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A RANDOM VARIABLE WHOSE DIGITS IN THE $\widetilde{L}$-REPRESENTATION HAVE THE MARKOVIAN DEPENDENCE

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Abstract. The distribution of the random variable

$$\theta = \frac{1}{\theta_1} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\theta_1(\theta_1+1)\dots\theta_{n-1}(\theta_{n-1}+1)\theta_n}$$

is studied where (θ_n) is a homogeneous Markov chain assuming only positive integer values and having the initial distribution $(p_1, p_2, \dots, p_n, \dots)$ and transition matrix $\|p_{ik}\|$. The Lebesgue structure of the distribution (discrete, absolutely continuous, and singular components) is studied and topological, metric and fractal properties of the spectrum (the minimal closed support of the distribution) is investigated.

Introduction

Recall [7] that the $\widetilde{L}$-representation of a real number $x \in (0;1]$ is its representation via an alternating Lüroth series [1, 2]:

$$x = \frac{1}{a_1} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{a_1(a_1+1)\dots a_{n-1}(a_{n-1}+1)a_n} \equiv \Delta^{\widetilde{L}}_{a_1a_2\dots a_n\dots}$$

or

$$x = \frac{1}{a_1} + \sum_{n=2}^{k} \frac{(-1)^{n-1}}{a_1(a_1+1)\dots a_{n-1}(a_{n-1}+1)a_n} \equiv \Delta^{\widetilde{L}}_{a_1a_2\dots a_k(0)}$$

if x is represented via a finite sum. We write in this case $x = \Delta^{\widetilde{L}}_{a_1a_2\dots a_n\dots}$, where $a_n \in \mathbb{N} = \{1, 2, 3, \dots\}$.

The criterion for a number to be rational in terms of its $\widetilde{L}$-representation is known [1]: *a real number $x \in (0,1]$ is rational if and only if its $\widetilde{L}$-representation is finite or periodic.* By analogy with rational and irrational numbers, we say that [6] a number is *$\widetilde{L}$-irrational* if its $\widetilde{L}$-representation is infinite; otherwise we say that a number is *$\widetilde{L}$-rational.* Every $\widetilde{L}$-rational number has two formally different representations

$$\Delta^{\widetilde{L}}_{a_1a_2\dots a_{k-1}(0)} = \Delta^{\widetilde{L}}_{a_1a_2\dots a_{k-2}[a_{k-1}+1](0)},$$

while every $\widetilde{L}$-irrational number has a unique representation. Thus the k^{th} digit in the $\widetilde{L}$-representation of a number x is a function of x, $a_k = a_k(x)$.

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In general use in number theory are the notions of a binary rational (more general, base s rational) number and binary irrational (base s irrational) number. Rational numbers have two representations and one of them is periodic with period (0). The notions of $\tilde{L}$-rational and $\tilde{L}$-irrational numbers are introduced above by analogy with the classical notions.

Note that the $\tilde{L}$-representation as well as the representation of numbers with the help of continued fractions are of the same topological nature but the metric theories for them are totally different.

In contrast, the so-called L-representation of numbers [4] based on the representation of numbers by a Lüroth series with positive terms (described for the first time in [2]) and $\tilde{L}$-representation have the same metric relation and thus similar metric theories but topologies for them are different.

In our earlier paper [3], we studied properties of the distribution of the random variable

$$\xi = \frac{1}{\eta_1} + \sum_{k=2}^{\infty} \frac{(-1)^{n-1}}{\eta_1(\eta_1+1)\dots\eta_{k-1}(\eta_{k-1}+1)\eta_k} \equiv \Delta^{\widetilde{L}}_{\eta_1\eta_2\dots\eta_k\dots}$$

with independent elements ($\widetilde{L}$-digits) η_k in the representation by a sign alternating Lüroth series. The Lebesgue structure of the distribution of ξ is studied in [3] in full detail, namely we proved that the distribution is always pure and found criteria for the distribution to belong to any of the three pure types. The topological, metric, and fractal properties of the spectrum are also investigated in [3].

In the current paper, we consider the random variable θ whose $\widetilde{L}$-digits are random variables with the Markov dependence. We are interested in the Lebesgue structure of its distribution as well as in spectral properties, in particular in fractal properties of supports of this distribution.

Let (θ_n) be a sequence of discrete random variables that assume only positive integer values. Assume that (θ_n) is a homogeneous Markov chain with the initial distribution $p_1, p_2, \dots, p_m, \dots$ and matrix of transient probabilities $\|p_{ik}\|$, that is,

$$\mathsf{P}\{\theta_1 = m\} = p_m > 0, \qquad \sum_{m=1}^{\infty} p_m = 1;$$

$$\mathsf{P}\{\theta_{k+1} = j \mid \theta_k = i\} = p_{ij} \geq 0, \qquad \sum_{j=1}^{\infty} p_{ij} = 1, \qquad \forall i \in \mathbb{N}.$$

Consider a random variable θ whose $\widetilde{L}$-representation is infinite and whose digits are random variables θ_n, that is,

$$\theta = \frac{1}{\theta_1} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\theta_1(\theta_1+1)\dots\theta_{n-1}(\theta_{n-1}+1)\theta_n} \equiv \Delta^{\widetilde{L}}_{\theta_1\theta_2\dots\theta_n\dots}.$$

Remark 1. It is obvious that the random variable θ does not assume values in the subset $\mathbb{Q}'$ of rational numbers whose $\widetilde{L}$-representation is finite. Thus θ is defined in the probability space $((0;1)^*, \mathfrak{B}^*, \mathsf{P})$, where $(0;1)^* = (0;1) \setminus \mathbb{Q}'$, and $\mathfrak{B}^*$ is the σ-algebra of Borel subsets of $(0;1)^*$.

Remark 2. If all the rows of the matrix of transient probabilities $\|p_{ik}\|$ are the same and coincide with a vector $(p_1, p_2, \dots, p_n, \dots)$, then the random variables θ_n are independent and identically distributed (recall that this case has been studied in the paper [3]). Moreover if $p_k = \frac{1}{k(k+1)}$ for all $k \in \mathbb{N}$, then the distribution of the random variable θ is uniform in the interval $[0, 1]$.

1. The criterion for the existence of atoms

Since we consider only those real numbers $x \in (0;1]$ that have infinite representations, the number $a_k(x)$, the k^{th} digit of the infinite $\widetilde{L}$-representation of a number x is uniquely defined. This makes clear the statement of the following result.

Lemma 1.1. *Let (c_n) be an arbitrary sequence of positive integer numbers. Then*

$$\mathsf{P}\left\{\theta = \Delta^{\widetilde{L}}_{c_1c_2\dots c_n\dots}\right\} = p_{c_1}\prod_{k=1}^{\infty} p_{c_kc_{k+1}}$$

and

$$\mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{c_1c_2\dots c_n}\right\} = p_{c_1}\prod_{k=1}^{n-1} p_{c_kc_{k+1}},$$

where $\Delta^{\widetilde{L}}_{c_1c_2\dots c_n} \equiv \{x\colon a_i(x) = c_i, i = 1,\dots,n\}$ is a cylinder of rank n with base $c_1c_2\dots c_n$.

Theorem 1.1. *The distribution of the random variable θ has atoms if and only if there exists a sequence of positive integer numbers $(a_1, a_2, \dots, a_k, \dots)$ such that*

$$H(a_n) \equiv p_{a_1}\prod_{k=1}^{\infty} p_{a_ka_{k+1}} > 0.$$

The set

$$D_\theta = \{x\colon H(a_n(x)) > 0\}$$

is the point spectrum (the collection of atoms) of the random variable θ.

Proof. The uniqueness of the $\widetilde{L}$-representation of a number $x \in (0;1)^*$ implies

$$\mathsf{P}\{\theta = x\} = p_{a_1(x)}\prod_{k=1}^{\infty} p_{a_k(x)a_{k+1}(x)}. \tag{1}$$

Thus x is an atom of the distribution of θ if and only if there exists at least one sequence (a_n) such that

$$p_{a_1(x)}\prod_{k=1}^{\infty} p_{a_k(x)a_{k+1}(x)} > 0.$$

This observation proves Theorem 1.1. □

Corollary 1.1. *The distribution of the random variable θ is continuous, that is, the probability measure of every single point set is equal to zero, if and only if $H(a_n) = 0$ for all sequences of positive integer numbers (a_k).*

Corollary 1.2. *Let there exist a collection of positive integer numbers $(i_1, i_2, \dots, i_k)$ such that*

$$p_{i_1i_2} = p_{i_2i_3} = \dots = p_{i_ki_1} = 1.$$

Then a point x is an atom of the distribution of θ with mass p_{i_1} if its $\widetilde{L}$-representation $\Delta^{\widetilde{L}}_{(i_1i_2\dots i_k)}$ is periodic.

Corollary 1.3. *If the entries of the matrix of transient probabilities $\|p_{ik}\|$ are separated from the unity, then the distribution of the random variable θ is continuous.*

Corollary 1.4. *If*

$$M \equiv \prod_{i=1}^{\infty} \max_k\{p_{ik}\} = 0,$$

then the distribution of the random variable θ is continuous.

Indeed, $\sum_{k=1}^{\infty} p_{ik} = 1$ for all $i \in \mathbb{N}$ and thus every row of the matrix $\|p_{ik}\|$ contains a maximal element. Then

$$\mathsf{P}\{\theta = x\} = p_{a_1(x)} \prod_{k=1}^{\infty} p_{a_k(x)a_{k+1}(x)} \le M = 0$$

for all $x \in (0,1)^*$. Therefore the distribution is continuous by definition.

Corollary 1.5. *The inequality*

$$M > 0$$

is a necessary condition for the existence of atoms for the distribution of θ.

Remark 3. The condition $M > 0$ is not sufficient for the existence of atoms. Indeed, if

$$p_{jk_j} \equiv \max_k \{p_{jk}\} = p_{jj}, \qquad j = 1, 2, 3, \dots,$$

then there are no atoms for the distribution of the random variable θ.

2. The distribution function

Lemma 2.1. *The distribution function* $F_\theta(x)$ *of the random variable* θ *is given by*

$$F_\theta(x) = \beta_1(x) + p_{a_1(x)} \sum_{k=2}^{\infty} (\beta_k(x) \prod_{i=1}^{k-2} p_{a_i(x)a_{i+1}(x)}) \tag{2}$$

for an arbitrary $x \in (0;1)^*$, *where*

$$\beta_1(x) = \sum_{j=a_1(x)+1}^{\infty} p_j, \qquad \beta_k(x) = \begin{cases} \sum_{j=a_k(x)+1}^{\infty} p_{a_{k-1}(x)j}, & \text{if } k = 2m-1, \\ \sum_{j=1}^{a_k(x)-1} p_{a_{k-1}(x)j}, & \text{if } k = 2m, \ m \in \mathbb{N}, \end{cases}$$

and $a_k(x)$ *is the* k^{th} *digit in the* $\widetilde{L}$*-representation of a number* x. *The distribution function for other arguments* $x \in \mathbb{R}$ *is defined by using the property of its left continuity.*

Proof. Recall that $F_\theta(x) = \mathsf{P}\{\theta < x\}$. The random event $\{\theta < x\}$, where $x = \Delta^{\widetilde{L}}_{a_1a_2\dots a_k\dots}$, is a union of disjoint events, namely

$$\begin{aligned}\{\theta < x\} = \{\theta_1 > a_1(x)\} &\cup \{\theta_1 = a_1(x) \wedge \theta_2 < a_2(x)\} \cup \dots \\ &\cup \{\theta_1 = a_1(x) \wedge \theta_2 = a_2(x) \wedge \dots \wedge \theta_{2k-2} = a_{2k-2}(x) \wedge \theta_{2k-1} > a_{2k-1}(x)\} \\ &\cup \{\theta_1 = a_1(x) \wedge \theta_2 = a_2(x) \wedge \dots \wedge \theta_{2k-1} = a_{2k-1}(x) \wedge \theta_{2k} < a_{2k}(x)\} \cup \dots\end{aligned}$$

and

$$\begin{aligned}&\mathsf{P}\{\theta_1 = a_1(x), \dots, \theta_{k-1} = a_{k-1}(x), \theta_k > a_k(x)\} \\ &\quad = \mathsf{P}\{\theta_1 = a_1(x)\} \dots \mathsf{P}\{\theta_{k-1} = a_{k-1}(x) \mid \theta_{k-2} = a_{k-2}(x)\} \\ &\qquad \times \mathsf{P}\{\theta_k > a_k(x) \mid \theta_{k-1} = a_{k-1}(x)\} \\ &\quad = p_{a_1(x)} \prod_{i=1}^{k-2} p_{a_i(x)a_{i+1}(x)} \cdot \sum_{j=a_k(x)+1}^{\infty} p_{a_{k-1}(x)j},\end{aligned}$$

while

$$\begin{aligned}&\mathsf{P}\{\theta_1 = a_1(x), \dots, \theta_{k-1} = a_{k-1}(x), \theta_k < a_k(x)\} \\ &\quad = \mathsf{P}\{\theta_1 = a_1(x)\} \dots \mathsf{P}\{\theta_{k-1} = a_{k-1}(x) \mid \theta_{k-2} = a_{k-2}(x)\} \\ &\qquad \times \mathsf{P}\{\theta_k < a_k(x) \mid \theta_{k-1} = a_{k-1}(x)\} \\ &\quad = p_{a_1(x)} \prod_{i=1}^{k-2} p_{a_i(x)a_{i+1}(x)} \cdot \sum_{j=1}^{a_k(x)-1} p_{a_{k-1}(x)j}.\end{aligned}$$

Thus

$$F_\theta(x) = \sum_{j=a_1(x)+1}^{\infty} p_j + p_{a_1(x)} \cdot \sum_{j=1}^{a_2(x)-1} p_{a_1(x)j} + \dots$$
$$+ p_{a_1(x)} \prod_{i=1}^{2k-3} p_{a_i(x)a_{i+1}(x)} \sum_{j=a_{2k-1}(x)+1}^{\infty} p_{a_{2k-2}(x)j}$$
$$+ p_{a_1(x)} \prod_{i=1}^{2k-2} p_{a_i(x)a_{i+1}(x)} \sum_{j=1}^{a_{2k}(x)-1} p_{a_{2k-1}(x)j} + \dots .$$

Using the above notation $\beta_k(x)$ we prove (2). □

Corollary 2.1. *If all entries of the matrix of transient probabilities $\|p_{ik}\|$ are positive, then the distribution function F_θ of the random variable θ is strictly increasing in the interval $[0,1]$ and its spectrum does not have fractal properties. If $p_{\tau\varsigma=0}$, then $F_\theta(x)$ is constant in every interval*

$$\nabla_{a_1a_2\dots a_k\tau\varsigma} \equiv \left(\Delta_{a_1a_2\dots a_k\tau\varsigma(1)}; \Delta_{a_1a_2\dots a_k\tau(\varsigma+1)(1)}\right).$$

Lemma 2.2. *If the derivative $F'_\theta(x_0)$ of the distribution function $F_\theta(x_0)$ exists at a point x_0, then*

$$F'_\theta(x_0) = p_{a_j(x_0)} \prod_{k=1}^{\infty} \left[a_k(x_0)\left(a_k(x_0) \cdot p_{a_k(x_0)a_{k+1}(x_0)} + 1\right)\right]. \tag{3}$$

Proof. Indeed, if the derivative $F'_\theta(x_0)$ exists, then

$$F'_\theta(x_0) = \lim_{k\to\infty} \frac{\mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{a_1(x_0)\dots a_k(x_0)}\right\}}{\left|\Delta^{\widetilde{L}}_{a_1(x_0)\dots a_k(x_0)}\right|}.$$

Further,

$$\mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{a_1(x_0)\dots a_k(x_0)}\right\} = p_{a_1(x_0)} \prod_{j=1}^{k} p_{a_j(x_0)a_{j+1}(x_0)}$$

and

$$\left|\Delta^{\widetilde{L}}_{a_1(x_0)\dots a_k(x_0)}\right| = \prod_{j=1}^{k} \frac{1}{a_j(x_0)(a_j(x_0)+1)}.$$

This implies (3). □

3. The Lebesgue structure of the distribution

Recall that [5], according to the classical Lebesgue theorem on the decomposition of any function of bounded variation, an arbitrary distribution function F can be represented as follows:

$$F(x) = \alpha_1 F_d(x) + (1-\alpha_1)F_c(x) \tag{4}$$
$$= \alpha_1 F_d(x) + \alpha_2 F_{ac}(x) + \alpha_3 F_s(x), \tag{5}$$

where $\alpha_i \geq 0$, $\alpha_1+\alpha_2+\alpha_3 = 1$, $0 \leq \alpha_1 \leq 1$, and where $F_d(x)$ is discrete (jump function), $F_c(x)$ continuous, $F_{ac}(x)$ absolutely continuous, and $F_s(x)$ singular components of the distribution function F.

Each of the equalities (4) and (5) is called the *Lebesgue structure of the distribution function F*; equality (4) means that a distribution function is a mixture of discrete and continuous components, while (5) extends this result by stating that the continuous component is in turn a mixture of absolutely continuous and singular components.

If one of the coefficients α_i equals 1 (and thus all others are 0), then the distribution function is called pure. Depending on which coefficient equals one, the distribution function is either purely discrete, or purely absolutely continuous, or purely singular. If any of the coefficients does not equal one, then the distribution function is called a mixture of those components whose coefficients are non-zero. In particular, if $0 < \alpha_1 < 1$, then the distribution function is called a mixture of discrete and continuous components.

Theorem 3.1. *If every entry of the matrix $\|p_{ik}\|$ equals either zero or one, then the distribution function of the random variable θ is purely discrete with a single atom at every cylinder of rank one whose mass is equal to the corresponding initial probability. In particular,*

1) *if $p_{jj} = 1$ for all $j \in \mathbb{N}$, then the point spectrum coincides with the collection of points $\Delta^{\widetilde{L}}_{(j)}$;*
2) *if $p_{jk} = 1$ for all $j \in \mathbb{N}$ and some fixed $k \in \mathbb{N}$, then the point spectrum coincides with the collection of points $\Delta^{\widetilde{L}}_{j(k)}$;*
3) *if $p_{j(j+k)} = 1$ for all $j \in \mathbb{N}$ and some fixed $k \in \mathbb{N}$, then the point spectrum coincides with the collection of points $\Delta^{\widetilde{L}}_{j[j+k][j+k][j+2k][j+2k][j+3k]\dots}$.*

Proof. It suffices to show that

$$\mathsf{P}\{\theta \in \Delta_j\} = \sum_{x \in D_\theta \cap \Delta_j} \mathsf{P}\{\theta = x\} \tag{6}$$

for all positive integer numbers j.

Let j be an arbitrary positive integer number. By Lemma 1.1,

$$\mathsf{P}\{\theta \in \Delta^{\widetilde{L}}_j\} = p_j.$$

Consider separately the following exclusive cases: $p_{jj} = 1$ and $p_{jj} \neq 1$.

If $p_{jj} = 1$, then, by Theorem 1.1, the point $\Delta^{\widetilde{L}}_{(j)}$ is a unique atom of the distribution of the random variable θ in the cylinder $\Delta^{\widetilde{L}}_j$. Since the mass concentrated at this point is p_j, equality (6) holds.

If $p_{jj} \neq 1$, then there exists a number $j_1 \neq j$ such that $p_{jj_1} = 1$. Then either $p_{j_1j_1} = 1$ or $p_{j_1j_1} \neq 1$.

If $p_{j_1j_1} = 1$, then a point with a periodic representation $\Delta^{\widetilde{L}}_{j(j_1)}$ is an atom with mass p_j. Then the cylinder $\Delta^{\widetilde{L}}_j$ contains only one atom with mass p_j. Thus equality (6) holds.

If $p_{j_1j_1} \neq 1$, then there exists a positive integer number $j_2 \neq j_1$ such that $p_{j_1j_2} = 1$. Again we consider two cases $p_{j_2j_2} = 1$ and $p_{j_2j_2} \neq 1$.

If there exists j_k that equals j_m, $m < k$, and such that $p_{j_kj_m} = 1$, then a point with a periodic representation is an atom whose mass equals p_j.

If there are no k and j_k, then one may find a sequence (j_n) such that $p_{j_nj_{n+1}} = 1$ for all n (moreover $j_n \neq j_m$ for $m < n$). Then the point $\Delta^{\widetilde{L}}_{jj_1j_2\dots j_n\dots}$ is an atom of the distribution θ and it belongs to $\Delta^{\widetilde{L}}_j$. Its mass is given by

$$\mathsf{P}\left\{\theta = \Delta^{\widetilde{L}}_{jj_1j_2\dots j_n\dots}\right\} = p_j \prod_{n=1}^{\infty} p_{j_nj_{n+1}} = p_j.$$

Therefore equality (6) holds.

Now we consider some partial cases.

1) The part proved above for the case of $p_{jj} = 1$ with some j shows that the point $\Delta^{\widetilde{L}}_{(j)}$ is an atom of the distribution and its mass equals p_j. Since $\sum_{j=1}^{\infty} p_j = 1$, the total mass of these atoms equals one. This means that the point spectrum coincides with the collection of points $\Delta^{\widetilde{L}}_{(j)}$.

2) If $p_{jk} = 1$ for all $j \in \mathbb{N}$ and some fixed $k \in \mathbb{N}$, then $p_{kk} = 1$ and thus the points $\Delta^{\widetilde{L}}_{j(k)}$ are the atoms of the distribution and their masses are p_j, respectively. Since the sum of all p_j with $j \in \mathbb{N}$ equals one, the point spectrum coincides with the collection of these points.

3) If $p_{j(j+k)} = 1$ for all $j \in \mathbb{N}$ and some fixed $k \in \mathbb{N}$, then Lemma 1.1 implies that

$$\mathsf{P}\left\{\theta = \Delta^{\widetilde{L}}_{m[m+k][m+k][m+2k][m+2k][m+3k]\ldots}\right\} = p_m \prod_{i=m}^{\infty} p_{i(i+k)} = p_m$$

for all $m \in \mathbb{N}$. Since the sum of all initial probabilities equals one, the point spectrum coincides with the set of points $\Delta^{\widetilde{L}}_{j[j+k][j+k][j+2k][j+2k][j+3k]\ldots}$. □

Remark 4. The distribution of θ may have atoms even if the matrix of transient probabilities does not contain zeros at all. For example, if $p_{ik} > 0$ for all $i \in \mathbb{N}$, $k \in \mathbb{N}$, but

$$p_{i(i+1)} = 1 - \frac{1}{(i+1)^2},$$

then the point spectrum of the distribution coincides with the tail set of the $\widetilde{L}$-representation. One of its members is the point $x_0 = \Delta^{\widetilde{L}}_{a_1 a_2 \ldots a_n \ldots}$, where $a_n = n$, $n = 1, 2, \ldots$. Indeed,

$$\mathsf{P}\{\theta = x_0\} = \prod_{i=1}^{\infty}\left(1 - \frac{1}{(i+1)^2}\right) = \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) > 0,$$

since the series $\sum_{n=2}^{\infty} n^{-2}$ converges.

Lemma 3.1. *Let there exist $m \in \mathbb{N}$ such that $p_{jj} = 1$ and $p_{[m+i]j} = 0$ for all $j \le m$. Assume that*

$$\prod_{i=1}^{\infty} \max_k \{p_{[m+i][m+k]}\} = 0$$

for all $k \in \mathbb{N}$. Then the Lebesgue structure of the distribution function of the random variable θ is given by

$$F_\theta(x) = pF_d(x) + qF_c(x), \tag{7}$$

where

$$p = \sum_{j=1}^{m} p_j, \qquad q = 1 - p, \qquad F_d(x) = \frac{1}{p}\sum_{x_j < x} p_j, \qquad F_c(x) = \frac{1}{q}\left(F_\theta(x) - pF_d(x)\right).$$

Proof. By Lebesgue's theorem, the distribution function of the random variable θ can be written in the form of (7), where $F_d(x)$ is a discrete distribution function, $F_c(x)$ is a continuous distribution function, p is the total mass of atoms, and $q = 1 - p$.

Under the assumptions of the lemma, it is easy to see that $p_{j(k+1)} = 0$ for all $k \in \mathbb{N}$ and the distribution function of θ has m atoms, namely the points $x_j = \Delta^{\widetilde{L}}_{(j)}$ are the atoms with masses p_j, respectively. Thus

$$p = \sum_{j=1}^{m} p_j, \qquad q = \left(1 - \sum_{j=1}^{m} p_j\right).$$

Then

$$F_d(x) = \frac{1}{p} \sum_{x_j < x} p_j.$$

This equality together with Lemma 2.1 yields the distribution function $F_\theta(x)$, and thus equality (7) implies that $F_c(x) = \frac{1}{q}\left(F_\theta(x) - pF_d(x)\right)$. □

Corollary 3.1. *Depending on the structure of the matrix of transient probabilities $\|p_{ik}\|$, the distribution of the random variable θ can be purely discrete, or purely singular, or a mixture of discrete and singular types, that is, it may contain both a non-zero discrete and continuous components. Theorem* 3.1 *provides examples of purely discrete distributions or purely continuous distribution* (*see Corollary* 1.1). *Lemma* 3.1 *provides an example of a mixture of discrete and singular types.*

4. Spectral properties of the distribution

Let $D[\widetilde{L}, \overline{ij}]$ be the set of real numbers whose $\widetilde{L}$-representation does not contain a given sequence of digits i and j, that is,

$$D[\widetilde{L}, \overline{ij}] = \left\{x \colon x = \Delta^{\widetilde{L}}_{a_1 \ldots a_n \ldots}, a_k a_{k+1} \neq ij, \forall k \in N\right\}.$$

Lemma 4.1. *The set*

$$D[\widetilde{L}, \overline{ij}] = \left\{x \colon x = \Delta^{\widetilde{L}}_{a_1 a_2 \ldots a_n \ldots}, a_k a_{k+1} \neq ij, \forall k \in \mathbb{N}\right\}$$

is a nowhere dense set whose Lebesgue measure equals zero.

Proof. First we prove that $D[\widetilde{L}, \overline{ij}]$ is a nowhere dense set. Let (a, b) be an arbitrary subinterval of $[0, 1]$. It is easy to find a cylinder $\Delta^{\widetilde{L}}_{c_1 c_2 \ldots c_m} \subset (a, b)$. Then the interior of the cylinder $\Delta^{\widetilde{L}}_{c_1 c_2 \ldots c_m ij}$ does not contain any point of the set $D[\widetilde{L}, \overline{ij}]$. Thus $D[\widetilde{L}, \overline{ij}]$ is a nowhere dense set by definition.

Then we prove that $\lambda\big(D[\widetilde{L}, \overline{ij}]\big) = 0$, where λ is the Lebesgue measure. Let F_{2k} be the union of all cylinders of rank $2k$ whose interiors contain points of the set $D[\widetilde{L}, \overline{ij}]$, $F_0 = (0, 1]$. Let the set $\overline{F}_{2(k+1)}$ be defined by

$$\overline{F}_{2(k+1)} = F_{2k} \setminus F_{2(k+1)}.$$

Then $\lambda(\overline{F}_{2(k+1)}) = \lambda(F_{2k}) - \lambda(F_{2(k+1)})$ and

$$\frac{\lambda(\overline{F}_{2(k+1)})}{\lambda(F_{2k})} = 1 - \frac{\lambda(F_{2(k+1)})}{\lambda(F_{2k})}. \tag{8}$$

It is clear that $F_{2k} \supset F_{2(k+1)} \supset D[\widetilde{L}, \overline{ij}]$ for all $k \in \mathbb{N}$ and

$$D[\widetilde{L}, \overline{ij}] = \bigcap_{k=1}^{\infty} F_{2k} = \lim_{k \to \infty} F_{2k}.$$

The property of continuity of the Lebesgue measure implies that

$$\lambda\big(D[\widetilde{L}, \overline{ij}]\big) = \lim_{k \to \infty} \lambda(F_{2k}).$$

Then

$$\lambda\big(D[\widetilde{L}, \overline{ij}]\big) = \lim_{k \to \infty} \frac{\lambda(F_{2k})}{\lambda(F_{2(k-1)})} \cdot \frac{\lambda(F_{2(k-1)})}{\lambda(F\lambda_{2(k-2)})} \cdot \ldots \cdot \frac{\lambda(F_2)}{\lambda(F_0)} = \prod_{k=1}^{\infty} \frac{\lambda(F_{2k})}{\lambda(F_{2(k-1)})}.$$

Now equality (8) implies

$$\lambda\big(D[\widetilde{L},\overline{ij}]\big) = \prod_{k=1}^{\infty}\left[1 - \frac{\lambda(\overline{F}_{2k})}{\lambda(F_{2(k-1)})}\right].$$

The latter infinite product diverges to zero if and only if the series

$$\sum_{k=1}^{\infty}\frac{\lambda(\overline{F}_{2k})}{\lambda(F_{2(k-1)})} \tag{9}$$

diverges.

Next we establish some estimates of the ratio $\frac{\lambda(\overline{F}_{2k})}{\lambda(F_{2(k-1)})}$. Let $\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{2k}}$ be a cylinder of F_{2k}. Consider two cases

1) $c_{2k} = i$,
2) $c_{2k} \neq i$.

If $c_{2k} = i$, then $\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{2k}j} \cap D[\widetilde{L},\overline{ij}] = \varnothing$ and

$$\frac{\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{2k}j}}{\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{2k}}} = \frac{1}{j(j+1)}.$$

If $c_{2k} \neq i$, then $\operatorname{int}\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{2k}ij} \cap D[\widetilde{L},\overline{ij}] = \varnothing$ and

$$\frac{\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{2k}ij}}{\Delta^{\widetilde{L}}_{c_1c_2\ldots c_{2k}}} = \frac{1}{i(i+1)j(j+1)}.$$

Thus

$$0 < \frac{1}{i(i+1)j(j+1)} \leq \frac{\lambda(\overline{F}_{2k})}{\lambda(F_{2(k-1)})} \leq \frac{1}{j(j+1)} < 1.$$

Therefore series (9) diverges and $\lambda\big(D[\widetilde{L},\overline{ij}]\big) = 0$. □

Recall that the *spectrum* S_θ of the distribution of the random variable θ is the set of all points of growth of the distribution function F_θ, that is, the minimal closed set where the distribution of the random variable θ is concentrated.

Lemma 4.2. *The spectrum of the distribution of the random variable θ coincides with closure of the set*

$$E = \left\{x\colon x = \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k\ldots}, p_i > 0\,\forall i \in \mathbb{N}, p_{a_ka_{k+1}} > 0\,\forall k \in \mathbb{N}\right\}.$$

Proof. 1. First we show that $E \subset S_\theta$. Let $\Delta^{\widetilde{L}}_{a_1a_2\ldots a_k\ldots} = x \in E$. Then

$$\mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}\right\} = p_{a_1}\prod_{i=1}^{k-1}p_{a_ia_{i+1}} > 0$$

for an arbitrary $k \in \mathbb{N}$. The properties of a cylinder imply that, given an arbitrary positive ε, there exists a number k such that

$$\Delta^{\widetilde{L}}_{a_1a_2\ldots a_k} \subset (x-\varepsilon, x+\varepsilon).$$

Thus

$$\mathsf{P}\{\theta \in (x-\varepsilon,x+\varepsilon)\} \geq \mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}\right\} > 0,$$

that is, $x \in S_\theta$, whence $E \subset S_\theta$.

2. Next we prove that $S_\theta \subset \overline{E}$. Let $x \in S_\theta$, that is,

$$\mathsf{P}\{\theta \in (x-\varepsilon,x+\varepsilon)\} > 0 \quad \text{for all } \varepsilon > 0. \tag{10}$$

Assume that there exists k such that $p_{a_{k-1}a_k} = 0$, where $\Delta^{\widetilde{L}}_{a_1a_2\ldots a_k\ldots} = x$. Then

$$\mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}\right\} = p_{a_1}\prod_{i=1}^{k-1} p_{a_ia_{i+1}} = 0.$$

Consider separately the following two cases:

(1) there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}$;

(2) $(x-\varepsilon, x+\varepsilon) \not\subset \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}$ for any $\varepsilon > 0$.

In the first case,

$$\mathsf{P}\{\theta \in (x-\varepsilon, x+\varepsilon)\} \le \mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}\right\} = 0,$$

which contradicts (10).

In the second case, x is a one-sided limit point of the set S_θ. For the sake of definiteness, assume that x is a left limit point. Then there exists $\varepsilon > 0$ such that $(x-\varepsilon, x) \subset \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}$ and $\mathsf{P}\{\theta \in (x, x+\varepsilon)\} = 0$. In this case,

$$\mathsf{P}\{\theta \in (x-\varepsilon, x+\varepsilon)\} = \mathsf{P}\{\theta \in (x-\varepsilon, x)\} \le \mathsf{P}\left\{\theta \in \Delta^{\widetilde{L}}_{a_1a_2\ldots a_k}\right\} = 0,$$

which contradicts (10).

This contradiction proves that $p_{a_{k-1}a_k} > 0$ for all $k \in \mathbb{N}$, that is, $x \in E$. Therefore $S_\theta = E$ and this is what had to be proved. □

Theorem 4.1. *If the matrix of transient probabilities contains at least one zero, then the Lebesgue measure of the spectrum of the distribution of the random variable θ equals zero.*

Proof. If $p_{ij} = 0$, then $S_\theta \subset D[L, \overline{ij}]$ by Lemma 4.2. Thus $\lambda(S_\theta) = \lambda(D[L, \overline{ij}]) = 0$ in view of Lemma 4.1. □

Corollary 4.1. *If all entries of the matrix of transient probabilities $\|p_{ik}\|$ are separated from unity and it contains at least one zero, then the distribution of the random variable θ is a Cantor type singular distribution.*

Corollary 4.2. *If the matrix of transient probabilities $\|p_{ik}\|$ contains at least one zero and for any sequence (a_n), $a_n \in \mathbb{N}$, the right hand side of (1) equals zero, then the distribution of θ is a Cantor type singular distribution.*

5. Fractal properties of the spectrum

Theorem 5.1. *Assume that $p_{ii} > 0$, $p_{i(i+1)} > 0$, and $p_{ii} + p_{i(i+1)} = 1$ for all $i \in \mathbb{N}$. Then the spectrum of the distribution of the random variable θ is an anomalous fractal set, that is, its Hausdorff–Besicovitch dimension* [6] *equals zero.*

Proof. It is obvious that

$$S_\theta \subset E = \left\{x\colon x = \Delta^{\widetilde{L}}_{a_1a_2\ldots a_n\ldots}, \text{ where } a_{n+1} - a_n \in \{0;1\}\, \forall n \in \mathbb{N}\right\}.$$

We are going to show that $\alpha_0(E) = 0$. It suffices to prove that

$$\alpha_0(E) < \alpha$$

for all $\alpha > 0$.

Since

$$S_\theta = \bigcup_{i=1}^{\infty} \Delta'_i,$$

where $\Delta'_i = \Delta_i^{\widetilde{L}} \cap S_\theta$, $i = 1, 2, \dots$, and $S_\theta \overset{\frac{1}{2}}{\sim} \Delta'_i$, it suffices to prove that

$$\alpha_0(\Delta'_1) = 0.$$

It is easy to see that the dimension of the set Δ'_1 does not exceed the dimension of the self-similar set

$$C = C[\widetilde{L}, V], \quad \text{where } V = \{1, 2\}.$$

The latter dimension is the solution of the equation

$$\frac{1}{2^x} + \frac{1}{6^x} = 1$$

and thus the dimension of the set Δ'_1 does not exceed 0.61.

Taking into account that

$$\Delta'_1 = \bigcup_{n=1}^{\infty} \Delta'_{\underbrace{11\dots1}_{n}2} \cup \{\Delta_{(1)}\}, \quad \text{where } \Delta'_{12} \overset{k}{\sim} \Delta'_{\underbrace{1\dots1}_{n}2}, \; k = \frac{1}{2^{n-1}},$$

and $\alpha_0(\Delta'_{12}) \le \alpha_0(C[\widetilde{L}, V])$, where $V = \{2, 3\}$, we see that $\alpha_0(\Delta'_{12})$ does not exceed the solution of the equation

$$\frac{1}{6^x} + \frac{1}{12^x} = 1,$$

that is, it does not exceed the number 0.34.

Analogously,

$$\Delta'_1 = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \Delta'_{\underbrace{11\dots1}_{n}\underbrace{22\dots2}_{m}3} \cup \{\Delta_{1(2)}\} \cup \{\Delta_{11(2)}\},$$

where

$$\Delta'_{123} \overset{k}{\sim} \Delta'_{\underbrace{11\dots1}_{n}\underbrace{22\dots2}_{m}3}, \qquad k = \frac{1}{2^{n+m-2}},$$

whence $\alpha_0(\Delta'_{123}) \le \alpha_0(C[\widetilde{L}, V])$, where $V = \{3, 4\}$. This proves that $\alpha_0(\Delta'_{123})$ does not exceed the solution of the equation

$$\frac{1}{12^x} + \frac{1}{20^x} = 1,$$

that is, the number 0.26.

A similar reasoning proves that the Hausdorff–Besicovitch dimension of the set Δ'_1 does not exceed the solution of the equation

$$\left(\frac{1}{m(m+1)}\right)^x + \left(\frac{1}{(m+1)(m+2)}\right)^x = 1$$

with an arbitrary $m \in \mathbb{N}$, that is, $\alpha_0(E) = \alpha_0(S_\theta) = 0$. □

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