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ANALYTIC PROPERTIES OF INFINITE-HORIZON SURVIVAL PROBABILITY IN A RISK MODEL WITH ADDITIONAL FUNDS

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Abstract. We consider a generalization of the classical risk model where an insurance company gets additional funds whenever a claim arrives. We investigate the properties of continuity and differentiability of the infinite-horizon survival probability and derive an integro-differential equation. We find a closed form solution of this equation in the case where the claim sizes and additional funds are exponentially distributed.

1. Introduction

Let all random variables considered below be defined on a common probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. We consider a risk model with continuous time that generalizes the classical model.

The classical risk model (see, for example, [1, 2, 3, 4]) assumes that an insurance company starts with an initial surplus $x \ge 0$ and experiences premiums arriving with a constant rate $c > 0$. The sizes of claims arriving at the company form a sequence $(\xi_i)_{i\ge 1}$ of non-negative independent identically distributed random variables with the distribution function $F_1(y) = \mathsf{P}[\xi_i \le y]$ and finite expectation $\mathsf{E}[\xi_i] = \mu_1$. The moment when an i^{th} claim arrives is denoted by τ_i. Put $\tau_0 = 0$. The total number of claims arriving at the company in the time interval $[0, t]$ is modelled by a homogeneous Poisson process $(N_t)_{t\ge 0}$ with intensity $\lambda > 0$.

In contrast to the classical risk model we assume that an insurance company gets an additional fund η_i at the moment τ_i when an i^{th} claim arrives. This money can be interpreted as an investment income or any other additional income. We assume that $(\eta_i)_{i\ge 1}$ is a sequence of non-negative independent identically distributed random variables with the distribution function $F_2(y) = \mathsf{P}[\eta_i \le y]$ and finite expectation $\mathsf{E}[\eta_i] = \mu_2$. In addition, we assume that the sequences $(\xi_i)_{i\ge 1}$ and $(\eta_i)_{i\ge 1}$ and the process $(N_t)_{t\ge 0}$ are jointly independent. Put $\zeta_i = \eta_i - \xi_i$, $i \ge 1$.

Let $X_t(x)$ be the capital of an insurance company at moment t provided its initial surplus equals x. Then the above assumptions lead to

$$X_t(x) = x + ct - \sum_{i=1}^{N_t} (\xi_i - \eta_i), \qquad t \ge 0. \tag{1}$$

Here and throughout below the sums equal zero if the upper limit of summation is less than the lower limit. In particular, $\sum_{i=1}^{0} (\xi_i - \eta_i) = 0$ in (1) if $N_t = 0$.

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Then the infinite-horizon ruin probability as a function of the initial surplus is defined by

$$\psi(x) = \mathsf{P}\left[\inf_{t\geq 0} X_t(x) < 0\right]$$

for all $x \geq 0$. Correspondingly, the infinite-horizon survival probability as a function of x is equal to

$$\varphi(x) = 1 - \psi(x).$$

It is clear that $\psi(x)$ is a non-increasing function, while $\varphi(x)$ is a non-decreasing function. On the other hand, such analytic properties of $\psi(x)$ and $\varphi(x)$ as differentiability or even continuity are not obvious and cannot easily be derived from the definitions of these functions. It turns out that these functions are not necessarily differentiable for all $x \geq 0$. Moreover, the results of the paper [5] imply that $\psi(x)$ and $\varphi(x)$ can even be discontinuous under certain conditions in some models.

The properties of continuity and differentiability of the ruin (survival) probabilities are needed, in the first place, to derive integro-differential equations for these functions that, in turn, are used to find exact or approximate expressions for the solutions as well as to study their asymptotic behavior or to construct estimators, or solve optimization problems for them, etc. Properties of continuity and differentiability for the survival probabilities in various non-classical risk models are studied in detail in [6, 7, 8, 9]. These problems are also considered in [1] for the classical risk model by using a different approach. General properties of homogeneous processes with independent increments and negative jumps are studied, for example, in [10, 11].

The paper is organized as follows. Some auxiliary results are given in Section 2. The continuity and differentiability of the survival probability are studied in Section 3 and an integro-differential equation is derived for this function. The case of the exponential random variables ξ_i and η_i, $i \geq 1$, is considered in Section 4.

Throughout below we assume that

$$\mathsf{P}[\xi_i - \eta_i > 0] > 0 \quad \text{and} \quad \mathsf{P}[\xi_i - \eta_i < 0] > 0.$$

The cases $\mathsf{P}[\xi_i - \eta_i \leq 0] = 1$ and $\mathsf{P}[\xi_i - \eta_i \geq 0] = 1$ are trivial. Indeed, the ruin never happens in the first case, while the second one reduces to the classical risk model.

2. Auxiliary results

Lemma 2.1. *Let the evolution of the capital of an insurance company be described by equality* (1).

(i) *If $c - \lambda\mu_1 + \lambda\mu_2 \leq 0$, then $\varphi(x) = 0$ for all $x \geq 0$.*
(ii) *If $c - \lambda\mu_1 + \lambda\mu_2 > 0$, then $\lim_{x\to+\infty} \varphi(x) = 1$.*

We need the following properties of random walks to prove Lemma 2.1.

Theorem 2.1 ([1, Theorem 6.3.1, p. 233])**.** *Let $(\tilde{\zeta}_i)_{i\geq 1}$ be a sequence of independent identically distributed random variables such that $\mathsf{P}[\tilde{\zeta}_i > 0] > 0$, $\mathsf{P}[\tilde{\zeta}_i < 0] > 0$, and $\mathsf{E}[\tilde{\zeta}_i] < \infty$. Let $S_n = \sum_{i=1}^n \tilde{\zeta}_i$.*

(i) *If $\mathsf{E}[\tilde{\zeta}_i] > 0$, then $\mathsf{P}\big[\lim_{n\to\infty} S_n = +\infty\big] = 1$.*
(ii) *If $\mathsf{E}[\tilde{\zeta}_i] < 0$, then $\mathsf{P}\big[\lim_{n\to\infty} S_n = -\infty\big] = 1$.*
(iii) *If $\mathsf{E}[\tilde{\zeta}_i] = 0$, then*

$$\mathsf{P}\left[\limsup_{n\to\infty} S_n = +\infty\right] = 1 \quad \textit{and} \quad \mathsf{P}\left[\liminf_{n\to\infty} S_n = -\infty\right] = 1.$$

The proof of Theorem 2.1 is rather easy for the first two cases. Indeed, the strong law of large numbers implies that $\mathsf{P}\big[\lim_{n\to\infty} S_n/n = \mathsf{E}[\tilde{\zeta}_i]\big] = 1$, whence

$$\mathsf{P}\Big[\lim_{n\to\infty} S_n = +\infty\Big] = 1 \quad \text{if } \mathsf{E}[\tilde{\zeta}_i] > 0 \quad \text{or} \quad \mathsf{P}\Big[\lim_{n\to\infty} S_n = -\infty\Big] = 1 \quad \text{if } \mathsf{E}[\tilde{\zeta}_i] < 0.$$

The case of $\mathsf{E}[\tilde{\zeta}_i] = 0$ is much more complicated and is proved, for example, in [1].

Proof of Lemma 2.1. We follow the lines of the proof in [1, pp. 153, 162] for the classical risk model. Consider random variables

$$\tilde{\zeta}_i = \xi_i - \eta_i - c(\tau_i - \tau_{i-1}), \qquad i \ge 1.$$

Note that the assumptions of Theorem 2.1 hold for the random variables $\tilde{\zeta}_i$ and

$$\mathsf{E}[\tilde{\zeta}_i] = \mu_1 - \mu_2 - c/\lambda.$$

Let $S_n = \sum_{i=1}^n \tilde{\zeta}_i$ and $M = \sup_{t\ge 0}\big(\sum_{i=1}^{N_t}(\xi_i - \eta_i) - ct\big)$. It is easy to see that M can be represented as $M = \sup_{n\ge 1} S_n$. This implies

$$\psi(x) = \mathsf{P}[M > x]. \tag{2}$$

If $c - \lambda\mu_1 + \lambda\mu_2 \le 0$, then $\mathsf{E}[\tilde{\zeta}_i] \ge 0$ and $\mathsf{P}\big[\limsup_{n\to\infty} S_n = +\infty\big] = 1$ in view of Theorem 2.1. Thus $\psi(x) = 1$ for all $x \ge 0$ in view of (2) and this is equivalent to $\varphi(x) = 0$.

If $c - \lambda\mu_1 + \lambda\mu_2 > 0$, then $\mathsf{E}[\tilde{\zeta}_i] < 0$ and $\mathsf{P}\big[\lim_{n\to\infty} S_n = -\infty\big] = 1$ by Theorem 2.1. Thus $\mathsf{P}[M < \infty] = 1$, whence $\lim_{x\to+\infty}\psi(x) = 0$ in view of (2) which is equivalent to $\lim_{x\to+\infty}\varphi(x) = 1$. The lemma is proved. □

Remark 2.1. The condition $c - \lambda\mu_1 + \lambda\mu_2 > 0$ is an analog of the assumption that the income is positive in the classical risk model. This condition means that an insurance company collects more premiums than pays for claims in the mean sense.

Recall that $\zeta_i = \eta_i - \xi_i$, $i \ge 1$. It is obvious that $(\zeta_i)_{i\ge 1}$ is a sequence of independent identically distributed random variables. Denote by $F(y)$ their common distribution function. We introduce the distribution functions $G_1(y)$ and $G_2(y)$ as follows: $G_1(y)$ is the distribution function of the random variable ζ_i given $\zeta_i \ge 0$; $G_2(y)$ is the distribution function of the random variable $-\zeta_i$ given $-\zeta_i > 0$.

Lemma 2.2. *Under the above assumptions,*

$$F(y) = \int_{(-y\vee 0)_-}^{+\infty} F_2(y+u)\, dF_1(u), \qquad y \in \mathbb{R}, \tag{3}$$

$$G_1(y) = \begin{cases} \frac{F(y) - F(0_-)}{1 - F(0_-)}, & y \ge 0,\\ 0, & y < 0, \end{cases} \tag{4}$$

$$G_2(y) = \begin{cases} \frac{F(0_-) - F((-y)_-)}{F(0_-)}, & y \ge 0,\\ 0, & y < 0. \end{cases} \tag{5}$$

Here and in what follows $F(y_-)$ means the limit on the left of the distribution function at point y and the symbol y_- in the integral means that the interval of integration contains a lower neighborhood of the point y and the radius of this neighborhood approaches zero.

Proof. For all $y \in \mathbb{R}$, we have

$$F(y) = \mathsf{P}[\zeta_i \le y] = \mathsf{P}[\eta_i \le \xi_i + y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{y+u} dF_2(v)\, dF_1(u)$$
$$= \int_{-\infty}^{+\infty} F_2(y+u)\, dF_1(u) = \int_{(-y\vee 0)_-}^{+\infty} F_2(y+u)\, dF_1(u)$$

and this proves (3). For the latter equality, we used the non-negativity of the random variables ξ_i and η_i.

Further, since $G_1(y) = \mathsf{P}\left[\zeta_i \le y \,/\, \zeta_i \ge 0\right]$ and $G_2(y) = \mathsf{P}\left[-\zeta_i \le y \,/\, -\zeta_i > 0\right]$, we conclude that $G_1(y) = 0$ and $G_2(y) = 0$ for all $y < 0$. Then

$$G_1(y) = \frac{\mathsf{P}[0 \le \zeta_i \le y]}{\mathsf{P}[\zeta_i \ge 0]} = \frac{F(y) - F(0_-)}{1 - F(0_-)}$$

and

$$G_2(y) = \frac{\mathsf{P}[0 < -\zeta_i \le y]}{\mathsf{P}[-\zeta_i > 0]} = \frac{\mathsf{P}[-y \le -\zeta_i < 0]}{\mathsf{P}[\zeta_i < 0]} = \frac{F(0_-) - F((-y)_-)}{F(0_-)}$$

for all $y \ge 0$. The lemma is proved. □

Lemma 2.3. *The process* $\big(X_t(x)\big)_{t\ge 0}$ *defined by equality* (1) *admits the following representation:*

$$X_t(x) = x + ct + \sum_{i=1}^{N_t^+} \zeta_i^+ - \sum_{i=1}^{N_t^-} \zeta_i^-, \qquad t \ge 0, \tag{6}$$

where $(N_t^+)_{t\ge 0}$ *and* $(N_t^-)_{t\ge 0}$ *are homogeneous Poisson processes with intensities*

$$\lambda\big(1 - F(0_-)\big) \quad \textit{and} \quad \lambda F(0_-),$$

respectively, $(\zeta_i^+)_{i\ge 1}$ *and* $(\zeta_i^-)_{i\ge 1}$ *are sequences of independent identically distributed random variables with the distribution functions* $G_1(y)$ *and* $G_2(y)$*, defined by equalities* (4) *and* (5)*, respectively. Moreover, all random variables and processes on the right hand side of* (6) *are independent.*

Proof. Considering only non-negative ζ_i, $i \ge 1$, we obtain a thinned process $(N_t)_{t\ge 0}$. As a result we obtain the new sequence of random variables $(\zeta_i^+)_{i\ge 1}$ and new process $(N_t^+)_{t\ge 0}$. It is obvious that $(\zeta_i^+)_{i\ge 1}$ is a sequence of independent identically distributed random variables with the distribution function $G_1(y)$. Then $(N_t^+)_{t\ge 0}$ is a homogeneous Poisson process with intensity $\lambda\,\mathsf{P}[\zeta_i \ge 0]$ (see, for example, Problem 5 in [12, p. 209]). Moreover, the process $(N_t^+)_{t\ge 0}$ does not depend on the sequence $(\zeta_i^+)_{i\ge 1}$. Note that $\lambda\,\mathsf{P}[\zeta_i \ge 0] = \lambda\big(1 - F(0_-)\big)$.

In the same fashion, consider only negative ζ_i, $i \ge 1$, multiply them by -1, and construct the thinned process $(N_t)_{t\ge 0}$. As a result, we obtain a new sequence of random variables $(\zeta_i^-)_{i\ge 1}$ and a new process $(N_t^-)_{t\ge 0}$. It is easy to see that $(\zeta_i^-)_{i\ge 1}$ is a sequence of independent identically distributed random variables with the distribution function $G_2(y)$. Then $(N_t^-)_{t\ge 0}$ is a homogeneous Poisson process with intensity $\lambda\,\mathsf{P}[\zeta_i < 0]$ that does not depend on the sequence $(\zeta_i^-)_{i\ge 1}$. In addition, $\lambda\,\mathsf{P}[\zeta_i < 0] = \lambda F(0_-)$.

It is clear that every random variable ζ_i, $i \ge 1$, is involved in only one thinning procedure described above. Thus

$$-\sum_{i=1}^{N_t}(\xi_i - \eta_i) = \sum_{i=1}^{N_t} \zeta_i = \sum_{i=1}^{N_t^+} \zeta_i^+ - \sum_{i=1}^{N_t^-} \zeta_i^-, \qquad t \ge 0,$$

and moreover $(\zeta_i^+)_{i\ge 1}$, $(\zeta_i^-)_{i\ge 1}$ and $(N_t^+)_{t\ge 0}$, $(N_t^-)_{t\ge 0}$ are independent, since $(\zeta_i)_{i\ge 1}$ and $(N_t)_{t\ge 0}$ are independent. The lemma is proved. □

3. Integro-differential equation for the survival probability

Put $\lambda_1 = \lambda\big(1 - F(0_-)\big)$ and $\lambda_2 = \lambda F(0_-)$. Note that $\lambda_1 + \lambda_2 = \lambda$.

Theorem 3.1. *Let the evolution of the capital of an insurance company be described by equality* (1)*.*

(1) *The function $\varphi(x)$ is continuous in the interval $[0, +\infty)$.*
(2) *If $c - \lambda\mu_1 + \lambda\mu_2 > 0$, then*
   (i) *the function $\varphi(x)$ is continuously differentiable everywhere in the interval $[0, +\infty)$ except the points of discontinuity of $G_2(y)$;*
   (ii) *if $x > 0$ is a point of discontinuity of $G_2(y)$ and $G_2(x) - G_2(x_-) = p$, then $\varphi(x)$ has the left and right hand side derivatives $\varphi'_-(x)$ and $\varphi'_+(x)$ at this point, and moreover*

$$\varphi'_-(x) - \varphi'_+(x) = \frac{\lambda_2 p \varphi(0)}{c} > 0. \tag{7}$$

   (iii) *$\varphi(x)$ satisfies the following integro-differential equation:*

$$c\varphi'(x) = \lambda\varphi(x) - \lambda_1 \int_0^{+\infty} \varphi(x+y)\, dG_1(y) - \lambda_2 \int_0^x \varphi(x-y)\, dG_2(y) \tag{8}$$

   in the interval $[0, +\infty)$ with boundary condition $\lim_{x\to+\infty} \varphi(x) = 1$ (we use the right hand side derivative of $\varphi(x)$ at points of discontinuity of $G_2(y)$).

Proof. The proof follows the lines of a similar proof in [6, 7, 8] and uses representation (6) of the process $\big(X_t(x)\big)_{t\ge 0}$.

Let the first jump of the process $(N_t)_{t\ge0}$ happen at moment $\tau_1 = s$ and let $|\zeta_1| = y$. The ruin does not happen prior the moment τ_1. The ruin does not happen after this moment if and only if

(a) either the first jump of the process $(N_t^+)_{t\ge0}$ occurs at the moment τ_1 (the probability of this event equals λ_1/λ) and the ruin does not occur in the interval $[s, +\infty)$ if the initial surplus equals $x + cs + y$;
(b) or the first jump of the process $(N_t^-)_{t\ge0}$ occurs at the moment τ_1 (the probability of this event equals λ_2/λ) and the ruin does not occur in the interval $[s, +\infty)$ if the initial surplus equals $x + cs - y$.

Since τ_1 is an exponential random variable with parameter λ, the full probability formula implies

$$\begin{aligned}\varphi(x) = \int_0^{+\infty} e^{-\lambda s} \bigg(\lambda_1 \int_0^{+\infty} &\varphi(x + cs + y)\, dG_1(y) \\ &+\lambda_2 \int_0^{x+cs} \varphi(x + cs - y)\, dG_2(y)\bigg)\, ds\end{aligned} \tag{9}$$

for all $x \ge 0$.

Changing the variable $x + cs = u$ in the outer integral on the right hand side of (9) we obtain

$$\begin{aligned}\varphi(x) = \frac{e^{\lambda x/c}}{c} \int_x^{+\infty} e^{-\lambda u/c} \bigg(\lambda_1 \int_0^{+\infty} &\varphi(u + y)\, dG_1(y) \\ &+\lambda_2 \int_0^{u} \varphi(u - y)\, dG_2(y)\bigg)\, du.\end{aligned} \tag{10}$$

Since the inner integrals on the right hand side of (10) as functions of the variable u are non-decreasing and bounded, the integrand in the outer integral is an integrable function over the interval $[0, +\infty)$. Thus $\varphi(x)$ is integrable over the interval $[0, +\infty)$.

Applying the Lebesgue decomposition theorem for the non-decreasing function $G_2(y)$ we write $G_2(y) = G_2^c(y) + G_2^d(y)$, where $G_2^c(y)$ and $G_2^d(y)$ are continuous and discrete components of $G_2(y)$, respectively. Let $(y_k)_{k\geq 1}$ be the family of points of discontinuity (if they exist) of the function $G_2(y)$. As one knows the set of such points is at most countable. Put $p_k = G_2(y_k) - G_2((y_k)_-)$. Then (10) is rewritten as follows:

$$\varphi(x) = \varphi_1(x) + \varphi_2(x), \tag{11}$$

where

$$\varphi_1(x) = \frac{e^{\lambda x/c}}{c} \int_x^{+\infty} e^{-\lambda u/c} \big(\lambda_1 I_1(u) + \lambda_2 I_2(u)\big)\, du, \tag{12}$$

$$\varphi_2(x) = \frac{e^{\lambda x/c}}{c} \int_x^{+\infty} e^{-\lambda u/c} \left(\lambda_2 \sum_{k\colon y_k \leq u} p_k \varphi(u - y_k)\right) du, \tag{13}$$

$$I_1(u) = \int_0^{+\infty} \varphi(u + y)\, dG_1(y),$$

and

$$I_2(u) = \int_0^u \varphi(u - y)\, dG_2^c(y).$$

Now we prove the continuity from the left of the function $I_2(u)$ in the interval $(0, +\infty)$. Consider an arbitrary point $u_0 > 0$. Then, for $u \leq u_0$,

$$I_2(u_0) - I_2(u) = \int_0^u \big(\varphi(u_0 - y) - \varphi(u - y)\big)\, dG_2^c(y) + \int_u^{u_0} \varphi(u_0 - y)\, dG_2^c(y). \tag{14}$$

It is obvious that

$$0 \leq \lim_{u\uparrow u_0} \int_u^{u_0} \varphi(u_0 - y)\, dG_2^c(y) \leq \lim_{u\uparrow u_0} \big(G_2^c(u_0) - G_2^c(u)\big) = 0. \tag{15}$$

Since the function $\varphi(x)$ is continuous in the interval $[0, u_0]$, it is uniformly continuous in the same interval. Then, given an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|\varphi(u_0 - y) - \varphi(u - y)| < \varepsilon$ for $|(u_0 - y) - (u - y)| < \delta$, $u \in [0, u_0]$, $y \in [0, u]$. Thus

$$\left|\int_0^u \big(\varphi(u_0 - y) - \varphi(u - y)\big)\, dG_2^c(y)\right| < \varepsilon G_2^c(u_0) \leq \varepsilon \tag{16}$$

for $|u_0 - u| < \delta$.

Now (14)–(16) imply the continuity from the left of the function $I_2(u)$ at an arbitrary point $u_0 > 0$.

The continuity from the right of the function $I_2(u)$ in the interval $[0, +\infty)$ is proved similarly. The difference is that, for $u \geq u_0$,

$$I_2(u) - I_2(u_0) = \int_0^{u_0} \big(\varphi(u - y) - \varphi(u_0 - y)\big)\, dG_2^c(y) + \int_{u_0}^u \varphi(u - y)\, dG_2^c(y).$$

The continuity of the function $I_1(u)$ in the interval $[0, +\infty)$ follows, since

$$\begin{aligned}\big|I_1(u) - I_1(u_0)\big| &\leq \int_0^{+\infty} \big|\varphi(u + y) - \varphi(u_0 + y)\big|\, dG_1(y)\\ &\leq \big(1 - G_1(L)\big) + \sup_{y\in[0,L]} \big|\varphi(u + y) - \varphi(u_0 + y)\big|\end{aligned}$$

for all $u \geq 0$, $u_0 \geq 0$, and an arbitrary number $L > 0$. This allows one to make the term $1 - G_1(L)$ arbitrarily small by choosing a sufficiently large L and to make the second term sufficiently small by choosing u sufficiently close to u_0.

Hence the integrand on the right hand side of (12) is continuous in $[0, +\infty)$ and thus $\varphi_1(x)$ is differentiable in this interval and

$$\varphi_1'(x) = \frac{\lambda}{c}\varphi_1(x) - \frac{\lambda_1 I_1(x) + \lambda_2 I_2(x)}{c}. \tag{17}$$

Since the integrand on the right hand side of (13) is right continuous in the interval $[0, +\infty)$, the right derivative of the function $\varphi_2(x)$ (we denote it by $\big(\varphi_2(x)\big)_+'$) exists in this interval and

$$\big(\varphi_2(x)\big)_+' = \frac{\lambda}{c}\varphi_2(x) - \frac{\lambda_2}{c}\sum_{k:\, y_k \le x} p_k \varphi(x - y_k). \tag{18}$$

Since the integral is the same for functions that differ at a countable set of points, equality (13) can be rewritten as follows:

$$\varphi_2(x) = \frac{e^{\lambda x/c}}{c}\int_x^{+\infty} e^{-\lambda u/c}\left(\lambda_2 \sum_{k:\, y_k < u} p_k \varphi(u - y_k)\right) du. \tag{19}$$

The integrand on the right hand side of (19) is left continuous in $(0, +\infty)$ and thus the left derivative of the function $\varphi_2(x)$ (we denote it by $\big(\varphi_2(x)\big)_-'$) exists in this interval and

$$\big(\varphi_2(x)\big)_-' = \frac{\lambda}{c}\varphi_2(x) - \frac{\lambda_2}{c}\sum_{k:\, y_k < x} p_k \varphi(x - y_k). \tag{20}$$

Taking into account (18) and (20), we obtain

$$\big(\varphi_2(x)\big)_-' - \big(\varphi_2(x)\big)_+' = \frac{\lambda_2 p_k \varphi(0) \mathbb{I}_{\{x = y_k\}}}{c}, \tag{21}$$

where $\mathbb{I}_{\{\cdot\}}$ denotes the indicator of a set.

Since $c - \lambda\mu_1 + \lambda\mu_2 > 0$, we get $\lim_{x\to+\infty}\varphi(x) = 1$ in view of Lemma 2.1. Then equality (9) implies that $\varphi(0) > 0$. Substituting $x = 0$ in (9) we see that the integrand in the outer integral is strictly positive (at least starting with some $s > 0$), since $\varphi(x)$ is a continuous function $[0, +\infty)$ and $\lim_{x\to+\infty}\varphi(x) = 1$.

Therefore the right hand side of (9) is positive for $x = 0$, whence $\varphi(0) > 0$. Hence equality (21) implies that the function $\varphi_2(x)$ is differentiable at all points of the interval $[0, +\infty)$ except the points y_k, $k \ge 1$. Taking into account (11), we see that $\varphi(x)$ also is a differentiable function at all points of the interval $[0, +\infty)$ except the points of discontinuity of $G_2(y)$ and that (7) holds. Moreover, equalities (11), (17), and (18) imply that

$$\begin{aligned}\varphi'(x) &= \frac{\lambda}{c}\big(\varphi_1(x) + \varphi_2(x)\big) - \frac{\lambda_1}{c} I_1(x) - \frac{\lambda_2}{c}\left(I_2(x) + \sum_{k:\, y_k \le x} p_k \varphi(x - y_k)\right) \\ &= \frac{\lambda}{c}\varphi(x) - \frac{\lambda_1}{c}\int_0^{+\infty}\varphi(x + y)\, dG_1(y) - \frac{\lambda_2}{c}\int_0^x \varphi(x - y)\, dG_2(y),\end{aligned}$$

whence we derive integro-differential equation (8). Since the right hand side of (8) is a continuous function at all points of the interval $[0, +\infty)$ except the points of discontinuity of $G_2(y)$, $\varphi'(x)$ also is continuous at these points. The theorem is proved. □

Remark 3.1. One can determine the points of discontinuity of the function $G_2(y)$ by using the points of discontinuity of $F_1(y)$ and $F_2(y)$. For this, consider the sequences of points of discontinuity of the functions $F_1(y)$ and $F_2(y)$, namely $(y_{1,i})_{i\ge1}$ and $(y_{2,j})_{j\ge1}$ respectively, where $F_1(y_{1,i}) - F_1((y_{1,i})_-) = p_{1,i}$, $i \ge 1$, and $F_2(y_{2,j}) - F_2((y_{2,j})_-) = p_{2,j}$,

$j \ge 1$. Consider all possible differences $y_{1,i} - y_{2,j}$ and keep only positive among them. Every positive value $y_{1,i} - y_{2,j}$ is a point of discontinuity of the function $G_2(y)$ and the height of the jump at this point equals $p_{1,i}\,p_{2,j}$ if $y_{1,i} - y_{2,j}$ occurs only one time in the family of differences. If a value $y_{1,i} - y_{2,j}$ occurs several times, then we add the corresponding $p_{1,i}\,p_{2,j}$ to obtain the height of the jump at this point.

4. Exponential claim sizes and additional funds

Lemma 4.1. *If random variables ξ_i and η_i, $i \ge 1$, are exponential with expectations μ_1 and μ_2, respectively, then the random variables ζ_i^+ and ζ_i^-, $i \ge 1$, also are exponential with expectations μ_2 and μ_1, respectively. In addition,*

$$\lambda_1 = \frac{\lambda\mu_2}{\mu_1+\mu_2} \quad \text{and} \quad \lambda_2 = \frac{\lambda\mu_1}{\mu_1+\mu_2}.$$

Proof. By assumption, $F_1(y) = 1 - e^{-y/\mu_1}$ and $F_2(y) = 1 - e^{-y/\mu_2}$ for all $y \ge 0$. Using Lemma 2.2, we obtain the distribution function $F(y)$ of the random variables ζ_i, $i \ge 1$.

Considering (3) we obtain

$$\begin{aligned} F(y) &= \int_{-y\vee 0}^{+\infty} \left(1 - e^{-(y+u)/\mu_2}\right) \frac{1}{\mu_1}\, e^{-u/\mu_1}\, du \\ &= \int_{-y\vee 0}^{+\infty} \frac{1}{\mu_1}\, e^{-u/\mu_1}\, du - \frac{1}{\mu_1}\, e^{-y/\mu_2} \int_{-y\vee 0}^{+\infty} \exp\left\{-\frac{(\mu_1+\mu_2)u}{\mu_1\mu_2}\right\} du \\ &= e^{-(-y\vee 0)/\mu_1} - \frac{\mu_2}{\mu_1+\mu_2}\, e^{-y/\mu_2} \exp\left\{-\frac{(-y\vee 0)(\mu_1+\mu_2)}{\mu_1\mu_2}\right\} \\ &= e^{(y\wedge 0)/\mu_1} - \frac{\mu_2}{\mu_1+\mu_2} \exp\left\{\frac{(y\wedge 0)(\mu_1+\mu_2) - \mu_1 y}{\mu_1\mu_2}\right\} \end{aligned}$$

for all $y \in \mathbb{R}$.

Since

$$F(0) = 1 - \frac{\mu_2}{\mu_1+\mu_2} = \frac{\mu_1}{\mu_1+\mu_2},$$

we get $\lambda_1 = \lambda\big(1 - F(0_-)\big) = \frac{\lambda\mu_2}{\mu_1+\mu_2}$ and $\lambda_2 = \lambda F(0_-) = \frac{\lambda\mu_1}{\mu_1+\mu_2}$ by Lemma 2.3.

Taking into account (4) and (5) we get

$$G_1(y) = \frac{1 - \frac{\mu_2}{\mu_1+\mu_2}\, e^{-y/\mu_2} - \frac{\mu_1}{\mu_1+\mu_2}}{1 - \frac{\mu_1}{\mu_1+\mu_2}} = \frac{\mu_2 - \mu_2 e^{-y/\mu_2}}{\mu_2} = 1 - e^{-y/\mu_2}$$

and

$$G_2(y) = \frac{\frac{\mu_1}{\mu_1+\mu_2} - e^{-y/\mu_1} + \frac{\mu_2}{\mu_1+\mu_2}\, e^{-y/\mu_1}}{\frac{\mu_1}{\mu_1+\mu_2}} = \frac{\mu_1 - \mu_1 e^{-y/\mu_1}}{\mu_1} = 1 - e^{-y/\mu_1}$$

for all $y \ge 0$, whence the lemma follows. □

Theorem 4.1. *Let the evolution of the capital of an insurance company be described by equality* (1) *and let the random variables ξ_i and η_i, $i \ge 1$, be exponential with expectations μ_1 and μ_2, respectively. If $c - \lambda\mu_1 + \lambda\mu_2 > 0$, then*

$$\varphi(x) = 1 + \frac{\lambda\mu_1(1-\alpha\mu_2)}{(c\alpha-\lambda)(1-\alpha\mu_2)(\mu_1+\mu_2) + \lambda\mu_2}\, e^{\alpha x} \tag{22}$$

for all $x \ge 0$, where

$$\alpha = \frac{\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2 - \sqrt{c^2(\mu_1^2+\mu_2^2) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2)}}{2c\mu_1\mu_2}.$$

Remark 4.1. It will be shown that $\alpha < 0$ and

$$-1 < \frac{\lambda\mu_1(1-\alpha\mu_2)}{(c\alpha-\lambda)(1-\alpha\mu_2)(\mu_1+\mu_2)+\lambda\mu_2} < 0$$

in the course of the proof of Theorem 4.1. Thus the function $\varphi(x)$ defined by equality (22), possesses all natural properties of the survival probability. In particular, it is non-decreasing and bounded from below by zero and from above by one.

Proof. By Theorem 3.1 and Lemma 4.1, the function $\varphi(x)$ is differentiable in the interval $[0,+\infty)$ and thus equation (8) is rewritten as follows:

$$\begin{aligned} c\varphi'(x) = \lambda\varphi(x) &- \frac{\lambda\mu_2}{\mu_1+\mu_2}\int_0^{+\infty}\varphi(x+y)\,\frac{1}{\mu_2}\,e^{-y/\mu_2}\,dy \\ &- \frac{\lambda\mu_1}{\mu_1+\mu_2}\int_0^{x}\varphi(x-y)\,\frac{1}{\mu_1}\,e^{-y/\mu_1}\,dy. \end{aligned} \tag{23}$$

Changing the variables $x+y=u$ and $x-y=u$ in the first and second integral on the right hand side of (23), respectively, we obtain

$$\begin{aligned} c\varphi'(x) = \lambda\varphi(x) &- \frac{\lambda}{\mu_1+\mu_2}\,e^{x/\mu_2}\int_x^{+\infty}\varphi(u)\,e^{-u/\mu_2}\,du \\ &- \frac{\lambda}{\mu_1+\mu_2}\,e^{-x/\mu_1}\int_0^{x}\varphi(u)\,e^{u/\mu_1}\,du. \end{aligned} \tag{24}$$

It is easy to see that the right hand side of (24) is a differentiable function in the interval $[0,+\infty)$. Thus the second derivative of the function $\varphi(x)$ exists for all $x \ge 0$. Differentiating (24), we get

$$\begin{aligned} c\varphi''(x) = \lambda\varphi'(x) &- \frac{\lambda}{\mu_2(\mu_1+\mu_2)}\,e^{x/\mu_2}\int_x^{+\infty}\varphi(u)\,e^{-u/\mu_2}\,du \\ &+ \frac{\lambda}{\mu_1(\mu_1+\mu_2)}\,e^{-x/\mu_1}\int_0^{x}\varphi(u)\,e^{u/\mu_1}\,du. \end{aligned} \tag{25}$$

It is obvious that the right hand side of (25) is a differentiable function in the interval $[0,+\infty)$. Thus the third derivative of the function $\varphi(x)$ exists in this interval. Differentiating (25), we obtain

$$\begin{aligned} c\varphi'''(x) = \lambda\varphi''(x) + \frac{\lambda}{\mu_1\mu_2}\,\varphi(x) &- \frac{\lambda}{\mu_2^2(\mu_1+\mu_2)}\,e^{x/\mu_2}\int_x^{+\infty}\varphi(u)\,e^{-u/\mu_2}\,du \\ &+ \frac{\lambda}{\mu_1^2(\mu_1+\mu_2)}\,e^{-x/\mu_1}\int_0^{x}\varphi(u)\,e^{u/\mu_1}\,du. \end{aligned} \tag{26}$$

Consider the functions

$$H_1(x) = \frac{\lambda}{\mu_1+\mu_2}\,e^{x/\mu_2}\int_x^{+\infty}\varphi(u)\,e^{-u/\mu_2}\,du$$

and

$$H_2(x) = \frac{\lambda}{\mu_1+\mu_2}\,e^{-x/\mu_1}\int_0^{x}\varphi(u)\,e^{u/\mu_1}\,du.$$

Then equalities (24), (25), and (26) are rewritten in the following form:

$$c\varphi'(x) = \lambda\varphi(x) - H_1(x) - H_2(x), \tag{27}$$

$$c\varphi''(x) = \lambda\varphi'(x) - \frac{1}{\mu_2}\,H_1(x) + \frac{1}{\mu_1}\,H_2(x), \tag{28}$$

and

$$c\varphi'''(x) = \lambda\varphi''(x) + \frac{\lambda}{\mu_1\mu_2}\,\varphi(x) - \frac{1}{\mu_2}\,H_1(x) + \frac{1}{\mu_1}\,H_2(x). \tag{29}$$

Now (27) implies

$$H_2(x) = -c\varphi'(x) + \lambda\varphi(x) - H_1(x). \tag{30}$$

Substituting $H_2(x)$ from (30) into (28), we get after a simple algebra

$$H_1(x) = \frac{\mu_1\mu_2}{\mu_1+\mu_2}\left(-c\varphi''(x) + \lambda\varphi'(x) - \frac{c}{\mu_1}\,\varphi'(x) + \frac{\lambda}{\mu_1}\,\varphi(x)\right). \tag{31}$$

Substituting $H_2(x)$ from (30) into (29) we get

$$c\varphi'''(x) = \lambda\varphi''(x) + \frac{c}{\mu_1^2}\,\varphi'(x) + \left(\frac{\lambda}{\mu_1\mu_2} - \frac{\lambda}{\mu_1^2}\right)\varphi(x) + \left(\frac{1}{\mu_1^2} - \frac{1}{\mu_2^2}\right)H_1(x). \tag{32}$$

Substituting $H_1(x)$ from (31) into (32) we get after a simple algebra

$$c\varphi'''(x) - \frac{\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2}{\mu_1\mu_2}\,\varphi''(x) - \frac{c - \lambda\mu_1 + \lambda\mu_2}{\mu_1\mu_2}\,\varphi'(x) = 0. \tag{33}$$

The linear homogeneous differential equation (33) with constant coefficients can be solved for $\varphi(x)$ by the well-known methods. The corresponding characteristic equation is given by

$$c\mu_1\mu_2\alpha^3 - (\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2)\alpha^2 - (c - \lambda\mu_1 + \lambda\mu_2)\alpha = 0. \tag{34}$$

Solving equation (34), we obtain

$$\alpha_1 = 0,$$

$$\alpha_2 = \frac{\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2 - \sqrt{c^2(\mu_1^2+\mu_2^2) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2)}}{2c\mu_1\mu_2},$$

and

$$\alpha_3 = \frac{\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2 + \sqrt{c^2(\mu_1^2+\mu_2^2) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2)}}{2c\mu_1\mu_2}.$$

Since $c - \lambda\mu_1 + \lambda\mu_2 > 0$, we conclude that

$$c^2(\mu_1^2+\mu_2^2) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2) > 0.$$

Moreover, it is not complicated to check that

$$|\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2| < \sqrt{c^2(\mu_1^2+\mu_2^2) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2)},$$

which yields $\alpha_2 < 0$ and $\alpha_3 > 0$.

The general solution of equation (33) is given by

$$\varphi(x) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + A_3 e^{\alpha_3 x}, \tag{35}$$

where A_1, A_2, and A_3 are unknown constants to be determined from the following three conditions:

$$\lim_{x\to+\infty} \varphi(x) = 1, \tag{36}$$

$$c\varphi'(0) = \lambda\varphi(0) - \frac{\lambda}{\mu_1+\mu_2}\int_0^{+\infty} \varphi(u)\,e^{-u/\mu_2}\,du, \tag{37}$$

and

$$c\varphi''(0) = \lambda\varphi'(0) - \frac{\lambda}{\mu_2(\mu_1+\mu_2)}\int_0^{+\infty} \varphi(u)\,e^{-u/\mu_2}\,du. \tag{38}$$

Condition (36) is satisfied by Lemma 2.1, while (37) and (38) are obtained by substituting $x = 0$ into (24) and (25), respectively.

Since $\lim_{x\to+\infty} e^{\alpha_2 x} = 0$ and $\lim_{x\to+\infty} e^{\alpha_3 x} = +\infty$, equalities (35) and (36) imply $A_1 = 1$ and $A_3 = 0$. Then (35) is rewritten as follows:

$$\varphi(x) = 1 + A_2 e^{\alpha_2 x}. \tag{39}$$

Now equalities $\varphi(0) = 1 + A_2$ and $\varphi'(0) = A_2\alpha_2$ follow from (39). Then (37) becomes of the following form:

$$cA_2\alpha_2 = \lambda(1 + A_2) - \frac{\lambda}{\mu_1 + \mu_2}\int_0^{+\infty} \left(1 + A_2 e^{\alpha_2 u}\right) e^{-u/\mu_2}\, du,$$

whence

$$A_2 = \frac{\lambda\mu_1(1 - \alpha_2\mu_2)}{(c\alpha_2 - \lambda)(1 - \alpha_2\mu_2)(\mu_1 + \mu_2) + \lambda\mu_2}. \tag{40}$$

Substituting $\varphi'(0) = A_2\alpha_2$ and $\varphi''(0) = A_2\alpha_2^2$ into (38) we get

$$cA_2\alpha_2^2 = \lambda A_2\alpha_2 - \frac{\lambda}{\mu_2(\mu_1 + \mu_2)}\int_0^{+\infty} \left(1 + A_2 e^{\alpha_2 u}\right) e^{-u/\mu_2}\, du,$$

whence

$$A_2 = \frac{-\lambda(1 - \alpha_2\mu_2)}{\alpha_2(c\alpha_2 - \lambda)(1 - \alpha_2\mu_2)(\mu_1 + \mu_2) + \lambda}. \tag{41}$$

Since α_2 is a root of the characteristic equation (34), we easily prove that A_2 given by (40) and A_2 given by (41) are the same and (22) follows. Note that we write α instead of α_2 in the statement of the theorem.

Further, since $\alpha_2 < 0$, we have $1 - \alpha_2\mu_2 > 0$ and

$$(\lambda - c\alpha_2)(1 - \alpha_2\mu_2)(\mu_1 + \mu_2) > \lambda \cdot 1 \cdot \mu_2 = \lambda\mu_2.$$

Thus (40) yields $A_2 < 0$. Since

$$\begin{aligned}\frac{\lambda\mu_1(1 - \alpha_2\mu_2)}{(\lambda - c\alpha_2)(1 - \alpha_2\mu_2)(\mu_1 + \mu_2) - \lambda\mu_2} &< \frac{\lambda\mu_1(1 - \alpha_2\mu_2)}{\lambda(1 - \alpha_2\mu_2)(\mu_1 + \mu_2) - \lambda\mu_2} \\ &= \frac{\lambda\mu_1(1 - \alpha_2\mu_2)}{\lambda\mu_1(1 - \alpha_2\mu_2) - \lambda\alpha_2\mu_2^2} < 1,\end{aligned}$$

equality (40) implies $A_2 > -1$.

Therefore the function $\varphi(x)$ defined by equality (22) possesses all the natural properties of the survival probability.

The solution found above is equal to the survival probability. Indeed, the survival probability is a solution of equation (8) derived without any extra assumption (on the differentiability of $\varphi(x)$, for example) and that equation reduces to equation (33). Thus the survival probability is a solution of equation (33). We have shown that equation (33) has a unique solution satisfying conditions (36)–(38) and this proves that it indeed equals the survival probability. The theorem is proved. □

Remark 4.2. If $c - \lambda\mu_1 > 0$ and claim sizes are exponential with expectation μ_1 in the classical risk model, then

$$\varphi(x) = 1 - \frac{\lambda\mu_1}{c}\exp\left\{\frac{(\lambda\mu_1 - c)x}{c\mu_1}\right\}$$

for all $x \ge 0$ (see, for example, [1, 2, 3, 4]).

Put

$$A(\lambda, c, \mu_1, \mu_2) = c^2\left(\mu_1^2 + \mu_2^2\right) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2).$$

Then, for the risk model considered above,

$$\begin{aligned}\lim_{\mu_2\downarrow 0}\alpha &= \lim_{\mu_2\downarrow 0}\frac{\lambda\mu_1\mu_2+c\mu_1-c\mu_2-\sqrt{A(\lambda,c,\mu_1,\mu_2)}}{2c\mu_1\mu_2}\\ &= \lim_{\mu_2\downarrow 0}\frac{\bigl(\lambda\mu_1\mu_2+c\mu_1-c\mu_2\bigr)^2-A(\lambda,c,\mu_1,\mu_2)}{2c\mu_1\mu_2\bigl(\lambda\mu_1\mu_2+c\mu_1-c\mu_2+\sqrt{A(\lambda,c,\mu_1,\mu_2)}\bigr)}\\ &= \lim_{\mu_2\downarrow 0}\frac{-4c\mu_1\mu_2(c-\lambda\mu_1+\lambda\mu_2)}{2c\mu_1\mu_2\cdot 2c\mu_1}=\frac{\lambda\mu_1-c}{c\mu_1}\end{aligned}$$

and

$$\lim_{\mu_2\downarrow 0}\frac{\lambda\mu_1(1-\alpha\mu_2)}{(c\alpha-\lambda)(1-\alpha\mu_2)(\mu_1+\mu_2)+\lambda\mu_2}=\lim_{\mu_2\downarrow 0}\frac{\lambda\mu_1}{(c\alpha-\lambda)\mu_1}=\frac{\lambda}{(\lambda\mu_1-c)/\mu_1-\lambda}=-\frac{\lambda\mu}{c}.$$

This means that if μ_2 is small (this is the case if additional funds are negligible), then the survival probability in the risk model considered above is close to the corresponding survival probability in the classical risk model.

Example 4.1. Let $c=10$, $\lambda=4$, $\mu_1=2$, and $\mu_2=0.1$. Then

$$\varphi(x)=1-0.8\,e^{-0.1x}$$

for the classical risk model. On the other hand,

$$\varphi(x)\approx 1-0.760473\,e^{-0.119763x}$$

for the risk model considered above.

5. Concluding remarks

A generalization of the classical risk model is considered in the paper. Our model assumes that each time a claim arrives at an insurance company it gets additional funds. Properties of continuity and differentiability of the infinite-horizon survival probability are studied. An integro-differential equation is established for this function. For the case where the claim sizes and additional funds are exponential, we find an exact solution of this equation and prove that if the additional funds are negligible, then the survival probability in our risk model is close to the corresponding survival probability in the classical risk model. An example is considered.

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