tag{26}$$

and equality (19) is proved for the distribution function $\widetilde{F}$.

Then we apply (8) with $\widetilde{R}(t) = \ln t$. Thus (26) implies

$$\liminf_{t\to\infty} \frac{N^e(t)\psi(\exp(t))}{\exp(t)} = 0 \quad \text{a.s.}$$

For the general case, we substitute $t = R(z)$ and apply (8) once more in the preceding equality. This proves equality (19). □

Remark 2. The proof above implies that if all the assumptions (ii) in Theorem 2 hold, except limit relation (18), then

$$\liminf_{t\to\infty} \frac{N(t)}{\exp(R(t))\psi^*\big(\exp(R(t))\big)} = 0 \quad \text{a.s.},$$

where $\psi^*(t)$ is a conjugate function to $\psi(t)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

3. Asymptotic behavior of the counting process. Discrete time

Throughout this section, ξ denotes a discrete random variable with the distribution (k, p_k), $k \geq 1$, that is,

$$\mathsf{P}(\xi = k) = p_k > 0, \qquad \sum_{k=1}^{\infty} p_k = 1,$$

$$R(k) = -\ln\Bigl(\sum_{i \geq k} p_i\Bigr), \qquad r(k) = R(k) - R(k-1).$$

Let ξ_i, $i \geq 1$, be independent copies of ξ and let z_n and $N(t)$ be defined by equality (1). The definition of the process N implies that

$$N(t) = N(k+1) \quad \text{for } k < t \leq k+1 \quad \text{a.s.}$$

This means that the process $N(t)$ can be considered only for $t = k$, $k \geq 1$.

The results given below show that analogs of Theorems 1 and 2 in the discrete time hold only under some additional restrictions imposed on the rate of increase of the function $R(t)$.

Theorem 3. *If ξ is a discrete random variable with the distribution (k, p_k), then*

(i)

$$\limsup_{k\to\infty} \frac{N(k)}{\exp(R(k)) \ln R(k)} \leq 1 \quad a.s. \tag{27}$$

(ii) *If*

$$r(k) \to 0, \qquad k \to \infty, \tag{28}$$

then

$$\limsup_{k\to\infty} \frac{N(k)}{\exp(R(k)) \ln R(k)} = 1 \quad a.s. \tag{29}$$

(iii) *If*

$$\exp(-r(k)) \ln R(k) \to 0, \qquad k \to \infty, \tag{30}$$

and if

$$\sum_{k>1} R(k)^{-b} < \infty \tag{31}$$

for $0 < b < 1$, then

$$\limsup_{k\to\infty} \frac{N(k)}{\exp(R(k)) \ln R(k)} \leq b \quad a.s. \tag{32}$$

Proof of Theorem 3. (i) We need some representations of discrete random variables obtained in [10].

Construct a random variable ξ^c with the continuous distribution function $F^c(t)$ as follows. Put

$$F^c(1) = 0, \qquad F^c(k) = \sum_{i<k} p_i \quad \text{for } k > 1,$$

$$F^c(t) = F^c(k) + (t-k)p_k \quad \text{for } k \leq t < k+1,$$

and

$$R^c(t) = -\ln\left(1 - F^c(t)\right).$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

It is obvious that

$$R^c(k) = -\ln\left(\sum_{i\geq k} p_i\right) = R(k). \tag{33}$$

As usual, the symbol $[t]$ stands for the integer part of a number t.

Consider the random variable $\xi^d = [\xi^c]$. It is clear that

$$\mathsf{P}\left(\xi^d = k\right) = \mathsf{P}\left(\xi^c \in [k, k+1)\right) = F^c(k+1) - F^c(k) = p_k.$$

Hence the random variables ξ^d and ξ are identically distributed. Thus one can assume without loss of generality that $\xi = \xi^d = [\xi^c]$.

Let ξ_i^c be independent copies of ξ^c,

$$z_n^c = \max_{1\leq i\leq n} \xi_i^c, \qquad N^c(t) = \min\left(n \geq 1\colon z_n^c \geq t\right).$$

By construction, the distribution function $F^c(t)$ is continuous, strictly increasing, and such that $F^c(t) < 1$ for all $t \in \mathbb{R}$. According to Theorem 1 we get

$$\limsup_{t\to\infty} \frac{N^c(t)}{\exp(R^c(t)) \ln R^c(t)} = 1 \quad \text{a.s.} \tag{34}$$

This implies that

$$\limsup_{k\to\infty} \frac{N^c(k)}{\exp\left(R^c(k)\right) \ln R^c(k)} \leq 1 \quad \text{a.s.} \tag{35}$$

Taking into account the inequality

$$\{x \geq k\} = \{[x] \geq k\},$$

we obtain

$$N(k) = N^c(k) \quad \text{a.s.} \tag{36}$$

Inequality (27) follows from relations (33), (35), and (36).

(ii) Let $\varepsilon > 0$ be an arbitrary number. Equality (34) means that there exists a sequence $(t_n = t_n(\omega))$ such that

$$\limsup_{t_n\to\infty} \frac{N^c(t_n)}{\exp(R^c(t_n)) \ln R^c(t_n)} \geq 1-\varepsilon \quad \text{a.s.} \tag{37}$$

Put $k_n = [t_n]$. Then

$$N^c((k_n+1)) \geq N^c(t_n), \qquad R^c(k_n) \leq R^c(t_n).$$

This together with (37) establishes

$$\limsup_{n\to\infty} \frac{N^c(k_n+1)}{\exp(R^c(k_n)) \ln R^c(k_n)} \geq 1-\varepsilon \quad \text{a.s.}$$

Combining this result with (33) and (36), we get

$$\limsup_{n\to\infty} \frac{N(k_n+1)}{\exp(R(k_n)) \ln R(k_n)} \geq 1-\varepsilon \quad \text{a.s.}$$

According to condition (28),

$$\frac{\exp(R((k+1))}{\exp(R(k))} \to 1$$

as $k \to \infty$. Thus

$$\limsup_{n\to\infty} \frac{N(k_n)}{\exp(R(k_n)) \ln R(k_n)} \geq 1-\varepsilon \quad \text{a.s.}$$

Since ε is an arbitrary positive number, the latter bound together with inequality (27) proves equality (29).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

(iii) Assume that inequality (32) does not hold. Then there exists a number $b^{'} > b$ such that

$$\mathsf{P}(N(k) \geq h(k) \text{ i.o.}) = 1,$$

where $h(k) = [b' \exp(R(k)) \ln R(k)]$.

We conclude from the latter result and (15) that

$$\mathsf{P}(z_{h(k)} \leq k \text{ i.o.}) = 1. \tag{38}$$

Put $X_k = \max_{h(k-1)<i\leq h(k)} \xi_i$. The following equality is an immediate corollary of relation (38):

$$\mathsf{P}(X_k \leq k \text{ i.o.}) = 1.$$

Since X_k are independent random variables, the Borel–Cantelli lemma implies that the series

$$\sum_{k>1} \mathsf{P}(X_k \leq k) \tag{39}$$

diverges. Now we estimate from above the terms of series (39). Using bound (5) and condition (30) we obtain

$$\begin{aligned} \mathsf{P}(X_k \leq k) &= \big(1 - \exp(-R(k))\big)^{h(k)-h(k-1)} \\ &\leq \exp\big(-b'(\ln R(k) - \exp(-r(k) \ln R(k-1)) + o(1))\big) \\ &\leq \frac{C}{R(k)^{b'}}. \end{aligned}$$

In view of condition (31), the latter bound implies the convergence of series (39). This contradiction completes the proof. □

Theorem 4. *Let ξ be a discrete random variable with the distribution (k, p_k) and let $\psi(t)$ be a continuous strictly increasing function for $t > t_0$. Let*

$$r(k) = R(k) - R(k-1) \leq C < \infty. \tag{40}$$

(i) *If series* (16) *converges, then*

$$\lim_{k\to\infty} \frac{N(k)\psi\big(\exp(R(k))\big)}{\exp(R(k))} = \infty \quad a.s.$$

(ii) *If $\psi(t)$ is a slowly varying at infinity function, the series* (16) *diverges, and condition* (18) *holds, then*

$$\liminf_{k\to\infty} \frac{N(k)\psi\big(\exp(R(k))\big)}{\exp(R(k))} = 0 \quad a.s.$$

The proof of Theorem 4 follows from Theorem 2 by reasoning used in the proof of Theorem 3. Details are omitted.

The following result explains the complications arising when dealing with the max-scheme in the case of discrete time.

Put $a_n = \max\big(k\colon \sum_{i\geq k} p_i \geq 1/n\big)$.

Theorem 5. *Let ξ be a discrete random variable with the distribution (k, p_k) and let the function $r(k)$ increase and satisfy the following condition:*

$$\sum_{k\geq 1} r(k+1) \exp(-r(k)) < \infty. \tag{41}$$

Then

$$\mathsf{P}(\exists n_0, \forall n \geq n_0\colon A_n) = 1, \tag{42}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

where

$$A_n = \{z_n = a_n\} \cup \{z_n = a_n - 1\} \cup \{z_n = a_n + 1\}.$$

Remark 3. Below are typical examples where condition (41) holds:

$$p_k \sim \exp(-Ck \ln k + O(k)), \qquad C > 1, \tag{43}$$
$$p_k \sim \exp(-Ck^\alpha), \qquad \alpha > 1,\ C > 0,$$
$$p_k \sim \exp(-\exp(Ck^\alpha)), \qquad \alpha > 0,\ C > 0.$$

Note however that the Poisson distribution as well as the geometric distribution does not satisfy these conditions. For the Poisson distribution, condition (43) holds only for a single value $C = 1$ which is not sufficient for (41) in the general case.

Proof of Theorem 5. Put

$$R^c(t) = R(k) + (t-k)r(k+1) \quad \text{for } t \in [k, k+1),\ k \geq 1,$$
$$r^c(t) = r(k+1) \quad \text{for } t \in [k, k+1),\ k \geq 1.$$

It is clear that

$$R^c(t) = \int_1^t r^c(s)\,ds. \tag{44}$$

Consider the distribution function

$$F^c(t) = 1 - \exp(-R^c(t)), \qquad t \geq 1, \tag{45}$$
$$F^c(1) = 0.$$

Let ξ^c be a random variable with the distribution function $F^c(t)$ and $\xi^d = [\xi^c]$. Then, for all $k \geq 1$,

$$\mathsf{P}\left(\xi^d = k\right) = F^c(k+1) - F^c(k) = \exp(-R(k)) - \exp(-R(k+1)) = p_k,$$

that is, the random variables ξ^d and ξ are identically distributed. Thus one may assume without loss of generality that $\xi_i \equiv \xi_i^d$, where ξ_i^d are independent copies of ξ^d.

Next we prove under the assumptions of Theorem 5 that, for all $\varepsilon > 0$,

$$\int_1^\infty \frac{dF^c(t)}{1 - F^c(t-\varepsilon)} < \infty. \tag{46}$$

Bound (46) (see [3]) guarantees the asymptotic almost sure stability of extreme values, that is,

$$z_n^c - a_n^c \to 0 \quad \text{a.s.} \tag{47}$$

as $n \to \infty$, where z_n^c are constructed from independent identically distributed random variables with the distribution function $F^c(t)$ and where

$$a_n^c = \inf\left(y\colon F^c(y) \geq 1 - 1/n\right).$$

The function $F^c(t)$ is continuous and strictly increasing. Thus

$$F^c(a_n^c) = 1 - \frac{1}{n}.$$

This implies that

$$a_n^c = k + \frac{\ln n - R(k)}{r(k+1)} \quad \text{for } \ln n \in [R(k), R(k+1)) \tag{48}$$

and

$$[a_n^c] = k \quad \text{for } \ln n \in [R(k), R(k+1))$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

or

$$[a_n^c] = \max(k\colon \ln n \ge R(k)) = \max\left(k\colon \sum_{i\ge k} p_i \ge \frac{1}{n}\right) = a_n.$$

The latter equality and relation (47) lead to a conclusion that if n is sufficiently large, then only the following three cases are possible:

$$[z_n^c] = \begin{matrix} a_n, \quad \text{or} \\ a_n - 1, \quad \text{or} \\ a_n + 1. \end{matrix}$$

Since

$$[z_n^c] = z_n^d = z_n$$

almost surely for all n, equality (42) is proved.

It remains to prove equality (46). Using equalities (44) and (45) we have, for a sufficiently small $\varepsilon > 0$,

$$(49)\qquad \begin{aligned} \int_k^{k+1} \frac{dF^c(t)}{1 - F^c(t-\varepsilon)} &= r(k+1) \int_k^{k+1} \frac{\exp(-R^c(t))\,dt}{\exp(-R^c(t-\varepsilon))} \\ &= r(k+1) \int_k^{k+1} \exp\left(-\int_{t-\varepsilon}^t r^c(s)\,ds\right) dt \\ &\le r(k+1)((1-\varepsilon)\exp(-r(k+1)) + \varepsilon \exp(-r(k))). \end{aligned}$$

Now inequality (46) follows from condition (41) in view of bounds (49). □

The reasoning in the proof of Theorem 5 proves the following result.

Corollary 2. *Let a_n^c be defined by equality* (48) *and let there exist a subsequence of integer numbers (n_k) such that*

$$\lim_{k\to\infty} (a_{n_k}^c - a_{n_k}) = \alpha > 0.$$

Then

$$\mathsf{P}(\exists k_0, \forall k \ge k_0\colon z_{n_k} = a_{n_k}) = 1.$$

4. Examples

Below we consider several applications of the above results for some important distributions. Throughout below we assume that (ξ_n) is a sequence of independent copies of a random variable ξ and that the counting process $N(t)$ is constructed from the sequence (ξ_n) in a usual way by using equality (1).

Example 1 (Standard normal distribution)**.** Let $\xi = \gamma$, where γ is a random variable with the standard normal distribution function $\Phi(t)$,

$$\Phi(t) = \int_{-\infty}^t \varphi(s)\,ds, \qquad \varphi(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right).$$

The following relation for the tail of the function $\Phi(t)$ is well known (see, for example, [12, Chapter 1, §5]): as $t \to \infty$,

$$1 - \Phi(t) = \frac{\varphi(t)}{t}(1 + o(1)).$$

This relation implies that

$$(50)\qquad R(t) = \frac{t^2}{2} + \ln t + \ln\sqrt{2\pi} + o(1).$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

According to Theorem 1,

$$\limsup_{t\to\infty} \frac{N(t)}{\exp(t^2/2)\, t\ln t} = 2\sqrt{2\pi} \quad \text{a.s.}$$

If $\psi(t)$ is a strictly increasing slowly varying function such that condition (18) holds, then Theorem 2 yields

$$\liminf_{t\to\infty} \frac{N(t)\psi(t\exp(t^2/2))}{t\exp(t^2/2)} = \infty \quad \text{or} \quad = 0 \quad \text{a.s.}$$

according to how series (16) converges or diverges.

Example 2. Let γ be a standard normal random variable. Put $\xi = [\gamma]$. Then it is easy to check that

$$\mathsf{P}(\xi \geq k) = \mathsf{P}(\gamma \geq k) = \exp(-R(k)),$$

where $R(t)$ is defined by (50) and

$$r(k) = R(k) - R(k-1) = k - \frac{1}{2} + o(1).$$

Since assumptions (30) and (31) of Theorem 3 obviously hold if $b > \frac{1}{2}$, we conclude that

$$\limsup_{t\to\infty} \frac{N(k)}{\exp(k^2/2)\, k\ln k} \leq \sqrt{2\pi} \quad \text{a.s.}$$

If (γ_i) is a sequence of independent copies of the random variable (γ) and $z_n^\gamma = \max_{1\leq i\leq n} \gamma_i$, then

$$z_n^\gamma - \sqrt{2\ln n} \to 0 \quad \text{a.s.}$$

as $n \to \infty$ (see [2, §4.4]).

It is not complicated to check that condition (41) also holds. Moreover, $z_n = [z_n^\gamma]$. Then equality (42) holds by Theorem 5 with $a_n = [\sqrt{2\ln n}]$.

Remark 4. Put $x_n = \sqrt{2\ln n} - [\sqrt{2\ln n}]$. It is known that the sequence (x_n) is such that, given an arbitrary $\alpha \leq 1$, there exists a sequence (n_k) depending on α, $(n_k = n_k(\alpha))$, and such that

$$x_{n_k} \to \alpha, \qquad k \to \infty, \tag{51}$$

(see [14, Chapter 4, Problem 183]. If condition (51) with $0 < \alpha < 1$ holds for a subsequence (n_k), then

$$\mathsf{P}\left(\exists\, k_0 \forall\, k \geq k_0 \colon z_{n_k} = [\sqrt{2\ln n_k}]\right) = 1$$

by Corollary 2.

Example 3. Let $L(x)$ be a slowly varying at infinity function, $\beta > 1$,

$$\mathsf{P}(\xi = k) = p_k = \frac{C}{k^\beta L(k)}, \qquad k \geq 1,$$

and

$$C = \left(\sum_{k\geq 1} \frac{1}{k^\beta L(k)}\right)^{-1}.$$

It is known that (see [10])

$$R(k) = -\ln \sum_{i\geq k} p_i = -\ln C + \ln(\beta - 1) + (\beta - 1)\ln k + \ln L(k) + o(1)$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

and

$$R(k+1) - R(k) = (\beta - 1) \ln \left(1 + \frac{1}{k}\right) + \ln \frac{L(k+1)}{L(k)} + o(1) \to 0$$

as $k \to \infty$.

Thus condition (28) of Theorem 3 holds and hence

$$\limsup_{k\to\infty} \frac{N(k)}{k^{\beta-1} L(k) \ln\ln k} = \frac{\beta - 1}{C} \quad \text{a.s.}$$

It is also clear that condition (40) of Theorem 4 holds. Thus if $\psi(t)$ is a strictly increasing slowly varying at infinity function such that condition (18) of Theorem 2 holds, then one may conclude that

$$\liminf_{k\to\infty} \frac{N(k)\psi(k^{\beta-1}L(k))}{k^{\beta-1}L(k)} = \infty \quad \text{or} \quad = 0$$

almost surely according to how series (16) converges or diverges.

Example 4 (Geometric distribution)**.** Let

$$\mathsf{P}(\xi = k) = p_k = q(1-q)^{k-1}, \qquad k \geq 1,\ 0 < q < 1.$$

Here

$$R(k) = -\ln \sum_{i \geq k} q(1-q)^{k-1} = (k-1) \ln \frac{1}{1-q}$$

and

$$r(k) = R(k) - R(k-1) = \ln \frac{1}{1-q}.$$

It is obvious that condition (28) of Theorem 3 does not hold but condition (40) of Theorem 4 holds. Thus

$$\limsup_{k\to\infty} \frac{N(k)(1-q)^{k-1}}{\ln k} \leq 1 \quad \text{a.s.} \tag{52}$$

If $\psi(t)$ is the same function as in Example 3, then

$$\liminf_{k\to\infty} N(k)(1-q)^{k-1} \psi\left((1-q)^{-k+1}\right) = \infty \quad \text{or} \quad = 0$$

almost surely according to how series (16) converges or diverges.

Remark 5. We do not know at the moment whether or not the inequality in (52) can be changed by an equality.

A random variable with a geometric distribution can be represented as follows. Let τ^e be a standard exponential random variable and let (z_n^e) be the corresponding sequence of maximums. Consider the random variable $\xi = [\tau^e/\lambda]$, $\lambda > 0$. Then

$$\mathsf{P}(\xi = k) = \mathsf{P}(\tau^e/\lambda \in [k-1, k)) = (1-q)^{k-1} q,$$

where $q = 1 - \exp(-\lambda)$, that is, ξ is a geometric random variable.

This implies

$$z_n^e/\lambda = z_n + \theta, \qquad |\theta| \leq 1.$$

According to results of [5, 8],

$$\limsup_{n\to\infty} \frac{z_n^e - \ln n}{\ln\ln n} = 1 \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \frac{z_n^e - \ln n}{\ln\ln\ln n} = -1 \quad \text{a.s.}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Combining together the latter two equalities we obtain the following form of the law of the iterated logarithm for geometric random variables:

$$\limsup_{n\to\infty} \frac{z_n \ln \frac{1}{1-q} - \ln n}{\ln\ln n} = 1 \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \frac{z_n \ln \frac{1}{1-q} - \ln n}{\ln\ln\ln n} = -1 \quad \text{a.s.}$$

Bibliography

1. V. V. Buldygin, K.-H. Indlekofer, O. I. Klesov, and J. G. Steinebach, *Pseudo-Regularly Varying Functions and Generalized Renewal Processes*, 'TViMS", Kyiv, 2012. (Ukrainian)
2. J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, John Wiley & Sons, New York, 1978. MR489334 (80b:60040)
3. Ole Barndorff-Nielsen, *On the limit behaviour of extreme order statistics*, Ann. Math. Statist. **34** (1963), no. 3, 992–1002. MR0150889 (27:875)
4. J. Pickands, *An iterated logarithm law for the maximum in a stationary Gaussian sequence*, Z. Warhscheinlichkeitstheorie verw. Geb. **12** (1969), no. 3, 344–355. MR0251776 (40:5003)
5. L. de Haan and A. Hordijk, *The rate of growth of sample maxima*, Ann. Math. Statist. **43** (1972), 1185–1196. MR0312550 (47:1107)
6. M. J. Klass, *The Robbins–Siegmund criterion for partial maxima*, Ann. Probab. **13** (1985), 1369–1370. MR806233 (87b:60046)
7. L. de Haan and A. Ferreira, *Extreme Values Theory: An Introduction*, Springer, Berlin, 2006. MR2234156 (2007g:62008)
8. K. S. Akbash and I. K. Matsak, *An improvement of the law of the iterated logarithm for the max-scheme*, Ukr. Mat. Zh. **64** (2012), no. 8, 1132–1137; English transl in Ukrainian Math. J. **64** (2013), no. 8, 1290–1296. MR3104866
9. I. K. Matsak, *Asymptotic behavior of a counting process in the max-scheme*, Ukr. Mat. Zh. **65** (2013), no. 11, 1575–1579; English transl in Ukrainian Math. J. **65** (2014), no. 11, 1743–1748. MR3216868
10. I. Matsak and I. Rozora, *Asymptotic behaviour of counting process in the max-scheme. Discrete case*, Georg. Math. J. (to appear)
11. G. M. Fikhtengol'ts, *Course of Differential and Integral Calculus*, vol. 2, "Nauka", Moscow, 1969. (Russian)
12. M. R. Leadbetter, G. Lindgren, and H. Rootzen, *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, New York, Heidelberg, Berlin, 1983. MR691492 (84h:60050)
13. E. Seneta, *Regularly Varying Functions*, Springer-Verlag, New York–Heidelberg–Berlin, 1976. MR0453936 (56:12189)
14. G. Pólya and G. Szegö, *Problems and Theorems in Analysis I: Series, Integral Calculus, Theory of Functions*, Springer-Verlag, New York–Heidelberg–Berlin, 1972. MR0344042 (49:8782)

Faculty for Cybernetics, National Taras Shevchenko University, Academician Glushkov Avenue, 6, Kyiv 03127, Ukraine

E-mail address: ivanmatsak@univ.kiev.ua

Received 06/AUG/2014

Translated by N. SEMENOV

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use