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THE POINCARÉ INEQUALITY AND LOGARITHMIC SOBOLEV INEQUALITY FOR A SPHERICALLY CENSORED GAUSSIAN MEASURE

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ABSTRACT. For a one-dimensional projection of a spherically censored Gaussian measure on $\mathbf{R}^n$, the logarithmic Sobolev inequality is proved. As a consequence, we obtain the Poincaré inequality for a spherically censored Gaussian measure on $\mathbf{R}^n$, $n \geq 3$.

1. Introduction

The aim of this paper is to prove the Poincaré inequality for a spherically censored multi-dimensional Gaussian measure and logarithmic Sobolev inequality for the projection of this measure to the real line.

First we discuss some preliminary results. Let μ be a probability measure on $\mathbf{R}^n$ being absolutely continuous with respect to a Gaussian measure. In what follows we use the notation

$$\mathsf{E}_\mu f = \int f \, d\mu,$$

$$\operatorname{Var}_\mu f = \int f^2 \, d\mu - \left(\int f \, d\mu \right)^2$$

for the expectation and variance, respectively, of a function f with respect to the measure μ. The entropy of a non-negative function f with respect to the measure μ is denoted by

$$\operatorname{Ent}_\mu f = \int f \ln f \, d\mu - \int f \, d\mu \int \ln f \, d\mu.$$

We say that the logarithmic Sobolev inequality with a constant $c > 0$ holds for a measure μ if

$$\operatorname{Ent}_\mu f^2 \leq 2c \, \mathsf{E}_\mu \|\nabla f\|^2$$

for any absolutely continuous function f such that both f and ∇f are square integrable with respect to the measure μ.

We also say that the Poincaré inequality with a constant $c > 0$ holds for a measure μ if

$$\operatorname{Var}_\mu f \leq c \, \mathsf{E}_\mu \|\nabla f\|^2$$

for any absolutely continuous function f such that both f and ∇f are square integrable with respect to the measure μ.

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It is known that the logarithmic Sobolev inequality for a measure μ implies the Poincaré inequality with the same constant ([4, Proposition 2.1, p. 144]).

Let γ_n be a standard Gaussian measure on $\mathbf{R}^n$ and $V \subset \mathbf{R}^n$ an arbitrary measurable set of $\gamma_n(V) > 0$. We say that γ_n^V is a censored Gaussian measure if

$$\gamma_n^V(A) = \frac{\gamma_n(A \cap V)}{\gamma_n(V)}.$$

In other words, γ_n is a conditional Gaussian measure where V is a domain of admissible values and $\mathbf{R}^n \setminus V$ is a censored set.

The classical proofs of the logarithmic Sobolev inequality and Poincaré inequality, the Bakry–Emery method [3] for example, require a restriction imposed on the derivative of the logarithmic derivative of the measure and cannot directly be applied to censored measures (except the trivial case of $V = \mathbf{R}^n$). Nevertheless, the classical proofs are easy to generalize to extend the logarithmic Sobolev inequality and the Poincaré inequality for the measure γ_V^n if V is a convex set.

The case where V is not convex is more challenging; the first result concerning the non-convex case is obtained in [2], where only the simplest (in the sense of geometry) case, namely the case where V is a ball, is studied.

A measure γ_n^V with $\mathbf{R}^n \setminus V$ being a ball is called a *spherically censored* Gaussian measure. An approach to prove the Poincaré inequality for a spherically censored Gaussian measure and to estimate the corresponding constants c in terms of the radius R of the censored ball is proposed in the paper [2].

In the current paper, we obtain an upper bound (being polynomial with respect to the radius of the ball) for the constant c in the Poincaré inequality for a spherically censored measure. We use the method proposed in [1] and analyze the constant in the logarithmic Sobolev inequality for a one-dimensional projection of a spherically censored Gaussian measure to the real line passing through the center of the corresponding ball and the origin. This result yields a bound for the constant in the Poincaré inequality for such a measure and this allows one to apply the method of "decomposition of variance" introduced in [2].

An essential difference between the cases of measures with convex admissible sets and those with spherically censored ones is that, according to equations in [2], the corresponding constants depend on the radius of a censored ball in the second case. We also should like to mention that the dependence between the constant and radius of a censored ball is not yet studied in full detail. It is conjectured in [2] that the constant c does not exceed $c_1(1 + R^2)$ if R is sufficiently large. On the other hand, a weaker estimate $c \leq c_1 \exp\{c_2 R^2\}$ is proved in [2] for some particular cases (for example, if R is sufficiently large and the distance to the origin belongs to the interval $[0, R + \sqrt{2}]$).

2. Mainstream

A Gaussian measure is invariant with respect to rotations about the origin and thus one can, without loss of generality, study the Poincaré inequality only for a spherically censored Gaussian measure in the case where the center of the corresponding ball is of the form $(a, 0, 0, \dots, 0)$, where $a \in [0; R + \sqrt{2}]$. Denote by $\mu_n^{a,R}$ a spherically censored measure with this property and by $\mu_{n,a,R}$ its projection to the axis $(1, 0, 0, \dots, 0)$.

Note that the density of the measure $\mu_{n,a,R}$ is given by

$$\rho_{\mu_{n,a,R}}(x) = \begin{cases} \frac{c_{n-1} e^{-ax + \frac{a^2 - R^2}{2}}}{\sqrt{2\pi}(1 - B_n(a,R))} H_{n-2}\left(\sqrt{R^2 - (a-x)^2}\right), & x \in [a - R, a + R], \\ \frac{e^{-x^2/2}}{\sqrt{2\pi}(1 - B_n(a,R))}, & \text{otherwise}, \end{cases}$$

where $B_n(a,R) = \gamma_n(B((\underbrace{a,0,,\dots,0}_{n}); R)$,

$$H_n(x) = e^{\frac{x^2}{2}} \int_x^{+\infty} y^n e^{-\frac{y^2}{2}}\, dy,$$

$$c_n = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{[0,\pi]^{n-2}} \sin(\vartheta_1)\dots(\sin(\vartheta_{n-2}))^{n-2}\, d\vartheta_1 \dots \vartheta_{\vartheta-2}.$$

Theorem 2.1. *For every absolutely continuous function f such that both f and f' are square integrable with respect to the measure $\mu_{n,a,R}$,*

$$\mathrm{Ent}_{\mu_{n,a,R}} f^2 \le 2c_{n,R}\, \mathsf{E}_{\mu_{n,a,R}}(f')^2,$$

where

$$c_{n,R} = \frac{2\ln 2}{2\ln 2 - 1}\left(\sqrt{2}(1+R)\frac{H_{n-2}(R)}{H_{n-2}(0)} + \frac{2\sqrt{2}\pi + 4c_n H_{n-1}(\sqrt{2}R)}{c_{n-1}(n-3)!!} + \frac{1}{c_{n-1}H_{n-2}(0)}\right) + 1.$$

Remark 2.1. Note that

$$H_{2k+1}(x) = \sum_{m=0}^{k} \frac{(2k)!!}{(2(k-m))!!} x^{2(k-m)},$$

$$\begin{aligned} H_{2k}(x) &= \sum_{m=0}^{k-1} \frac{(2k-1)!!}{(2k-2m-1)!!} x^{2k-2m-1} + (2k-1)!!\, e^{\frac{x^2}{2}} \int_x^{+\infty} e^{-\frac{y^2}{2}}\, dy \\ &\le \sum_{m=0}^{k-1} \frac{(2k-1)!!}{(2k-2m-1)!!} x^{2k-2m-1} + (2k-1)!!\sqrt{\frac{\pi}{2}}, \end{aligned}$$

that is, the function H_n is bounded from above by a polynomial. As mentioned above, the constant

$$c_{n,R} \le \frac{2\ln 2}{2\ln 2 - 1}\left(\frac{2}{\sqrt{\pi}}(1+R)\frac{H_{n-2}(R)}{(n-3)!!} + \frac{2\sqrt{2}\pi + 4c_n H_{n-1}(\sqrt{2}R)}{c_{n-1}(n-3)!!} + \frac{\sqrt{\frac{2}{\pi}}}{c_{n-1}(n-3)!!}\right) + 1$$

in the logarithmic Sobolev inequality for the measure $\mu_{n,a,R}$ admits a polynomial bound with respect to R.

Corollary 2.1. *The Poincaré inequality holds for $\mu_{n,a,R}$:*

$$\mathrm{Var}_{\mu_{n,a,R}} f \le c_{n,R} \int |f'|^2\, d\mu_{n,a,R}.$$

The Poincaré inequality for a measure $\mu_n^{a,R}$ with a constant admitting a polynomial bound with respect to R is a corollary of the Poincaré inequality for the measure $\mu_{n,a,R}$. Below we state this result (its proof follows from Corollary 2.1 and equalities (4.1) and (4.3) of the paper [2]).

Theorem 2.2. *The Poincaré inequality holds for a Gaussian measure $\mu_n^{a,R}$ with a cut out ball*:

$$\mathrm{Var}_{\mu_n^{a,R}} f \le \left(c_{n,R} + 1 + \frac{R^2}{n-1}\right) \int \|\nabla f\|^2\, d\mu_n^{a,R}.$$

3. Proof of the main results

The condition used in the following result is sufficient for the logarithmic Sobolev inequality stated in Theorem 2.1 (this follows from [1]). Thus the proof of Theorem 2.1 reduces in fact to Lemma 3.1.

Lemma 3.1. *The following inequality holds*:

$$K_{\mu_{n,a,R}}(x) = \frac{\rho_{\gamma_1}\left(F_{\gamma_1}^{-1}(F_{\mu_{n,a,R}}(x))\right)}{\rho_{\mu_{n,a,R}(x)}} \le c_{n,R}$$

$$= \frac{2\ln 2}{2\ln 2 - 1}\left(\sqrt{2}(1+R)\frac{H_{n-2}(R)}{H_{n-2}(0)} + \frac{2\sqrt{2}\pi + 4c_n H_{n-1}(\sqrt{2}R)}{c_{n-1}(n-3)!!} + \frac{1}{c_{n-1}H_{n-2}(0)}\right) + 1,$$

where $F_{\mu_{n,a,R}}(x)$ and F_{γ_1} are the distribution functions of the corresponding measures.

We start with the proof of Lemma 3.1 for the case of $x \in (-\infty, a-R] \cup [a+R, \infty)$.

Lemma 3.2. *If $x \in (-\infty, a-R] \cup [a+R, \infty)$, then $K_{\mu_{n,a,R}}(x) \le 1$.*

Proof. We denote by $I(p) := \rho_{\gamma_1}(F_{\gamma_1}^{-1}(p))$ the isoperimetric function of the standard Gaussian measure. Note that $I'(p) = -F_{\gamma_1}^{-1}(p)$. Then

$$[I(cF_{\gamma_1}(x))]' = -F_{\gamma_1}^{-1}(cF_{\gamma_1}(x))c\rho_{\gamma_1}(x) \le (-x)c\rho_{\gamma_1}(x) = c\rho'_{\gamma_1}(x)$$

for all $c > 1$ and $x \le F_{\gamma_1}^{-1}(1/c)$, since $F_{\gamma_1}^{-1}$ is an increasing function. It is clear that both functions $I(cF_{\gamma_1}(x))$ and $\rho_{\gamma_1}(x)$ tend to zero as $x \to -\infty$, whence

$$I(cF_{\gamma_1}(x)) = \int_{-\infty}^{x} [I(cF_{\gamma_1}(y))]'\,dy \le c\int_{-\infty}^{x} \rho'_{\gamma_1}(y)\,dy = c\rho_{\gamma_1}(x), \qquad x \le F_{\gamma_1}^{-1}(1/c).$$

Note that

$$F_{\mu_{n,a,R}}(x) = \frac{1}{1-B_n(a,R)}F_{\gamma_1}(x), \qquad \rho_{\mu_{n,a,R}} = \frac{1}{1-B_n(a,R)}\rho_{\gamma_1}(x)$$

for $x \le a-R$ and that $(1-B_n(a,R))^{-1} > 1$. The subspace $\{y = (y_1, y_2, \dots, y_n) \colon y_1 \le x\}$ belongs to $\mathbf{R}^n \setminus B_n(a,R)$ and hence

$$F_{\gamma_1}(x) = \gamma_n(\{y = (y_1, y_2, \dots, y_n) \colon y_1 \le x\}) \le (1-B_n(a,R)) \Leftrightarrow x \le F_{\gamma_1}^{-1}\left(1 - B_n(a,R)\right).$$

Applying the inequality proved above we get

$$K_{\mu_{n,a,R}}(x) = \frac{I(F_{\mu_{n,a,R}}(x))}{\rho_{\mu_{n,a,R}}(x)} = \frac{I\left(\frac{1}{1-B_n(a,R)}F_{\gamma_1}(x)\right)}{\frac{1}{1-B_n(a,R)}\rho_{\gamma_1}(x)} \le 1, \qquad x < a-R.$$

If $x \ge a+R$, then

$$1 - F_{\mu_{n,a,R}(x)} = \frac{1}{1-B_n(a,R)}(1 - F_{\gamma_1}(x)).$$

Note that $I(p) = I(1-p)$ and $F_{\gamma_1}^{-1}(1 - F_{\gamma_1}(x)) = -x$. Therefore we can apply the same method to prove $\hat{K}_\mu(x) \le 1$ as in the case of $x \le a-R$, since

$$I(c(1-F_{\gamma_1}(x)) = -c\int_x^\infty F_{\gamma_1}^{-1}\bigl(c(1-F_{\gamma_1}(y))\bigr)\,dy \le c\int_x^\infty y\rho'_{\gamma_1}(y)\,dy = c\rho_{\gamma_1}(x)$$

for all $c > 1$. □

It remains to prove the inequality stated in Lemma 3.1 for the case of $x \in [a-R, a+R]$. We need an auxiliary result.

Lemma 3.3. *For* $x \in (0,1)$,

$$I(x) = \rho_{\gamma_1}\left(F_{\gamma_1}^{-1}(x)\right) \le \frac{2\ln 2}{2\ln 2 - 1}(1-x)\sqrt{\ln\frac{1}{1-x}}.$$

Proof. Put $g(y) = \exp\{-y^2/2\} - F_{\gamma_1}(-y)$. Note that

$$g'(y) = -y\exp\left\{-\frac{y^2}{2}\right\} - (-1)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\} = \exp\left\{-\frac{y^2}{2}\right\}\left(\frac{1}{\sqrt{2\pi}} - y\right).$$

This means that the function g has a single local minimum in the interval $[0,\infty)$ at the point 0 and decreases in the interval $[\sqrt{2\pi}, +\infty)$. We have $g(0) = 1 - \frac{1}{2} = \frac{1}{2}$ and $\lim_{y\to+\infty} g(y) = 0 - 0 = 0$. Thus, $\exp\{-y^2/2\} \ge F_{\gamma_1}(-y)$, $y \in [0,+\infty)$, whence

$$\exp\left\{-\frac{1}{2}\left(\sqrt{2\ln\frac{1}{x}}\right)^2\right\} = x \ge F_{\gamma_1}\left(-\sqrt{2\ln\frac{1}{x}}\right), \qquad x \in (0,1),$$

and

$$F_{\gamma_1}^{-1}(x) \ge -\sqrt{2\ln\frac{1}{x}}, \qquad x \in (0,1),$$

or

$$-F_{\gamma_1}^{-1}(x) \le \sqrt{2\ln\frac{1}{x}}, \qquad x \in (0,1).$$

Note that

$$\sqrt{2\ln\frac{1}{x}} \le \frac{2\ln 2}{2\ln 2 - 1}\left(\sqrt{2\ln\frac{1}{x}} - \frac{1}{\sqrt{2\ln\frac{1}{x}}}\right)$$

in the interval $x \in (0, \frac{1}{2})$. Thus

$$-F_{\gamma_1}^{-1}(x) \le \frac{2\ln 2}{2\ln 2 - 1}\left(\sqrt{2\ln\frac{1}{x}} - \frac{1}{\sqrt{2\ln\frac{1}{x}}}\right), \qquad x \in \left(0, \frac{1}{2}\right],$$

which yields

$$\left(I(x) - \frac{2\ln 2}{2\ln 2 - 1}x\sqrt{2\ln\frac{1}{x}}\right)' \le 0, \qquad x \in \left(0, \frac{1}{2}\right].$$

Since

$$\lim_{x\to 0}\left(I(x) - \frac{2\ln 2}{2\ln 2 - 1}x\sqrt{2\ln\frac{1}{x}}\right) = 0,$$

we get

$$I(x) \le \frac{2\ln 2}{2\ln 2 - 1}x\sqrt{2\ln\frac{1}{x}}, \qquad x \in \left(0, \frac{1}{2}\right].$$

Let $x \in [\frac{1}{2}, 1]$. Then

$$I(x) = I(1-x) \le \frac{2\ln 2}{2\ln 2 - 1}(1-x)\sqrt{2\ln\frac{1}{1-x}}.$$

To prove the inequality for the case of $x \in (0, \frac{1}{2}]$, we use

$$(1-x)\sqrt{\ln\frac{1}{1-x}} \ge x\sqrt{\ln\frac{1}{x}}, \qquad x \in \left(0, \frac{1}{2}\right].$$

This inequality is valid, since the function $g_1(x) = (1-x)^2 \ln \frac{1}{1-x} - x^2 \ln \frac{1}{x}$ equals 0 at the points $\frac{1}{2}$ and 1 and is convex, since

$$g_1''(x) = 2(\ln x - \ln(1-x)) \le 0, \qquad x \in \left[\frac{1}{2}, 1\right].$$

Thus

$$I(x) \le \frac{2\ln 2}{2\ln 2 - 1} x \sqrt{2 \ln \frac{1}{x}} \le \frac{2\ln 2}{2\ln 2 - 1}(1-x)\sqrt{2\ln \frac{1}{1-x}}, \qquad x \in \left(0, \frac{1}{2}\right]. \qquad \square$$

Now we can estimate the function as follows:

$$K_{\mu_{n,a,R}}(x) \le \frac{2\ln 2}{2\ln 2 - 1} \frac{(1 - F_{\mu_{a,n,R}}(x))}{\rho_{\mu_{n,a,R}}(x)} \sqrt{2 \ln \frac{1}{1 - F_{\mu_{a,n,R}}(x)}}, \qquad x \in \mathbf{R}.$$

The following result implies Lemma 3.1 for $x \in [a-R, a+R]$ and $a \in [0; R]$.

Lemma 3.4. *Let $a \in [0; R]$. Then*

$$\frac{(1 - F_{\mu_{a,n,R}}(x))}{\rho_{\mu_{n,a,R}}(x)} \sqrt{2 \ln \frac{1}{1 - F_{\mu_{a,n,R}}(x)}} \le \frac{\sqrt{2} R H_{n-2}(R)}{H_{n-2}(0)} + \frac{2\sqrt{2\pi} c_n H_{n-1}(R) + 1}{c_{n-1} H_{n-2}(0)}$$

for $x \in [a-R, a+R]$.

Proof. Put

$$\begin{aligned} K_1(x) &= \frac{1 - F_{\mu_{a,n,R}}(x)}{\rho_{\mu_{a,n,R}}(x)} \\ &= \frac{\int_x^{a+R} e^{-ay} H_{n-2}(\sqrt{R^2 - (a-y)^2})\, dy}{e^{-ax} H_{n-2}\big(\sqrt{R^2 - (a-x)^2}\big)} + \frac{e^{\frac{R^2 - a^2}{2}} \int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\, dy}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2 - (a-x)^2}\big)} \end{aligned}$$

and

$$A := \frac{\int_x^{a+R} e^{-ay} H_{n-2}(\sqrt{R^2 - (a-y)^2})\, dy}{e^{-ax} H_{n-2}\big(\sqrt{R^2 - (a-x)^2}\big)}.$$

Since

$$\int_x^{\infty} e^{-\frac{y^2}{2}}\, dy < \frac{1}{x} e^{-\frac{x^2}{2}}$$

for $x > 0$, we conclude that

$$B := \frac{e^{\frac{R^2 - a^2}{2}} \int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\, dy}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2 - (a-x)^2}\big)} < \frac{\frac{1}{a+R} e^{-aR - a^2}}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2 - (a-x)^2}\big)}.$$

Note that

$$A \le \frac{1 - e^{a(x-a-R)}}{a} \frac{H_{n-2}(R)}{H_{n-2}(0)} \le \frac{1 - e^{-2aR}}{a} \frac{H_{n-2}(R)}{H_{n-2}(0)}.$$

Put

$$
\begin{aligned}
K_2(x) &= \sqrt{\ln \frac{1}{1-F_{\mu_{a,n,R}}(x)}} \\
&= \sqrt{\ln \frac{\sqrt{2\pi}\big(1-\gamma_n(B(a;R))\big)}{c_{n-1}e^{\frac{a^2-R^2}{2}}\int_x^{a+R} e^{-ay}H_{n-2}(\sqrt{R^2-(a-y)^2})\,dy + \int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\,dy}} \\
&\le \sqrt{\ln \frac{\sqrt{2\pi}\big(1-\gamma_n(B(0;R-a))\big)}{c_{n-1}e^{\frac{a^2-R^2}{2}}\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy + \int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\,dy}} \\
&= \sqrt{\ln \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)e^{-\frac{(R-a)^2}{2}}}{c_{n-1}e^{\frac{a^2-R^2}{2}}\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy + \int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\,dy}} \\
&= \sqrt{2aR+\ln M_1(x)} \le \sqrt{2aR}+\sqrt{\ln M_1(x)} \le \sqrt{2aR}+M_1(x),
\end{aligned}
$$

where

$$
M(x) = \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)}{c_{n-1}e^{a^2+aR}\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy + e^{\frac{(a+R)^2}{2}}\int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\,dy},
$$

and

$$
C := \sqrt{2aR},
$$

$$
D := \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)}{c_{n-1}e^{a^2+aR}\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy + e^{\frac{(a+R)^2}{2}}\int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\,dy}.
$$

Inequality 1.

$$
\begin{aligned}
AC &\le \frac{1-e^{-2aR}}{a}\sqrt{2aR}H_{n-2}(R) = \sqrt{1-e^{-2aR}}\sqrt{\frac{1-e^{-2aR}}{2aR}}\sqrt{2}R\frac{H_{n-2}(R)}{H_{n-2}(0)} \\
&\le \sqrt{2}R\frac{H_{n-2}(R)}{H_{n-2}(0)}.
\end{aligned}
$$

Inequality 2.

$$
\begin{aligned}
AD &= \frac{\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy}{e^{-ax}H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\quad\times \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)}{c_{n-1}e^{a^2+aR}\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy + e^{\frac{(a+R)^2}{2}}\int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\,dy} \\
&\le \frac{\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy}{e^{-ax}H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\quad\times \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)}{c_{n-1}e^{a^2+aR}\int_x^{a+R} e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy} \\
&= \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)}{c_{n-1}e^{-ax+a^2+aR}H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \le \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)}{c_{n-1}H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\le \frac{\sqrt{2\pi}c_nH_{n-1}(R-a)}{c_{n-1}H_{n-2}(0)} \le \frac{\sqrt{2\pi}c_nH_{n-1}(R)}{c_{n-1}H_{n-2}(0)}.
\end{aligned}
$$

Inequality 3.

$$
\begin{aligned}
BC &\le \frac{\frac{1}{a+R} e^{-aR-a^2}}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \sqrt{2aR} \\
&= \frac{e^{a(x-(a+R))}}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \frac{\sqrt{2aR}}{a+R} \le \frac{1}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\le \frac{1}{c_{n-1} H_{n-2}(0)}.
\end{aligned}
$$

Inequality 4.

$$
\begin{aligned}
BD &= \frac{\int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\,dy}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\quad \times \frac{\sqrt{2\pi} c_n H_{n-1}(R-a)}{c_{n-1} e^{a^2+aR} \int_x^{a+R} e^{-ay} H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy + e^{\frac{(a+R)^2}{2}} \int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\,dy} \\
&\le \frac{\int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\,dy}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \frac{\sqrt{2\pi} c_n H_{n-1}(R-a)}{e^{\frac{(a+R)^2}{2}} \int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\,dy} \\
&\le \frac{\sqrt{2\pi} c_n H_{n-1}(R-a) e^{\frac{ax-(a+R)^2}{2}}}{c_{n-1} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \le \frac{\sqrt{2\pi} c_n H_{n-1}(R-a)}{c_{n-1} H_{n-2}(0)} \le \frac{\sqrt{2\pi} c_n H_{n-1}(R)}{c_{n-1} H_{n-2}(0)}.
\end{aligned}
$$

Hence, for $a \in [0;R]$,

$$
\begin{aligned}
K_{\mu_{n,a,R}}(x) &\le \frac{2\ln 2}{2\ln 2 - 1} K_1(x) K_2(x) = \frac{2\ln 2}{2\ln 2 - 1}(AC + AD + BC + BD) \\
&\le \frac{2\ln 2}{2\ln 2 - 1} \left(\sqrt{2} R \frac{H_{n-2}(R)}{H_{n-2}(0)} + 2 \frac{\sqrt{2\pi} c_n H_{n-1}(R)}{c_{n-1} H_{n-2}(0)} + \frac{1}{c_{n-1} H_{n-2}(0)} \right) \le c_{n,R}
\end{aligned}
$$

and the right hand side is a polynomial of degree $n-1$ with respect to R. ☐

The following result completes the proof in the general case, namely it implies Lemma 3.1 for $x \in [a-R, a+R]$ and $a \in [R; R+\sqrt{2}]$.

Lemma 3.5. *Let* $a \in [R; R+\sqrt{2}]$. *Then*

$$
\frac{(1 - F_{\mu_{a,n,R}}(x))}{\rho_{\mu_{n,a,R}}(x)} \sqrt{2 \ln \frac{1}{1 - F_{\mu_{a,n,R}}(x)}} \le \sqrt{2} \frac{H_{n-2}(R)}{H_{n-2}(0)} + \frac{2\sqrt{2\pi} + 1}{c_{n-1} H_{n-2}(0)}
$$

for $x \in [a-R, a+R]$.

Proof. Put

$$
\begin{aligned}
K_1(x) &= \frac{1 - F_{\mu_{a,n,R}}(x)}{\rho_{\mu_{a,n,R}}(x)} \\
&= \frac{\int_x^{a+R} e^{-ay} H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy}{e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} + \frac{e^{\frac{R^2-a^2}{2}} \int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\,dy}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)}
\end{aligned}
$$

and

$$
A := \frac{\int_x^{a+R} e^{-ay} H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy}{e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)}.
$$

Since

$$\int_x^\infty e^{-\frac{y^2}{2}}\,dy < \frac{1}{x}e^{-\frac{x^2}{2}}$$

for $x > 0$, we conclude that

$$\begin{aligned} B &:= \frac{e^{\frac{R^2-a^2}{2}}\int_{a+R}^\infty e^{-\frac{y^2}{2}}\,dy}{c_{n-1}e^{-ax}H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\ &< \frac{\frac{1}{a+R}e^{-aR-a^2}}{c_{n-1}e^{-ax}H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)}. \end{aligned}$$

Note that

$$A \le \frac{1-e^{a(x-a-R)}}{a}\frac{H_{n-2}(R)}{H_{n-2}(0)} \le \frac{1-e^{-2aR}}{a}\frac{H_{n-2}(R)}{H_{n-2}(0)}.$$

Put

$$\begin{aligned} K_2(x) &= \sqrt{\ln\frac{1}{1-F_{\mu_{a,n,R}}(x)}} \\ &= \sqrt{\ln\frac{\sqrt{2\pi}(1-\gamma_n(B(a;R)))}{c_{n-1}e^{\frac{a^2-R^2}{2}}\int_x^{a+R}e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy+\int_{a+R}^\infty e^{-\frac{y^2}{2}}\,dy}} \\ &\le \sqrt{\ln\frac{\sqrt{2\pi}}{c_{n-1}e^{\frac{a^2-R^2}{2}}\int_x^{a+R}e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy+\int_{a+R}^\infty e^{-\frac{y^2}{2}}\,dy}} \\ &= \sqrt{\frac{(a+R)^2}{2}+\ln M_2(x)} \le \frac{a+R}{\sqrt{2}}+\sqrt{\ln M_2(x)} \le \frac{a+R}{\sqrt{2}}+M_2(x), \end{aligned}$$

where

$$M_2(x) = \frac{\sqrt{2\pi}}{c_{n-1}e^{a^2+aR}\int_x^{a+R}e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\,dy+e^{\frac{(a+R)^2}{2}}\int_{a+R}^\infty e^{\frac{-y^2}{2}}\,dy},$$

$C := (a+R)/\sqrt{2}$, and

$$D := \frac{\sqrt{2\pi}}{c_{n-1}e^{a^2+aR}\int_x^{a+R}e^{-ay}H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)dy+e^{\frac{(a+R)^2}{2}}\int_{a+R}^\infty e^{\frac{-y^2}{2}}dy}.$$

Inequality 1.

$$\begin{aligned} AC &\le \frac{1-e^{-2aR}}{a}\frac{a+R}{\sqrt{2}}\frac{H_{n-2}(R)}{H_{n-2}(0)} \\ &\le \frac{a+R}{\sqrt{2}a}\frac{H_{n-2}(R)}{H_{n-2}(0)} = \left(\frac{1}{\sqrt{2}}+\frac{R}{\sqrt{2}a}\right)\frac{H_{n-2}(R)}{H_{n-2}(0)} \\ &\le \sqrt{2}\frac{H_{n-2}(R)}{H_{n-2}(0)}. \end{aligned}$$

Inequality 2.

$$
\begin{aligned}
AD &= \frac{\int_x^{a+R} e^{-ay} H_{n-2}(\sqrt{R^2-(a-y)^2})\, dy}{e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\quad \times \frac{\sqrt{2\pi}}{c_{n-1} e^{a^2+aR} \int_x^{a+R} e^{-ay} H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\, dy + e^{\frac{(a+R)^2}{2}} \int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\, dy} \\
&\le \frac{\int_x^{a+R} e^{-ay} H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\, dy}{e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\quad \times \frac{\sqrt{2\pi}}{c_{n-1} e^{a^2+aR} \int_x^{a+R} e^{-ay} H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\, dy} \\
&= \frac{\sqrt{2\pi}}{c_{n-1} e^{-ax+a^2+aR} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \le \frac{\sqrt{2\pi}}{c_{n-1} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\le \frac{\sqrt{2\pi}}{c_{n-1} H_{n-2}(0)}.
\end{aligned}
$$

Inequality 3.

$$
\begin{aligned}
BC &\le \frac{\frac{1}{a+R} e^{-aR-a^2}}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \frac{a+R}{\sqrt{2}} = \frac{e^{a(x-(a+R))}}{\sqrt{2} c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\le \frac{1}{\sqrt{2} c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \le \frac{1}{\sqrt{2} c_{n-1} H_{n-2}(0)}.
\end{aligned}
$$

Inequality 4.

$$
\begin{aligned}
BD &= \frac{\int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\, dy}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \\
&\quad \times \frac{\sqrt{2\pi}}{c_{n-1} e^{a^2+aR} \int_x^{a+R} e^{-ay} H_{n-2}\big(\sqrt{R^2-(a-y)^2}\big)\, dy + e^{\frac{(a+R)^2}{2}} \int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\, dy} \\
&\le \frac{\int_{a+R}^{\infty} e^{-\frac{y^2}{2}}\, dy}{c_{n-1} e^{-ax} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \frac{\sqrt{2\pi}}{e^{\frac{(a+R)^2}{2}} \int_{a+R}^{\infty} e^{\frac{-y^2}{2}}\, dy} \\
&\le \frac{\sqrt{2\pi} e^{\frac{ax-(a+R)^2}{2}}}{c_{n-1} H_{n-2}\big(\sqrt{R^2-(a-x)^2}\big)} \le \frac{\sqrt{2\pi}}{c_{n-1} H_{n-2}(0)} \le \frac{\sqrt{2\pi}}{c_{n-1} H_{n-2}(0)}.
\end{aligned}
$$

Then, for $x \in [R; R+\sqrt{2}]$,

$$
\begin{aligned}
K_{\mu_{n,a,R}}(x) &\le \frac{2\ln 2}{2\ln 2 - 1} K_1(x) K_2(x) = \frac{2\ln 2}{2\ln 2 - 1}(AC + AD + BC + BD) \\
&\le \frac{2\ln 2}{2\ln 2 - 1} \left(\sqrt{2} \frac{H_{n-2}(R)}{H_{n-2}(0)} + 2 \frac{\sqrt{2\pi}}{c_{n-1} H_{n-2}(0)} + \frac{1}{c_{n-1} H_{n-2}(0)} \right) \\
&\le c_{n,R}. \qquad \square
\end{aligned}
$$

As mentioned above, Lemma 3.1 implies Theorems 2.1. and 2.2 and this is what had to be proved.

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