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THE DISTRIBUTIONS OF RANDOM INCOMPLETE SUMS OF A SERIES WITH POSITIVE TERMS SATISFYING THE PROPERTY OF NON-LINEAR HOMOGENEITY

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Abstract. The Lebesgue type as well as topological, metric, and fractal properties of the spectrum of the distribution of the random variable

$$\xi = \sum_{n=1}^{\infty} a_n \xi_n$$

are studied, where $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + r_n$ is a convergent series with positive terms such that $r_{n+1} = a_{n+1} a_n$ for any $n \in \mathbb{N}$ and (ξ_n) is a sequence of independent random variables taking only two values, 0 and 1, with probabilities p_{0n} and p_{1n}, respectively. We describe the point spectrum in the discrete case, and we prove that the distribution of ξ is of a Cantor singular type with an anomalous fractal spectrum in the continuous case. We also prove that the n-fold convolution of the random variable ξ with itself has an anomalous fractal distribution.

1. Introduction

Consider a random variable

$$\xi = \sum_{n=1}^{\infty} a_n \xi_n, \tag{1}$$

where

$$r_0 = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + r_n \tag{2}$$

is a convergent series with positive terms satisfying the following condition of homogeneity:

$$r_{n+1} = a_{n+1} a_n, \quad \forall n \in \mathbb{N}. \tag{3}$$

Here (ξ_n) is a sequence of independent random variables with the distributions

$$\mathsf{P}\{\xi_n = 0\} = p_{0n} \geq 0, \qquad \mathsf{P}\{\xi_n = 1\} = p_{1n} \geq 0, \qquad p_{0n} + p_{1n} = 1.$$

The properties of the distribution of the random variable ξ are uniquely determined by the sequence of terms (a_n) and by the stochastic matrix $\|p_{in}\|$. Note that the distribution of the random variable ξ belongs to the class of infinite Bernoulli convolutions studied over the last century or so. The interest to Bernoulli convolutions increased in recent

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years; see [1, 2, 6, 7, 8, 9, 12, 14, 15, 16]. This is explained, in particular, by the studies of the important fractal properties of Bernoulli convolutions.

There are many complicated problems of probabilistic nature in the theory of infinite Bernoulli convolutions. One of these problems is to extend a theorem due to Jessen and Wintner stating that the distribution of an infinite sum of a random series with independent discrete terms is pure; that is, it is either purely discrete, or continuous, or singular. It is worthwhile mentioning that the Jessen–Wintner theorem does not tell us when each of these three cases occur.

Another problem is related to the topological, metric, and fractal properties of the spectrum of a probability distribution, that is, to the set of growth of a distribution function.

The third problem deals with the behavior at infinity of the absolute value of a characteristic function. None of these problems is completely solved in the general case but partial solutions are known for several particular cases. One of these particular cases is considered in the current paper; namely, we study the random variable ξ defined by (1).

The Jessen–Wintner theorem [3] implies that the random variable ξ possesses the distribution of a pure Lebesgue type. In other words, its distribution function is either purely discrete, or purely absolutely continuous, or purely singular. The latter case means that the distribution function is continuous but its derivative equals zero almost surely with respect to the Lebesgue measure. A well known Lévy theorem [5] together with the Jessen–Wintner theorem provides necessary and sufficient conditions for the distribution of ξ to be either discrete or continuous.

2. Homogeneity condition

We say that the homogeneity condition with respect to n holds for series (2) if there exists a positive integer number k and a function f such that

$$r_n \vee f(a_n, a_{n-1}, a_{n-2}, \dots, a_{n-k+1})$$

for all natural numbers $n \geq k$, where the symbol "$\vee$" means one of the following signs: "$>$", "$<$", "$\leq$", "$\geq$", or "$=$".

We consider some properties of series (2) possessing the homogeneity condition (3).

Lemma 2.1. *If a_1 and a_2 are positive real numbers, then*

$$a_{n+2} = \frac{a_{n+1}}{1+a_{n+1}} a_n, \qquad n = 1, 2, 3, \dots, \tag{4}$$

is an infinitesimal sequence and, moreover, the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Equality (4) implies that

$$q_n \equiv \frac{a_{n+2}}{a_n} = \frac{a_{n+1}}{1+a_{n+1}} < 1.$$

Thus the sequences (a_{2n-1}) and (a_{2n}) are decreasing. Moreover, since $a_{n+1} > a_{n+3}$ for all $n \in \mathbb{N}$, we have

$$q_n - q_{n+2} = \frac{a_{n+1}}{1+a_{n+1}} - \frac{a_{n+3}}{1+a_{n+3}} = \frac{a_{n+1} - a_{n+3}}{(1+a_{n+1})(1+a_{n+3})} > 0;$$

that is, (q_{2n-1}) and (q_{2n}) are decreasing sequences. Then

$$a_{2n+1} = a_1 \prod_{k=1}^{n} q_{2k-1}, \qquad a_{2n+2} = a_2 \prod_{k=1}^{n} q_{2k},$$

whence

$$a_{2n-1} \leq a_1 q_1^{n-1} \to 0 \quad \text{and} \quad a_{2n} \leq a_2 q_2^{n-1} \to 0 \quad \text{as } n \to \infty.$$

Hence, the sequences (a_{2n-1}) and (a_{2n}) are infinitesimal and thus so is the sequence (a_n). Therefore, the necessary condition for the convergence of a series holds. Since

$$\sum_{n=1}^{\infty} a_{2n-1} \le a_1 \sum_{n=1}^{\infty} q_1^{n-1} = a_1(1+a_2) \quad \text{and} \quad \sum_{n=1}^{\infty} a_{2n} \le a_2 \sum_{n=1}^{\infty} q_2^{n-1} = a_2(1+a_3),$$

the series converges. □

Theorem 2.1. *Homogeneity condition* (3) *holds for series* (2) *if and only if equality* (4) *holds for its terms.*

Proof. Necessity. If a series converges and satisfies homogeneity condition (3), then, for all $n \in \mathbb{N}$,

$$\begin{cases} a_{n+1}a_n = r_{n+1} = a_{n+2} + a_{n+3} + a_{n+4} + \dots, \\ a_{n+2}a_{n+1} = r_{n+2} = a_{n+3} + a_{n+4} + \dots . \end{cases}$$

Subtracting the second equality from the first one, we obtain

$$a_{n+1}a_n - a_{n+2}a_{n+1} = a_{n+2}, \tag{5}$$

whence equality (4) follows.

Sufficiency. Now we prove that (4) implies (3). Since the series converges (see Lemma 2.1), equality (4) implies (5). Recalling $a_{n+2} = r_{n+1} - r_{n+2}$, we get

$$r_{n+1} - r_{n+2} = a_{n+1}a_n - a_{n+2}a_{n+1},$$

which is equivalent to

$$r_{n+1} - a_{n+1}a_n = r_{n+2} - a_{n+2}a_{n+1}, \qquad n = 1, 2, 3, \dots,$$

whence $r_{n+1} - a_{n+1}a_n = r_{n+1+l} - a_{n+1+l}a_{n+l} = \text{const}$ for all $l \in \mathbb{N}$. Since

$$\lim_{n\to\infty}(r_{n+1} - a_{n+1}a_n) = \lim_{n\to\infty} r_{n+1} - \lim_{n\to\infty} a_{n+1}a_n = 0,$$

we conclude that

$$r_{n+1} - a_{n+1}a_n = 0 \Leftrightarrow r_{n+1} = a_{n+1}a_n \quad \forall n \in \mathbb{N}.$$

The sufficiency as well as the theorem is proved. □

Lemma 2.2. *Let series* (2) *be such that its terms satisfy homogeneity condition* (3). *Then its sum is equal to*

$$r_0 = a_1 + a_2 + a_1 a_2. \tag{6}$$

Proof. Equality (3) implies

$$a_1 a_2 = a_3 + a_4 + a_5 + \dots .$$

Adding $a_1 + a_2$ to both sides of the latter equality, we prove (6). □

Lemma 2.3. *Assume that the terms of series* (2) *satisfy homogeneity condition* (3). *Then*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0, \tag{7}$$

$$\lim_{n\to\infty} \frac{a_n}{q^n} = 0, \tag{8}$$

where q is an arbitrary real number of the interval $(0,1)$, *and*

$$\lim_{n\to\infty} a_n n! = 0. \tag{9}$$

Proof. Since the series converges, equality (3) implies that

$$a_n = \frac{r_{n+1}}{a_{n+1}} = \frac{a_{n+2} + a_{n+3} + a_{n+4} + \dots}{a_{n+1}} = \sum_{m=2}^{\infty} \frac{a_{n+m}}{a_{n+1}}.$$

The latter series converges for any $n \in \mathbb{N}$ and its sum as well as a_n approaches zero as $n \to \infty$; that is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \dots \right) = 0.$$

Thus $\lim_{n\to\infty} a_{n+m}/a_{n+1} = 0$, $m = 2, 3, 4, \dots$, and equality (7) is proved.

Equality (7) yields that there exists $n_0 \in \mathbb{N}$ such that either $a_{n+1}/a_n < \varepsilon$ or $a_{n+1} < \varepsilon a_n$ for sufficiently small $\varepsilon > 0$ and for all $n > n_0$. Let $a_{m+s} < 1$ for all $s = 0, 1, 2, \dots$. Then

$$a_{m+s} < \varepsilon a_{m+s-1} < \varepsilon^2 a_{m+s-2} < \dots < \varepsilon^s a_m < \varepsilon^s.$$

Hence either $a_{m+s} < \varepsilon^s$ or $a_n < \varepsilon^{n-m} = \frac{1}{\varepsilon^m}\varepsilon^n = c\varepsilon^n$, where $s = n - m$, $c = \text{const}$. Choosing $\varepsilon < q$, we prove equality (8).

Now equality (7) yields that there exists an integer number m such that $a_{n+1} < a_n < 1$ for all $n \geq m$. Below we estimate the terms of the series:

$$a_{m+2} = \frac{a_{m+1}a_m}{1 + a_{m+1}} < a_m a_{m+1} < a_m^2,$$

$$a_{m+3} = \frac{a_{m+2}a_{m+1}}{1 + a_{m+2}} < a_{m+2}a_{m+1} < a_m^2 a_{m+1} < a_m^3,$$

$$a_{m+4} = \frac{a_{m+3}a_{m+2}}{1 + a_{m+3}} < a_{m+3}a_{m+2} < a_m^3 a_m^2 = a_m^5,$$

$$a_{m+5} = \frac{a_{m+4}a_{m+3}}{1 + a_{m+4}} < a_{m+4}a_{m+3} < a_m^5 a_m^3 = a_m^8.$$

It is easy to see that the indices u_{j+1} of the terms $a_m^{u_{j+1}}$ that estimate a_{m+j} form the classical Fibonacci sequence with the general term

$$u_j = \frac{1}{\sqrt{5}} \left((\varphi)^j - (\psi)^j \right) = \frac{1}{\sqrt{5}} \left(1 - \left(\frac{\psi}{\varphi} \right)^j \right) \cdot \varphi^j \to \frac{1}{\sqrt{5}} \varphi^j, \qquad j \to \infty,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Hence

$$a_n = a_{m+j} < a_m^{u_{j+1}} = a_m^{u_{n-m+1}} = p^{\varphi^n},$$

where $p = a_m^{\frac{\varphi^{1-m}}{\sqrt{5}}}$ is a number of the interval $(0, 1)$.

To prove equality (9) it suffices to show that $\lim_{n\to\infty}(p^{\varphi^n} \cdot n!) = 0$. The latter property follows, since $n! < n^n$ for all $n > 2$ and since $\lim_{n\to\infty}(p^{\varphi^n} \cdot n^n) = 0$ (moreover, the series with the general term $b_n = p^{\varphi^n} \cdot n^n$ converges). □

Corollary 2.1. *Given an arbitrary $q \in (0, 1)$, there exists a number n_0 such that $a_n < q^n$ for all $n > n_0$.*

3. The criterion for the discrete case. Point spectrum

If M is a subset of the set of positive integer numbers $\mathbb{N}$, then $\sum_{n\in M\subset\mathbb{N}} a_n$ is called a *sub-series of series* (2), and its sum $x = x(M)$ is called an *incomplete sum of series* (2). The set of all incomplete sums of series (2) is denoted by $E(a_n)$.

Lemma 3.1. *If the property $r_n < a_n$ holds for all $n \in \mathbb{N}$, then the incomplete sums (of series (2)) $x_1 = x(M_1)$ and $x_2 = x(M_2)$ corresponding to sub-series over two different sets M_1 and M_2 are different, as well.*

Proof. Let

$$x_1 = \sum_{n\in M_1} a_n = \sum_{n=1}^{\infty} \varepsilon_n a_n, \qquad x_2 = \sum_{n\in M_2} a_n = \sum_{n=1}^{\infty} \varepsilon'_n a_n.$$

Since $M_1 \neq M_2$, there exists $m \in \mathbb{N}$ such that $\varepsilon_m \neq \varepsilon'_m$ but $\varepsilon_j = \varepsilon'_j$ for $j < m$. Without loss of generality, assume that $\varepsilon_m = 1$, $\varepsilon'_m = 0$. Consider the difference

$$x_1 - x_2 = \varepsilon_m a_m + r_m^{(1)} - r_m^{(2)},$$

where $r_m^{(1)} = \varepsilon_{m+1}a_{m+1} + \varepsilon_{m+2}a_{m+2} + \dots$ and $r_m^{(2)} = \varepsilon'_{m+1}a_{m+1} + \varepsilon'_{m+2}a_{m+2} + \dots$.

Since $r_m^{(2)} \leq r_m$ and $a_m > r_m$, we get $x_1 - x_2 = a_m + r_m^{(1)} - r_m^{(2)} \geq a_m - r_m > 0$, whence $x_1 \neq x_2$, which had to be proved. □

Lemma 3.2. *Let the terms (a_n) of series (2) satisfy condition (3). Then there exists a number $n_0 \in \mathbb{N}$ such that $a_n > r_n$ for all $n \geq n_0$.*

Proof. Since $r_n = a_n a_{n-1}$ and $a_n \to 0$ as $n \to \infty$, we conclude that there exists a number $n_0 \in \mathbb{N}$ such that $a_{n-1} < 1$ for all $n \geq n_0$, whence the inequality $a_n > r_n$ follows. □

Theorem 3.1. *The distribution of the random variable ξ defined by (1) is purely discrete if and only if*

$$L = \prod_{n=1}^{\infty} \max\{p_{0n}, p_{1n}\} > 0.$$

In the case of a purely discrete distribution, if $a_n > r_n$ for all $n \in \mathbb{N}$ for series (2), then the point spectrum D_ξ contains a point x_0 such that

$$x_0 = \sum_{n=1}^{\infty} c_n a_n, \quad \text{where } p_{c_n n} = \max\{p_{0n}, p_{1n}\} \equiv p_n^*, \tag{10}$$

and all points x such that

$$x = \sum_{n=1}^{m} \varepsilon_n a_n + \sum_{n=m+1}^{\infty} c_n a_n, \tag{11}$$

where $\varepsilon_n \in \{0, 1\}$ and $p_{\varepsilon_n n} \neq 0$ for $n \leq m$.

Proof. The first part of Theorem 3.1 follows directly from the Jessen–Wintner [3] and Lévy [5] theorems. It remains to prove the second part.

Let the distribution of the random variable ξ be purely discrete; that is, $L > 0$. This condition means that the sequence $p_n^* = \max\{p_{0n}, p_{1n}\}$ converges to 1 rather quickly. Thus there exists $l \in \mathbb{N}$ such that $p_n^* > \frac{1}{2}$ for all $n > l$.

Note that a point x_0 may not be uniquely determined by condition (10). This happens if $p_{0n} = \frac{1}{2} = p_{1n}$. We take $c_n = 0$ in such a case.

Since a representation of an element of the set of incomplete sums of series (2) is unique under the condition of homogeneity (3) (this follows from $r_n < a_n$ and Lemma 3.1), we have

$$\mathsf{P}\left\{\xi = \sum_{n=1}^{\infty} c_n a_n\right\} = \prod_{n=1}^{\infty} p_{c_n n} = L > 0;$$

that is, $x_0 \in D_\xi$.

Let $A_0 = \{x_0\}$ and let A_m, $m = 1, 2, 3, \dots$, be the set of all points x of the form (11). Then $A_m \subset A_{m+1}$ for all $m \in Z_0$.

Since

$$\mathsf{P}\{\xi \in A_m\} = \left(\prod_{n=m+1}^{\infty} p_{c_n n}\right) \cdot \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_m=0}^{1} \prod_{j=1}^{m} p_{\varepsilon_j j} = \prod_{n=m+1}^{\infty} p_{c_n n} \to 1, \qquad m \to \infty,$$

we obtain

$$\lim_{m\to\infty} A_m = \bigcup_{m=0}^{\infty} A_m = D_\xi,$$

which had to be proved. □

Remark 3.1. If the condition of homogeneity (3) is satisfied for a series (2) and if the condition $x_1(M_1) \neq x_2(M_2)$ does not hold for some pair of sets $M_1 \neq M_2$, that is there are two sets $M_1 \neq M_2$ such that $x_1(M_1) = x_2(M_2)$, then the two preceding lemmas imply that there exists n_0 such that this condition holds for the sub-series $\sum_{n=n_0+1}^{\infty} a_n$. Therefore the point spectrum D_ξ is the vector sum of the point spectra $D_{\hat{\xi}^{(1)}}$ and $D_{\hat{\xi}^{(2)}}$ of the random variables

$$\hat{\xi}^{(1)} = \sum_{n=1}^{n_0} \xi_n a_n \quad \text{and} \quad \hat{\xi}^{(2)} = \sum_{n=n_0+1}^{\infty} \xi_n a_n,$$

respectively. The set $D_{\hat{\xi}^{(2)}}$ is determined by condition 3.1, while

$$D_{\hat{\xi}^{(1)}} = \left\{x\colon x = \sum_{n=1}^{n_0} \varepsilon_n a_n,\ \text{where } p_{\varepsilon_n n} \neq 0\right\}.$$

The latter remark and Theorem 3.1 describe exclusively the point spectra of the distribution of the random variable ξ.

4. Topological and metric properties of the spectrum of the probability distribution of ξ

Recall that [14] the set of points of growth of the distribution function $F_\xi(x)$ of a random variable ξ is called the *spectrum* of $F_\xi(x)$ and is denoted by S_ξ (alternatively, S_ξ is called the minimal closed support); that is,

$$S_\xi = \big\{x\colon F_\xi(x+\varepsilon) - F_\xi(x-\varepsilon) = \mathsf{P}\{\xi \in (x-\varepsilon; x+\varepsilon)\} > 0, \forall \varepsilon > 0\big\}.$$

Lemma 4.1. *If $p_{in} > 0$ for all $i \in \{0,1\}$ and all $n \in \mathbb{N}$, then the spectrum S_ξ of the distribution of the random variable ξ coincides with the set $E(a_n)$ of all incomplete sums of series* (2); *that is,*

$$S_\xi = E(a_n) \equiv \left\{x\colon x = \sum_{n\in\mathbb{N}} a_n,\ M \in 2^{\mathbb{N}}\right\}.$$

Proof. The result follows from the definition of the spectrum of a distribution, since every incomplete sum of a series can be written as follows:

$$x(M) = \sum_{n=1}^{\infty} a_n \varepsilon_n, \quad \text{where } \varepsilon_n = \begin{cases} 1, & \text{if } n \in M, \\ 0, & \text{if } n \notin M, \end{cases}$$

and since the set of all incomplete sums of a series is a perfect set (in other words, it is a closed set without isolated points; see [4]). □

Corollary 4.1. *The spectrum S_ξ of the distribution of ξ is such that $S_\xi \subset E(a_n)$.*

In order to study the spectral properties of the distribution of ξ we investigate the topological, metric, and fractal properties of the set of incomplete sums $E(a_n)$ of series (2) for which the condition of homogeneity (3) holds.

Let series (2) be such that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. It is known in this case that if $r_n \geq a_n$ for all $n \in \mathbb{N}$, then the set of all incomplete sums coincides with the interval $[0, r_0]$ (see [4]). If $r_n \geq a_n$ for all sufficiently large n, then the set of incomplete sums is a finite union of intervals. If $r_n < a_n$ for all sufficiently large n, then the set of incomplete sums is nowhere dense [4, 14]. Less studied is the case where both inequalities $a_n \leq r_n$ and $a_n > r_n$ hold for infinitely many numbers n. In this case, it may happen that the set of incomplete sums is nowhere dense or it may contain intervals.

The topological and metric properties of the set of incomplete sums depend essentially on the rate of convergence of the series. The authors are not aware of any necessary and sufficient condition for this set to have a zero Lebesgue measure. The fractal properties of the set of incomplete sums are studied even less, although some results can be found in [7, 11, 12, 14, 16] for partial classes of series.

We recall the definition of the *Hausdorff α-measure* and *Hausdorff–Besicovitch dimension* of the set $E \subset \mathbb{R}^1$ that better characterize the "capacity" of null sets with respect to the Lebesgue measure.

Definition 4.1. Let $0 < \alpha$ be a fixed real number. Then

$$\mathcal{H}^\alpha(E) = \lim_{\varepsilon \to 0} m_\varepsilon^\alpha(E) = \sup_{\varepsilon > 0} m_\varepsilon^\alpha(E), \quad \text{where } m_\varepsilon^\alpha(E) = \inf_{|E_j| \leq \varepsilon} \left\{ \sum_j |E_j|^\alpha \right\},$$

is called the *Hausdorff α-measure* or *α-measure* of a set E, where lim inf is evaluated over all possible at-most-countable coverings of the set E by intervals E_i whose diameters $|E_i|$ do not exceed ε.

The non-negative number

$$\alpha_0(E) = \sup \{\alpha \colon \mathcal{H}^\alpha(E) = +\infty\} = \inf \{\alpha \colon \mathcal{H}^\alpha(E) = 0\}$$

is called the *Hausdorff–Besicovitch dimension* of the set E.

The Hausdorff–Besicovitch dimension possesses the following properties:

1) If $E_1 \subset E_2$, then $\alpha_0(E_1) \leq \alpha_0(E_2)$;
2) $\alpha_0 \left(\bigcup_i E_i \right) = \sup_i \alpha_0(E_i)$.

A set of cardinality larger than $\aleph_0$ whose Hausdorff–Besicovitch dimension equals zero is called *anomalous fractal*, while a Lebesgue null set whose Hausdorff–Besicovitch equals 1 is called *superfractal*.

Theorem 4.1. *The set of all incomplete sums of series* (2) *satisfying homogeneity condition* (3) *is a nowhere dense set whose Lebesgue measure and Hausdorff–Besicovitch dimension both equal zero.*

Proof. Since the series converges, $a_n \to 0$ and $\delta_n \equiv \frac{a_n}{r_n} = \frac{1}{a_{n-1}} \to +\infty$ as $n \to \infty$. According to Lemma 3.2, there exists a number n_0 such that the inequality $a_n > r_n$ holds for all $n \geq n_0$, and thus Theorem 2.8.3 of [14] implies that the set of incomplete sums E_1 of the series $\sum_{n=n_0}^{\infty} a_n$ is anomalous fractal.

Since the set $E(a_n)$ is a vector sum of the sets E_1 and E_2 and the cardinality of the set

$$E_2 = \left\{ x \colon x = \sum_{n=1}^{n_0-1} \varepsilon_n a_n,\, \varepsilon_n \in \{0,1\} \right\}$$

does not exceed 2^{n_0-1}, the fractal properties of the sets E_1 and $E(a_n)$ coincide. Hence $E(a_n)$ is an anomalous fractal set. □

Corollary 4.2. *The spectrum S_ξ of the distribution of the random variable ξ is an anomalous fractal set.*

Theorem 4.2. *In the continuous case, that is if $L = 0$, the distribution of the random variable ξ is a singular Cantor type distribution with an anomalous fractal spectrum.*

Proof. The distribution of the random variable ξ is continuous if $L = 0$. Its spectrum is a subset of the set of incomplete sums. The result obtained above concerning the geometric structure of the set of incomplete sums of series (2) shows that the spectrum of the distribution of the random variable ξ is an anomalous fractal set whose Lebesgue measure equals zero. □

5. Autoconvolutions of the distribution of the random variable ξ

Recall that the distribution of the random variable $\psi_2 = \xi^{(1)} + \xi^{(2)}$ is called the *autoconvolution of the distribution of the random variable* ξ, where $\xi^{(1)}$ and $\xi^{(2)}$ are independent copies of ξ. Then the *s-tuple convolution of the distribution of the random variable* ξ with itself is the distribution of the random variable

$$\psi_s = \xi^{(1)} + \xi^{(2)} + \cdots + \xi^{(s)},$$

where $\xi^{(j)}$ are independent identically distributed random variables with the same distribution as that of ξ.

It is well known that if the distribution of ξ is discrete, then ψ_s has a discrete distribution, as well. All three cases are possible; namely, the convolution of two singular distributions can be either singular, or absolutely continuous, or their mixture, but we are interested in the case where the distribution of ξ is singular.

Remark 5.1. The autoconvolution of two (more generally, of a finite number of) infinite Bernoulli convolutions cannot be a mixture, since the sum of two (of a finite number) of independent random variables of the Jessen–Wintner type is a Jessen–Wintner random variable, and its distribution belongs to a pure type.

Lemma 5.1. *The spectrum S_{ψ_s} of the distribution of the random variable ψ_s is a subset of the interval $[0, sr_0]$ and belongs to the union of $(s+1)^n$ isometric intervals of length sr_n, $n = 1, 2, 3, \ldots$.*

Proof. The random variable ψ_s can be represented as follows:

$$\psi_s = \eta_1 a_1 + \eta_2 a_2 + \cdots + \eta_n a_n + \cdots = \sum_{n=1}^{\infty} \eta_n a_n,$$

where

$$\eta_n = \xi_n^{(1)} + \xi_n^{(2)} + \cdots + \xi_n^{(s)}$$

are independent random variables whose distributions are

$$P\{\eta_n = i\} = C_s^i p_{1n}^i p_{0n}^{s-i}, \qquad i \in \{0, 1, 2, \dots, s\} = A_{s+1}.$$

Since $p_{in} > 0$, the spectrum of the random variable ψ_s coincides with the set

$$S_{\psi_s} = S_{\xi^{(1)}} \oplus S_{\xi^{(2)}} \oplus \dots \oplus S_{\xi^{(s)}} = \left\{ x \colon x = \sum_{n=1}^{\infty} \zeta_n a_n,\ (\zeta_n) \in A_{s+1}^{\infty} \right\}.$$

Let $(d_1, d_2, \dots, d_m)$ be a fixed ordered family of numbers of the set A_{s+1}, and let $\Delta'_{d_1 \dots d_m}$ be the set of all numbers of the following form:

$$\sum_{n=1}^{m} d_n a_n + \sum_{n=m+1}^{\infty} \zeta_n a_n, \quad \text{where } \zeta_n \in A_{s+1}.$$

It is easy to see that the set S_{ψ_s} belongs to the union of all intervals of the form

$$\Delta_{d_1 d_2 \dots d_m} = \left[\sum_{n=1}^{m} d_n a_n,\ s r_m + \sum_{n=1}^{m} d_n a_n \right] = \left[\inf \Delta'_{d_1 \dots d_m}, \sup \Delta'_{d_1 \dots d_m} \right]$$

that are called *cylindrical intervals of rank* m *with base* $d_1 d_2 \dots d_m$, $d_i \in A_{s+1}$. The diameter of such an interval equals $|\Delta_{d_1 \dots d_n}| = s r_n$. Since

$$\Delta'_{d_1 d_2 \dots d_m} = \Delta'_{d_1 d_2 \dots d_m 0} \cup \Delta'_{d_1 d_2 \dots d_m 1} \cup \dots \cup \Delta'_{d_1 d_2 \dots d_m s},$$

the total number of the corresponding cylindrical intervals of rank m is $(s+1)^m$.

Thus $S_{\psi_s} \subset G_{m+1} \subset G_m$ for all m and

$$S_{\psi_s} = \lim_{m \to \infty} G_m = \bigcap_{m=1}^{\infty} G_m, \quad \text{where } G_m = \bigcup_{(d_1 \dots d_m)} \Delta_{d_1 \dots d_m}. \qquad \square$$

Theorem 5.1. *Assume that the random variable ξ is continuous; that is, $L = 0$. Then, for an arbitrary natural number $s \geq 2$, the distribution of the random variable ψ_s is a singular Cantor type distribution with an anomalous fractal spectrum.*

Proof. According to the preceding lemma, the set S_{ψ_s} belongs to the union of $(s+1)^n$ cylindrical intervals of rank n whose diameters are $s r_n$ and thus

$$\lambda(S_{\psi_s}) \leq \lim_{n \to \infty} (s(s+1)^n r_n) = s \lim_{n \to \infty} ((s+1)^n a_n a_{n-1}).$$

By Corollary 2.1,

$$\lambda(S_{\psi_s}) \leq s \lim_{n \to \infty} (s+1)^n q^n q^{n-1} = \frac{s}{q} \lim_{n \to \infty} \left((s+1) q^2 \right)^n = 0$$

for all sufficiently small q, whence $\lambda(S_{\psi_s}) = 0$.

To find the Hausdorff–Besicovitch dimension of the set S_{ψ_s}, consider its ε-covering by cylindrical intervals of rank n, $\varepsilon = s r_n$. For an arbitrary $\alpha > 0$ and arbitrary $n \geq n_0$, we have

$$\begin{aligned} m_\varepsilon^\alpha(S_{\psi_s}) &\leq (s+1)^n (s r_n)^\alpha = (s+1)^n (s a_n a_{n-1})^\alpha \\ &\leq (s+1)^n \left(s q^n q^{n-1} \right)^\alpha = \left(\frac{s}{q} \right)^\alpha \left((s+1) q^{2\alpha} \right)^n. \end{aligned}$$

Since

$$(s+1) q^{2\alpha} < 1$$

and

$$\left(\frac{s}{q} \right)^\alpha \left((s+1) q^{2\alpha} \right)^n \to 0 \quad \text{as } n \to \infty,$$

for a sufficiently small q, we obtain

$$\mathcal{H}^\alpha(S_{\psi_s}) = \lim_{\varepsilon\to 0} m_\varepsilon^\alpha(S_{\psi_s}) = 0$$

for all $\alpha > 0$. Thus $\alpha_0(S_{\psi_s}) = 0$. □

Remark 5.2. One can generalize the results obtained above to the case of infinite Bernoulli convolutions such that

$$r_n/a_n \to 0, \qquad n \to \infty.$$

Indeed, the condition

$$\frac{r_n}{a_n} = \frac{a_{n+1} + a_{n+2} + a_{n+3} + \dots}{a_n} = \sum_{m=1}^{\infty} \frac{a_{n+m}}{a_n} \to 0, \qquad n \to \infty,$$

implies that $a_{n+m}/a_n \to 0$, $n \to \infty$, for all $m \in \mathbb{N}$. Thus, given an arbitrary $q \in (0,1)$, there exists $n_0 = n_0(q)$ such that

$$r_n < a_n < q^n, \qquad n > n_0(q).$$

The rest of the proof is analogous to that of Theorem 5.1.

Remark 5.3. The distribution of the random variable ξ belongs to the class of singular distributions $\mathfrak{M}_0$ according to the Zolotarev–Kruglov classification (see [13]).

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