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INTEGRAL EQUATIONS WITH RESPECT TO A GENERAL STOCHASTIC MEASURE

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V. M. RADCHENKO

Abstract. An integral with respect to a general stochastic measure is defined for random functions whose trajectories belong to a Besov space. The existence and uniqueness of solutions of some stochastic equations involving such integrals are established.

1. Introduction

The aim of this paper is to study the following stochastic equation:

$$u(x) = g(x) + \int_{(a,b]} f(x,y)u(y)\,d\mu(y),$$

where $u(x)$ is an unknown random function and μ is a general stochastic measure. The integral with respect to μ is defined as the limit of integral sums of a special form and the integrand is a random function whose trajectories belong to a Besov space.

Stochastic equations of this type are considered, for example, in [1, 2]. Some special conditions are imposed on the stochastic integrand in [1, 2]. The Fredholm type equations are studied in [1] for integrands possessing the generalized derivative in the space $\mathsf{L}_2([0,1]^d)$. The stochastic integral is defined in [1] as the sum of a certain series with respect to an orthonormal basis. A Volterra type equation is considered in [2] for processes of a finite p-variation, $0 < p < 2$. The integral is defined in [2] in the Young sense as the limit of the corresponding integral sums.

In the current paper, we assume that μ is σ-additive in probability on the Borel σ-algebra $\mathcal{B}\big((a,b]\big)$. The stochastic integral involved in the stochastic equations is defined as a limit of integral sums of a special form. This definition is similar to that considered in [2], however our approach is, in fact, to define the integral path-wise for a fixed elementary random event ω rather than to use probability properties of μ. We assume that the trajectories of the integrand belong to some Besov space.

Equations with integrals of a nonrandom function with respect to a stochastic measure are studied in [3, 4]. This paper is an attempt to study equations with integrals with respect to μ for random integrands.

The paper is constructed as follows. Some necessary definitions and results for stochastic measures and Besov spaces are discussed in Section 2. An integral with respect to μ is defined in Section 3; its additivity is proved, as well. In Section 4, we study a stochastic integral equation and prove the existence and uniqueness of its solution.

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2. Auxiliary results

Let X be a set, $\mathcal{B}$ a σ-algebra of subsets of X, and $(\Omega, \mathcal{F}, \mathsf{P})$ a complete probability space. Denote by $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ the family of all random variables (more precisely, L_0 is the family of P-equivalent classes of random variables). The convergence in L_0 means the convergence in probability.

Definition 1. Any σ-additive mapping $\mu\colon \mathcal{B} \to \mathsf{L}_0$ is called a *stochastic measure* on the σ-algebra $\mathcal{B}$.

We do not impose any restriction on μ like the nonnegativity or existence of moments. We also do not assume that μ is non-anticipating. Such an object μ is called a general stochastic measure in [5].

Consider some examples. If $M(t)$, $0 \le t \le T$, is a square integrable martingale, then

$$\mu(\mathsf{A}) := \int_{[0,T]} \mathbf{1}_{\mathsf{A}}(t)\, dM(t)$$

is a stochastic measure defined on the Borel σ-algebra $\mathcal{B}\big([0,T]\big)$. A fractional Brownian motion $B^H(x)$ defines a stochastic measure in a similar way if the Hurst index is such that $H > 1/2$ (this follows from inequality (1.5) of [6]). A particular case of stochastic measures considered in [7, Section 3] is represented by α-stable measures with independent values defined on a σ-algebra. Other examples as well as conditions under which the differences of a stochastic process with independent increments generate a stochastic measure are given in Sections 7 and 8 of [5].

An integral of the form $\int_{\mathsf{A}} h(x)\, d\mu(x)$ is constructed and studied in [8], where $h\colon \mathsf{X} \to \mathbb{R}$ is a measurable nonrandom function and $\mathsf{A} \in \mathcal{B}$. The integral is constructed in a standard way by using an approximation by simple functions. A similar construction is presented in Section 7 of [5] (also see [9]). In particular, any bounded measurable function h is integrable with respect to every stochastic measure μ. An analogue of the Lebesgue dominated convergence theorem is valid for such an integral (see Corollary 1.2 in [8] or Proposition 7.1.1 in [5]).

For further considerations we need the definition of the classical Besov spaces

$$B_{22}^{\alpha}([a,b]).$$

Recall that the norm in these classical Banach spaces can be defined in the following way if $0 < \alpha < 1$:

$$\|g\|_{B_{22}^{\alpha}([a,b])} = \|g\|_{\mathsf{L}_2([a,b])} + \left(\int_0^{b-a} \big(\omega_2(g,r)\big)^2 r^{-2\alpha-1}\, dr\right)^{1/2}, \tag{2.1}$$

where

$$\omega_2(g,r) = \sup_{0\le s\le r} \left(\int_a^{b-s} \big|g(y+s) - g(y)\big|^2\, dy\right)^{1/2}.$$

It is known that if $1/2 < \alpha < 1$, then each equivalency class of $B_{22}^{\alpha}([a,b])$ contains a function of the space $\mathbb{C}([a,b])$. Throughout below the functions of $B_{22}^{\alpha}([a,b])$, $1/2 < \alpha < 1$, are assumed to be continuous.

For $n \ge 0$, put

$$d_{kn} = a + k2^{-n}(b-a), \quad 0 \le k \le 2^n, \qquad \Delta_{kn} = \big(d_{(k-1)n}, d_{kn}\big], \quad 1 \le k \le 2^n$$

(we do not indicate the interval $(a,b]$ in the notation).

As shown in Theorem 1.1 of [11] for $1/2 < \alpha < 1$, the following is an equivalent norm in the space $B^{\alpha}_{22}([a,b])$:

$$\|g\|^{\alpha}_{22} = |g(a)| + \left(\sum_{n=1}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^n} \big(g\big(d_{kn}\big) - g\big(d_{(k-1)n}\big)\big)^2\right)^{1/2}. \tag{2.2}$$

Note that Lemma in [10] (also see Lemma 3.1 in [3]) implies that

$$\sum_{n=1}^{\infty} 2^{n(1-2\alpha)} \sum_{k=1}^{2^n} \big|\mu\big(\mathsf{A}_{kn}\big)\big|^2 < +\infty \quad \text{almost surely} \tag{2.3}$$

for sets $\mathsf{A}_{kn} \in \mathcal{B}((a,b])$ such that $\mathsf{A}_{k_1 n} \cap \mathsf{A}_{k_2 n} = \varnothing$ and for all n, $k_1 \neq k_2$ and $\alpha > 1/2$.

By C and $C(\omega)$, we denote the constants and random variables whose values do not matter in our reasoning; they may vary from line to line.

3. An integral with respect to a stochastic measure for a process belonging to a Besov space

In this section, we give a definition and study some properties of an integral

$$\int_{(a,b]} h(t,\omega)\, d\mu(t),$$

where μ is a stochastic measure defined on $\mathcal{B}\big((a,b]\big)$ and $h(t)$ is a stochastic process whose trajectories belong to the Besov space $B^{\alpha}_{22}([a,b])$, $1/2 < \alpha < 1$.

Definition 2. Put

$$\int_{(a,b]} h\, d\mu := \lim_{n\to\infty} \sum_{k=1}^{2^n} h\left(d_{(k-1)n}, \omega\right) \mu(\Delta_{kn}) \tag{3.1}$$

for a stochastic measure μ defined on the Borel σ-algebra $\mathcal{B}\big((a,b]\big)$ and for a function $h\colon [a,b] \times \Omega \to \mathbb{R}$ such that $h(\cdot,\omega) \in B^{\alpha}_{22}([a,b])$ for some $1/2 < \alpha < 1$ and all $\omega \in \Omega$, where the limit is considered for the almost sure convergence.

We show that this integral is well defined by proving that the above limit exists almost surely. Put

$$S_n = \sum_{k=1}^{2^n} h\left(d_{(k-1)n}, \omega\right) \mu(\Delta_{kn}), \qquad n \geq 0.$$

Since $\Delta_{k(n-1)} = \Delta_{(2k-1)n} \cup \Delta_{(2k)n}$, we conclude from the Cauchy inequality that

$$\begin{aligned}
\sum_{n=1}^{\infty} |S_n - S_{n-1}| &= \sum_{n=1}^{\infty} \left| \sum_{k=1}^{2^{n-1}} \big(h\big(d_{(2k-1)n}\big) - h\big(d_{(k-1)(n-1)}\big)\mu\left(\Delta_{(2k)n}\right) \right| \\
&\leq \left(\sum_{n=1}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^{n-1}} \left(h\big(d_{(2k-1)n}\big) - h\big(d_{(k-1)(n-1)}\big)^2\right)\right)^{1/2} \\
&\quad \times \left(\sum_{n=1}^{\infty} 2^{n(1-2\alpha)} \sum_{k=1}^{2^{n-1}} \big|\mu\big(\Delta_{(2k)n}\big)\big|^2\right)^{1/2} \\
&\leq \left(\|h\|^{\alpha}_{22} - h(a)\right) \left(\sum_{n=1}^{\infty} 2^{n(1-2\alpha)} \sum_{k=1}^{2^{n-1}} \big|\mu\big(\Delta_{(2k)n}\big)\big|^2\right)^{1/2} \overset{(2.3)}{<} +\infty
\end{aligned} \tag{3.2}$$

almost surely. In the latter conclusion, we neglect a random event of zero probability for which

$$\mu(\Delta_{k(n-1)}) \neq \mu(\Delta_{(2k-1)n}) + \mu(\Delta_{(2k)n})$$

for some k and n. This implies that limit (3.1) exists almost surely.

Since

$$\left|\int_{(a,b]} h\, d\mu\right| = \left|\lim_{n\to\infty} S_n\right| \le |S_0| + \sum_{n=1}^{\infty} |S_n - S_{n-1}|, \tag{3.3}$$

we obtain

$$\begin{aligned}\left|\int_{(a,b]} h\, d\mu\right| &\le |h(a)| \cdot |\mu((a,b])| \\ &+ \left(\|h\|_{22}^{\alpha} - |h(a)|\right) \left(\sum_{n=1}^{\infty} 2^{n(1-2\alpha)} \sum_{k=1}^{2^{n-1}} \left|\mu(\Delta_{(2k)n})\right|^2\right)^{1/2}\end{aligned} \tag{3.4}$$

almost surely.

Note that this integral coincides with the integral defined in [5] and [8] in the case of nonrandom functions $h \in B_{22}^{\alpha}([a,b])$. The integral of [5] and [8] equals the limit of integral sums (3.1) in view of an analogue of the Lebesgue dominated convergence theorem (Proposition 7.1.1 in [5]).

It is easy to see that if μ is a nonrandom nonnegative measure, then the integral introduced above coincides with the usual Lebesgue integral

$$\int_{(a,b]} h(x,\omega)\, d\mu(x)$$

defined for every fixed ω (properties of such an integral are studied in [12]).

The above integral is defined for intervals only. Since the partitions of the corresponding interval in (3.1) are binary, some properties of the integral are not obvious.

First we obtain a result about a relationship between integrals over different intervals. This result deals with partitions into 2^n equal parts of two intervals $(a,b]$ and $(c,d]$; the points of partitions are denoted by $d_{kn}^{(c,d]}$, $\Delta_{kn}^{(c,d]}$ (the notation d_{kn}, Δ_{kn}, correspond, as above, to partitions of the interval $(a,b]$).

Theorem 1. *Let $(c,d] \subset (a,b]$ and let $\{h(\cdot,\omega), \omega \in \Omega\} \subset B_{22}^{\alpha}([a,b])$ for $1/2 < \alpha < 1$. Then*

$$\int_{(c,d]} h(t)\, d\mu(t) = \lim_{n\to\infty} \sum_{k=1}^{2^n} h(d_{(k-1)n}, \omega)\mu(\Delta_{kn} \cap (c,d]) \quad \textit{almost surely.} \tag{3.5}$$

Proof. Consider an approximation of h by step wise linear functions on $[a,b]$. Let $h_j\colon [a,b] \to \mathbb{R}$ be functions being linear on $[d_{(k-1)j}, d_{kj}]$ and such that

$$h_j(d_{kj}) = h(d_{kj}), \qquad 1 \le k \le 2^j.$$

For some random constant $L_j(\omega)$,

$$|h_j(x) - h_j(y)| \le L_j(\omega)|x-y|, \tag{3.6}$$

whence we conclude that $h_j \in B^{\alpha}_{22}([a,b])$ are integrable with respect to μ. Then, on the interval $[a,b]$,

$$\begin{aligned}\left(\|h-h_j\|^{\alpha}_{22}\right)^2 &= \sum_{n=j}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^n} \left|(h-h_j)\left(d_{kn}\right) - (h-h_j)\left(d_{(k-1)n}\right)\right|^2 \\ &\le 2\sum_{n=j}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^n} \left|h(d_{kn}) - h(d_{(k-1)n})\right|^2 \\ &\quad + 2\sum_{n=j}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^n} \left|h_j(d_{kn}) - h_j(d_{(k-1)n})\right|^2 \\ &:= S_1 + S_2.\end{aligned}$$

Since $\|h(\cdot,\omega)\|^{\alpha}_{22} < +\infty$, we obtain from (2.2) that $S_1 < +\infty$ and thus $S_1 \to 0$ as $j \to \infty$.

For the term S_2, we use the linearity of the function h_j in the interval $[d_{(k-1)j}, d_{kj}]$ and obtain

$$\begin{aligned}&\sum_{n=j}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^n} \left|h_j(d_{kn}) - h_j(d_{(k-1)n})\right|^2 \\ &\qquad = \sum_{k=1}^{2^j} \left|h(d_{kj}) - h(d_{(k-1)j})\right|^2 \sum_{n=j}^{\infty} 2^{n(2\alpha-2)+j} \\ &\qquad = \frac{2^{j(2\alpha-1)}}{1-2^{2\alpha-2}} \sum_{k=1}^{2^j} \left|h(d_{kj}) - h(d_{(k-1)j})\right|^2.\end{aligned}$$

Since the norm (2.2) of h is finite, we get $S_2 \to 0$, $j \to \infty$.

By the definition of the Besov spaces,

$$\|h(\cdot,\omega) - h_j(\cdot,\omega)\|_{B^{\alpha}_{22}([c,d])} \le \|h(\cdot,\omega) - h_j(\cdot,\omega)\|_{B^{\alpha}_{22}([a,b])}.$$

In particular,

$$h_j(\cdot,\omega) \in B^{\alpha}_{22}\left([c,d]\right).$$

Since the norm (2.2) is equivalent to the norm in the Besov space (both considered in the interval $[c,d]$) we derive from inequality (3.4) that

$$\int_{(c,d]} (h-h_j)\, d\mu \to 0, \qquad j \to \infty, \tag{3.7}$$

almost surely. By the definition of the integral,

$$\int_{(c,d]} h_j\, d\mu - \sum_{k=1}^{2^n} h_j\left(d^{(c,d]}_{(k-1)n}\right) \mu\left(\Delta^{(c,d]}_{kn}\right) \to 0, \qquad n \to \infty, \tag{3.8}$$

almost surely for all fixed j.

Then

$$\begin{aligned}&\left|\sum_{k=1}^{2^n} h_j\left(d^{(c,d]}_{(k-1)n}\right) \mu\left(\Delta^{(c,d]}_{kn}\right) - \sum_{k=1}^{2^n} h_j(d_{(k-1)n})\mu(\Delta_{kn} \cap (c,d])\right| \\ &\qquad \le \sum_{1\le k,k'\le 2^n} \left|h_j\left(d^{(c,d]}_{(k-1)n}\right) - h_j\left(d_{(k'-1)n}\right)\right| \left|\mu\left(\Delta^{(c,d]}_{kn} \cap \Delta_{k'n}\right)\right| \qquad (3.9) \\ &\qquad \overset{(3.6)}{\le} L_j(\omega) 2^{-n}(b-a) \sum_{1\le k,k'\le 2^n} \left|\mu\left(\Delta^{(c,d]}_{kn} \cap \Delta_{k'n}\right)\right|\end{aligned}$$

for all $n \geq j$. Here k' is chosen such that $\Delta_{kn}^{(c,d]} \cap \Delta_{k'n} \neq \varnothing$. Thus

$$\left| d_{(k-1)n}^{(c,d]} - d_{(k'-1)n} \right| \leq 2^{-n}(b-a).$$

Since

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{1\leq k,k'\leq 2^n} 2^{-n} \left| \mu\left(\Delta_{kn}^{(c,d]} \cap \Delta_{k'n}\right) \right| \\
&\qquad \leq \left(\sum_{n=1}^{\infty} 2^{1+n(2\alpha-2)} \right)^{1/2} \\
&\qquad \times \left(\sum_{n=1}^{\infty} \sum_{1\leq k,k'\leq 2^n} 2^{n(1-2\alpha)} \left| \mu\left(\Delta_{kn}^{(c,d]} \cap \Delta_{k'n}\right) \right|^2 \right)^{1/2} \\
&\qquad \overset{(2.3)}{<} +\infty
\end{aligned} \tag{3.10}$$

almost surely, the right hand side of (3.9) tends to zero as $n \to \infty$.

Consider

$$T_n := \sum_{k=1}^{2^n} h_j(d_{(k-1)n}, \omega)\mu(\Delta_{kn} \cap (c,d]) - \sum_{k=1}^{2^n} h(d_{(k-1)n})\mu(\Delta_{kn} \cap (c,d]), \qquad n \geq 0.$$

The same reasoning as in (3.2) but for $h - h_j$ instead of h proves that

$$\begin{aligned}
|T_n| &\leq \sum_{n=1}^{\infty} |T_n - T_{n-1}| \\
&\leq \|h_j - h\|_{22}^{\alpha} \left(\sum_{n=1}^{\infty} 2^{n(1-2\alpha)} \sum_{k=1}^{2^{n-1}} \left| \mu\big(\Delta_{(2k)n} \cap (c,d]\big) \right|^2 \right)^{1/2} \overset{(2.3)}{<} +\infty
\end{aligned} \tag{3.11}$$

almost surely, since

$$(h - h_j)(a) = 0$$

and $T_0 = 0$.

Consider $\omega \in \Omega$ such that (3.7)–(3.11) hold for all n and j. If ω is fixed, one can choose a sufficiently large j such that (3.7) and (3.11) are sufficiently small. If j is fixed, one can choose a sufficiently large n to decrease (3.8) and (3.9). As a result, we complete the proof of (3.5). □

Corollary 1. *Let $a < c < b$ and $\{h(\cdot,\omega), \omega \in \Omega\} \subset B_{22}^{\alpha}([a,b])$ for $1/2 < \alpha < 1$. Then*

$$\int_{(a,b]} h\,d\mu = \int_{(a,c]} h\,d\mu + \int_{(c,b]} h\,d\mu \quad \textit{almost surely.}$$

Proof. The proof follows directly from (3.5). □

4. Integral equations with stochastic measures

Now we consider an equation with respect to an unknown random function u, namely

$$u(x) = g(x) + \int_{(a,b]} f(x,y)u(y)\,d\mu(y), \tag{4.1}$$

where $g, u\colon [a,b] \times \Omega \to \mathbb{R}$ are measurable random functions,

$$\{g(\cdot,\omega), u(\cdot,\omega), \omega \in \Omega\} \subset B_{22}^{\alpha}([a,b]), \qquad 1/2 < \alpha < 1,$$

and the integral is defined according to Definition 2.

In the set of functions $B_{22}^{\alpha}([a,b])$, consider the norm

$$\|h\|_2^{\alpha} = \left(\max_{x\in[a,b]} |h(x)|^2 + \sum_{n=1}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^n} \big(h\big(d_{kn}\big) - h\big(d_{(k-1)n}\big)\big)^2 \right)^{1/2}.$$

The convergence with respect to this norm is equivalent to the uniform convergence and to the convergence with respect to the usual norm $B_{22}^{\alpha}([a,b])$. Thus this space is complete with respect to the norm $\|h\|_2^{\alpha}$ and this allows one to apply some properties of contracting mappings to find a solution.

In what follows we use the notation

$$M_{\alpha}(\omega) = \left(\sum_{j=1}^{\infty} 2^{j(1-2\alpha)} \sum_{i=1}^{2^{j-1}} \big|\mu\big(\Delta_{(2i)j}\big)\big|^2 \right)^{1/2}.$$

Recall that (2.3) implies $M_{\alpha} < +\infty$ almost surely.

Lemma 1. *Let $\{u(\cdot,\omega), \omega\in\Omega\} \subset B_{22}^{\alpha}([a,b])$, $1/2<\alpha<1$, and let a random function $f(x,y,\omega)\colon [a,b]^2\times\Omega\to\mathbb{R}$ be such that*

$$|f(x_1,y) - f(x_2,y)| \le K_x(\omega)|x_1-x_2|^{\delta_x}, \qquad \delta_x>\alpha, \tag{4.2}$$

$$|f(x,y_1) - f(x,y_2)| \le K_y(\omega)|y_1-y_2|^{\delta_y}, \qquad \delta_y>\alpha, \tag{4.3}$$

$$|f(x_1,y_1) - f(x_1,y_2) - f(x_2,y_1) + f(x_2,y_2)| \le K_{xy}(\omega)|x_1-x_2|^{\delta_x}|y_1-y_2|^{\delta_y}. \tag{4.4}$$

Put $K_f(\omega) = \max_{x\in[a,b]}|f(x,\omega)|$. Then

$$\eta(x) = \int_{(a,b]} f(x,y)u(y)\,d\mu(y) \tag{4.5}$$

has a version $\tilde\eta(x)$ such that $\{\tilde\eta(\cdot,\omega), \omega\in\Omega\}\subset B_{22}^{\alpha}([a,b])$. Moreover,

$$\begin{aligned}\big(\|\tilde\eta\|_2^{\alpha}\big)^2 \le 3\big(\|u\|_2^{\alpha}\big)^2 & \\ \times\Bigg((b-a)^{2\delta_x} 2^{2\alpha}\big(2^{2\delta_x}-2^{2\alpha}\big)^{-1} & \\ \times\max\Big\{K_x^2|\mu((a,b])|^2 & \\ + K_{xy}^2(b-a)^{2\delta_y}2^{2\alpha-1}\big(2^{2\delta_y}-2^{2\alpha}\big)^{-1}M_{\alpha}^2, K_x^2M_{\alpha}^2\Big\} & \\ +\max\Big\{K_f^2|\mu((a,b])|^2 & \\ + K_y^2(b-a)^{2\delta_y}2^{2\alpha-1}\big(2^{2\delta_y}-2^{2\alpha}\big)^{-1}M_{\alpha}^2, & \\ K_f^2(b-a)^{2\delta_x}M_{\alpha}^2\Big\}\Bigg) &\end{aligned} \tag{4.6}$$

almost surely.

Proof. The Hölder condition (4.3) and boundedness of the functions f and u together with inclusion $u(\cdot,\omega)\in B_{22}^{\alpha}([a,b])$ and inequalities

$$\begin{aligned}&|f(x,y+s)u(y+s) - f(x,y)u(y)| \\ &\quad\le |f(x,y+s)) - f(x,y)|\cdot|u(y+s)| + |u(y+s)-u(y)|\cdot|f(x,y)|\end{aligned}$$

and (2.1) imply that $\{f(x,\cdot,\omega)u(\cdot,\omega), \omega\in\Omega, x\in[a,b]\}\subset B_{22}^{\alpha}([a,b])$. Thus the integral in (4.5) is well defined.

The integral in (4.5) is defined by (3.1) in such a way that the same version of μ is considered for all x. The version $\tilde{\eta}(x)$ obtained in this way satisfies inequality (3.4) almost surely. Moreover, the exceptional null-set is common for all $x \in [a,b]$.

Next we consider

$$\begin{aligned}\big(\|\tilde{\eta}\|_2^{\alpha}\big)^2 &= \max_{x\in[a,b]} |\tilde{\eta}(x)|^2 \\ &\quad + \sum_{n=1}^{\infty} 2^{n(2\alpha-1)} \sum_{k=1}^{2^n} \left| \int_{(a,b]} \big(f(d_{kn},y) - f(d_{(k-1)n},y)\big) u(y)\, d\mu(y) \right|^2 . \end{aligned} \tag{4.7}$$

Put

$$v_{kn}(y) = f(d_{kn},y) - f(d_{(k-1)n},y), \qquad K_u(\omega) = \max_{x\in[a,b]} |u(x,\omega)|.$$

Then

$$\begin{aligned} &\left| \int_{(a,b]} v_{kn}(y) u(y)\, d\mu(y) \right|^2 \\ &\overset{(3.3)}{\le} \Bigg(|v_{kn}(a)u(a)| \cdot |\mu((a,b])| \\ &\quad + \sum_{j=1}^{\infty} \sum_{i=1}^{2^{j-1}} \Big| \left(v_{kn}\left(d_{(2i-1)j}\right) u\left(d_{(2i-1)j}\right) - v_{kn}\left(d_{(i-1)(j-1)}\right) u\left(d_{(i-1)(j-1)}\right) \right) \\ &\qquad \times \mu\left(\Delta_{(2i)j}\right) \Big| \Bigg)^2 \\ &\le 3\big(|v_{kn}(a)u(a)| \cdot |\mu((a,b])|\big)^2 \\ &\quad + 3 \left(\sum_{j=1}^{\infty} \sum_{i=1}^{2^{j-1}} \big| \big(v_{kn}(d_{(2i-1)j}) - v_{kn}(d_{(i-1)(j-1)})\big) u(d_{(2i-1)j}) \mu\big(\Delta_{(2i)j}\big) \big| \right)^2 \\ &\quad + 3 \left(\sum_{j=1}^{\infty} \sum_{i=1}^{2^{j-1}} \big| \big(u(d_{(2i-1)j}) - u(d_{(i-1)(j-1)})\big) v_{kn}(d_{(i-1)(j-1)}) \mu\big(\Delta_{(2i)j}\big) \big| \right)^2 \\ &\overset{(4.2),(4.4)}{\le} 3K_x^2 (b-a)^{2\delta_x} 2^{-2n\delta_x} K_u^2 |\mu((a,b])|^2 \\ &\quad + 3K_u^2 \left(\sum_{j=1}^{\infty} \sum_{i=1}^{2^{j-1}} K_{xy} (b-a)^{\delta_x+\delta_y} 2^{-n\delta_x - j\delta_y} \big|\mu\big(\Delta_{(2i)j}\big)\big| \right)^2 \\ &\quad + 3K_x^2 (b-a)^{2\delta_x} 2^{-2n\delta_x} \left(\sum_{j=1}^{\infty} \sum_{i=1}^{2^{j-1}} \big| \big(u(d_{(2i-1)j}) - u(d_{(i-1)(j-1)})\big) \mu\big(\Delta_{(2i)j}\big) \big| \right)^2 \\ &\le 3(b-a)^{2\delta_x} 2^{-2n\delta_x} \\ &\quad \times \Big(K_x^2 K_u^2 |\mu((a,b])|^2 + K_u^2 K_{xy}^2 (b-a)^{2\delta_y} 2^{2\alpha-1} \big(2^{2\delta_y} - 2^{2\alpha}\big)^{-1} M_\alpha^2 \\ &\qquad + K_x^2 \left(\big(\|u\|_2^{\alpha}\big)^2 - K_u^2 \right) M_\alpha^2 \Big) \\ &\le 3\big(\|u\|_2^{\alpha}\big)^2 (b-a)^{2\delta_x} 2^{-2n\delta_x} \\ &\quad \times \max\big\{ K_x^2 |\mu((a,b])|^2 + K_{xy}^2 (b-a)^{2\delta_y} 2^{2\alpha-1} \big(2^{2\delta_y} - 2^{2\alpha}\big)^{-1} M_\alpha^2, K_x^2 M_\alpha^2 \big\}. \end{aligned}$$

Here we used the bound

$$
\begin{aligned}
&\left(\sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}} 2^{-j\delta_y}\left|\mu\big(\Delta_{(2i)j}\big)\right|\right)^2 \\
&\qquad \le \left(\sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}} 2^{j(2\alpha-2\delta_y-1)}\right)\left(\sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}} 2^{j(1-2\alpha)}\left|\mu\big(\Delta_{(2i)j}\big)\right|^2\right) \\
&\qquad = 2^{2\alpha-1}\big(2^{2\delta_y}-2^{2\alpha}\big)^{-1}M_\alpha^2.
\end{aligned}
\tag{4.8}
$$

Similarly,

$$
\begin{aligned}
|\tilde{\eta}(x)| &= \Big|\int_{(a,b]} f(x,y)u(y)\,d\mu(y)\Big| \\
&\overset{(3.3)}{\le} \Bigg(|f(x,a)u(a)|\cdot|\mu((a,b])| \\
&\quad + \sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}}\Big|\left(f\left(x,d_{(2i-1)j}\right)u\left(d_{(2i-1)j}\right) - f\big(x,d_{(i-1)(j-1)}\big)u\left(d_{(i-1)(j-1)}\right)\right) \\
&\qquad\qquad \times \mu\big(\Delta_{(2i)j}\big)\Big|\Bigg)^2 \\
&\le 3\big(|f(x,a)u(a)|\cdot|\mu((a,b])|\big)^2 \\
&\quad + 3\left(\sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}}\left|\left(f\left(x,d_{(2i-1)j}\right)-f\left(x,d_{(i-1)(j-1)}\right)\right)u\left(d_{(2i-1)j}\right)\mu\big(\Delta_{(2i)j}\big)\right|\right)^2 \\
&\quad + 3\left(\sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}}\left|\left(u\left(d_{(2i-1)j}\right)-u\left(d_{(i-1)(j-1)}\right)\right)f\left(x,d_{(i-1)(j-1)}\right)\mu\big(\Delta_{(2i)j}\big)\right|\right)^2 \\
&\overset{(4.2),(4.4)}{\le} 3K_u^2K_f^2|\mu((a,b])|^2 \\
&\quad + 3K_u^2K_y^2(b-a)^{2\delta_y}\left(\sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}} 2^{-j\delta_y}\left|\mu\big(\Delta_{(2i)j}\big)\right|\right)^2 \\
&\quad + 3K_f^2(b-a)^{2\delta_x}\left(\sum_{j=1}^{\infty}\sum_{i=1}^{2^{j-1}}\left|\left(u\left(d_{(2i-1)j}\right)-u\left(d_{(i-1)(j-1)}\right)\right)\mu\big(\Delta_{(2i)j}\big)\right|\right)^2 \\
&\overset{(4.8)}{\le} 3K_u^2K_f^2|\mu((a,b])|^2 \\
&\quad + 3K_u^2K_y^2(b-a)^{2\delta_y}2^{2\alpha-1}\big(2^{2\delta_y}-2^{2\alpha}\big)^{-1}M_\alpha^2 \\
&\quad + 3K_f^2(b-a)^{2\delta_x}\left(\big(\|u\|_2^\alpha\big)^2-K_u^2\right)M_\alpha^2 \\
&\le 3\big(\|u\|_2^\alpha\big)^2\max\Big\{K_f^2|\mu((a,b])|^2+K_y^2(b-a)^{2\delta_y}2^{2\alpha-1}\big(2^{2\delta_y}-2^{2\alpha}\big)^{-1}M_\alpha^2, \\
&\qquad\qquad K_f^2(b-a)^{2\delta_x}M_\alpha^2\Big\}
\end{aligned}
$$

for all $x\in[a,b]$. It remains to apply equality (4.7). □

In what follows we use the notation

$$\Omega_1 = \Big\{\omega \in \Omega \colon (b-a)^{2\delta_x} 2^{2\alpha} \big(2^{2\delta_x} - 2^{2\alpha}\big)^{-1} \times \max\Big\{K_x^2 |\mu((a,b])|^2 + K_{xy}^2 (b-a)^{2\delta_y} 2^{2\alpha-1} \big(2^{2\delta_y} - 2^{2\alpha}\big)^{-1} M_\alpha^2, K_x^2 M_\alpha^2\Big\} + \max\Big\{K_f^2 |\mu((a,b])|^2 + K_y^2 (b-a)^{2\delta_y} 2^{2\alpha-1} \big(2^{2\delta_y} - 2^{2\alpha}\big)^{-1} M_\alpha^2, K_f^2 (b-a)^{2\delta_x} M_\alpha^2\Big\} < \frac{1}{3}\Big\}.$$

The following assertion deals with the restriction of equation (4.1) to Ω_1. The conditions used to define the integral and to prove its properties still hold for this restriction. Formally speaking, one can put $\mu(\mathsf{A},\omega) = 0$, $\mathsf{A} \in \mathcal{B}\big((a,b]\big)$, $\omega \in (\Omega \setminus \Omega_1)$, and consider equation (4.1) for the changed stochastic measure.

Theorem 2. *Let conditions* (4.2)–(4.4) *hold for equation* (4.1). *Then equation* (4.1) *has a unique solution on* Ω_1 *except a null probability random event.*

Proof. As seen from (4.6), the right hand side of (4.1) is a contracting operator on Ω_1 acting almost surely in the space $B_{22}^\alpha([a,b])$ equipped with the norm $\|\cdot\|_2^\alpha$. □

Given a function f, it is possible that the set Ω_1 is empty. Nevertheless, the corresponding sets where solutions for the equations

$$u(x) = g(x) + \int_{(a,b]} \varepsilon f(x,y) u(y)\, d\mu(y)$$

exist are such that $\mathsf{P}(\Omega_1) \to 1$ as $\varepsilon \to 0$. This follows, since the coefficients K_x, K_y, K_{xy}, and K_f corresponding to the functions εf approach zero as $\varepsilon \to 0$.

Above we considered a stochastic integral Fredholm type equation. The corresponding results for the stochastic Volterra type equation

$$u(x) = g(x) + \int_{(a,x]} f(x,y) u(y)\, d\mu(y)$$

fail, since the trajectories of the stochastic integral do not belong to the Besov space $B_{22}^\alpha([a,b])$, $\alpha > 1/2$, in the general case.

For example, it is only known that the trajectories of the process

$$\mu((a,x]) = \int_{(a,x]} d\mu(y)$$

belong to the space $B_{22}^\alpha([a,b])$ for all $\alpha < 1/2$ (see the main result in [10]).

Based on the results obtained above, one can study the asymptotic behavior of a solution of equation (4.1) as $x, b \to \infty$ and to consider a large deviations problem for it. These results will be published elsewhere.

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Department of Mathematical Analysis, National Taras Shevchenko University, Kyiv 01601, Ukraine
E-mail address: vradchenko@univ.kiev.ua

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