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# RESTRICTED ISOMETRY PROPERTY FOR MATRICES WHOSE ENTRIES ARE RANDOM VARIABLES BELONGING TO SOME ORLICZ SPACES $L_U(\Omega)$

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V. B. TROSHKI

Abstract. A new approach to the signal processing called compressive sensing has been extensively developed during the last few years. There are many papers devoted to this topic but the problem of constructing the universal measurement matrix has not yet been solved. We propose to use a matrix whose entries are random variables belonging to some Orlicz spaces $L_U(\Omega)$ as a measurement matrix. We prove that the matrix with such entries satisfies the so-called restricted isometry property which is one of the main concepts in compressive sensing.

## 1. Introduction

Random matrices are widely used in compressive sensing for compressing a vector $x \in \mathbf{R}^N$ obtained from an observation $y = Ax$, where $A$ is an $n \times N$ random matrix for which the norm of each column equals one. The compressive sensing is a new approach in signal processing. Its main idea is to develop a scheme that restores signals even if only a small amount of observations is available. The origin of this approach is due to Kashin [1] and Garnaev and Gluskin [2]; however, an extensive development of this method began after the papers by Candes and Tao [3] and Donoho [4].

One of the main notions of this theory is the so-called property of restricted isometry. We say that a matrix $A$ satisfies the restricted isometry property of order $K$ if there exists $\delta \in (0;1)$ such that

$$(1-\delta)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta)\| \, x\|_2^2$$

for all $x \in \Sigma_K$, where $\Sigma_K$ is the set of those vectors of the space $\mathbf{R}^N$ that contains at most $K$ nonzero coordinates.

The problem of establishing the restricted isometry property has been considered in the literature in previous years for certain classes of random variables. Wojtaszczyk [5] considers normalized Gaussian random variables, while Baraniuk, Davenport, DeVore, and Wakin [6] proved the restricted isometry property for random matrices such that

$$\mathsf{P}\left\{\left|\|Ax\|_2^2 - \|x\|_2^2\right| > \delta\|x\|_2^2\right\} \le 2\exp\left\{-c_0(\delta)n\right\},$$

where $c_0(\delta)$ is a constant that depends on $\delta$. Later DeVore, Petrova, and Wojtaszczyk [7] showed that the matrices whose entries are sub-Gaussian random variables can also be used for compressing signals.

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We show in the current paper that a matrix $A$ whose entries are random variables belonging to Orlicz spaces $L_{U_\alpha}(\Omega)$ with

$$U_\alpha(x) = \begin{cases} \left(\frac{e\alpha}{2}\right)^{\frac{2}{\alpha}} x^2, & \text{for } |x| \le \left(\frac{2}{\alpha}\right)^{\frac{1}{\alpha}}, \\ \exp\{|x|^\alpha\}, & \text{for } |x| > \left(\frac{2}{\alpha}\right)^{\frac{1}{\alpha}}, \end{cases} \tag{1}$$

and $0 < \alpha < 1$ also satisfies the restricted isometry property. Moreover, we consider a family of random variables $S_{U_{1\alpha}}(\Omega)$ introduced in [8], where $U_{1\alpha}(x) = \exp\{|x|^\alpha\}$, $0 < \alpha \le 1$. We prove that this family constitutes a linear space. Also, we find necessary and sufficient conditions that random variables belong to this space and prove that matrices satisfy the restricted isometry property if their entries are independent random variables belonging to $S_{U_{1\alpha}}(\Omega)$. All necessary definitions and properties of Orlicz spaces can be found in the book [9].

## 2. Preliminaries

**Definition 2.1** ([9])**.** A continuous even convex function $U = \{U(x), x \in \mathbf{R}\}$ is called an Orlicz $C$-function if $U(x)$ increases for $x > 0$ and $U(0) = 0$.

**Definition 2.2** ([9])**.** An Orlicz $C$-function $U = \{U(x), x \in \mathbf{R}\}$ is called an Orlicz $N$-function if

$$\lim_{x\to 0} \frac{U(x)}{x} = 0,$$
$$\lim_{x\to \infty} \frac{U(x)}{x} = \infty.$$

**Definition 2.3** ([9])**.** Let $U$ be an arbitrary $C$-function. A family of random variables $L_U(\Omega)$ is called the Orlicz space of random variables if, for every $\xi \in L_U(\Omega)$, there exists a constant $r_\xi > 0$ such that

$$\mathsf{E}\, U\left(\frac{\xi}{r_\xi}\right) < \infty.$$

Every Orlicz space is a Banach space with respect to the norm

$$\|\xi\|_U = \inf\left\{ r > 0; \mathsf{E}\, U\left(\frac{\xi}{r}\right) \le 1 \right\}.$$

**Definition 2.4.** We say that condition **H** holds for an Orlicz space $L_U(\Omega)$ if, for all independent centered random variables $\xi_1, \xi_2, \dots, \xi_n$ of the space $L_U(\Omega)$,

$$\left\| \sum_{i=1}^n \xi_i \right\|_U^2 \le C_U \sum_{i=1}^n \|\xi_i\|_U^2,$$

where $C_U$ is a universal constant.

Let $L_{U_\alpha}(\Omega)$ be the Orlicz space generated by the function $U_\alpha(x)$. Let

$$U_{1\alpha}(x) = \exp\{|x|^\alpha\}, \qquad 0 < \alpha \le 1,$$

and let $S_{U_{1\alpha}}(\Omega)$ be a family of random variables $\xi$ such that $\mathsf{E}\, U_{1\alpha}(\xi/r) < \infty$ for some $r$.

Consider the functional

$$\langle\!\langle \xi \rangle\!\rangle_{U_{1\alpha}} = \inf\left\{ r > 0; \mathsf{E}\, U_{1\alpha}\left(\frac{\xi}{r}\right) \le 2 \right\}. \tag{2}$$

**Lemma 2.1** ([10])**.** *Let $\xi$ be a random variable. The inclusion $\xi \in L_{U_\alpha}(\Omega)$ holds if and only if $\xi \in S_{U_{1\alpha}}(\Omega)$ and*

$$\|\xi\|_{U_\alpha} \le \left(e^{2/\alpha+2}\right) \langle\langle \xi \rangle\rangle_{U_{1\alpha}};$$

$$\langle\langle \xi \rangle\rangle_{U_{1\alpha}} \le \|\xi\|_{U_\alpha} \left(e^{2/\alpha}+1\right)^{1/\alpha}.$$

**Lemma 2.2** ([8])**.** *Condition* **H** *holds for the Orlicz space $L_{U_\alpha}(\Omega)$, where the function $U_\alpha(x)$ is defined by equality* (1) *and where the constant $C_{\psi,\alpha}$ is given by*

$$C_{\psi,\alpha} = 4 \cdot 9^{\frac{1}{\alpha}} \left( \frac{e^{2/\alpha+2}\left(1+\frac{e^{1/12}}{\sqrt{2\pi}}\right)^{1/\alpha}}{\frac{1}{2^{1/\alpha}}\left(e^{2/\alpha}+1\right)^{-1/\alpha} \alpha^{1/\alpha}} \right)^2.$$

**Lemma 2.3** ([11, §15])**.** *Let $B_n$ be the unit sphere in $(\mathbf{R}^n, \|\cdot\|^*)$, where $\|\cdot\|^*$ is an arbitrary norm. Then, given an arbitrary $\varepsilon > 0$, there exists a set $Q_\varepsilon \subset B_n$ such that*

$$\#(Q_\varepsilon) \le \left(\frac{3}{\varepsilon}\right)^n,$$

*where the symbol $\#(Q_\varepsilon)$ denotes the total number of elements of the set $Q_\varepsilon$. Moreover, for an arbitrary $b \in B_n$, there exists $q \in Q_\varepsilon$ such that $\|b-q\|^* \le \varepsilon$.*

Let $A$ be a matrix constituted from $n$ rows and $N$ columns. Assume that its entries $a_{ij}$ are independent copies of a random variable $\xi$ such that $\mathsf{E}\,\xi = 0$ and $\mathsf{E}\,\xi^2 = \frac{1}{n}$.

**Definition 2.5** ([6])**.** Let $\Sigma_K$ be the set of those vectors in the space $\mathbf{R}^N$ that have at most $K$ nonzero coordinates. We say that a matrix $A$ satisfies the restricted isometry property of order $K$ if there exists $\delta \in (0;1)$ such that

$$(1-\delta)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta)\|x\|_2^2$$

for an arbitrary $x \in \Sigma_K$.

## 3. Main results

**Lemma 3.1.** *Let $\xi$ be a random variable The inclusion $\xi \in S_{U_{1\alpha}}(\Omega)$ holds if and only if there exist two constants $C > 0$ and $D > 0$ such that*

$$\mathsf{P}\{|\xi| > x\} \le C \exp\left\{-\frac{|x|^\alpha}{D^\alpha}\right\} \tag{3}$$

*for all $x > 0$. Moreover, $\langle\langle \xi \rangle\rangle_{U_{1\alpha}} \le (1+C)^{\frac{1}{\alpha}} D$.*

*Proof. Necessity.* Let $\xi \in S_{U_{1\alpha}}(\Omega)$. Then, by the definition of the family $S_{U_{1\alpha}}(\Omega)$, the Chebyshev inequality implies that

$$\begin{aligned}\mathsf{P}\{|\xi| \ge x\} &= \mathsf{P}\left\{\exp\left\{\left(\frac{|\xi|}{\langle\langle \xi \rangle\rangle_{U_{1\alpha}}}\right)^\alpha\right\} \ge \exp\left\{\left(\frac{x}{\langle\langle \xi \rangle\rangle_{U_{1\alpha}}}\right)^\alpha\right\}\right\} \\ &\le \frac{\mathsf{E}\exp\left\{\left(\frac{|\xi|}{\langle\langle \xi \rangle\rangle_{U_{1\alpha}}}\right)^\alpha\right\}}{\exp\left\{\left(\frac{x}{\langle\langle \xi \rangle\rangle_{U_{1\alpha}}}\right)^\alpha\right\}} \le 2\exp\left\{-\left(\frac{x}{\langle\langle \xi \rangle\rangle_{U_{1\alpha}}}\right)^\alpha\right\}.\end{aligned} \tag{4}$$

*Sufficiency.* Let there exist constants $C > 0$ and $D > 0$ such that inequality (3) holds. Let $F(x)$ be the distribution function of the random variable $|\xi|$ and let $a > D$. Consider

$$\mathsf{E}\exp\left\{\frac{|\xi|^\alpha}{a^\alpha}\right\} = \int_0^\infty \exp\left\{\frac{|u|^\alpha}{a^\alpha}\right\} dF(x) = -\int_0^\infty \exp\left\{\frac{|u|^\alpha}{a^\alpha}\right\} d(1-F(u)).$$

Integrating by parts yields

$$\mathsf{E}\exp\left\{\frac{|\xi|^\alpha}{a^\alpha}\right\} = -(1-F(u))\exp\left\{\left(\frac{u}{a}\right)^\alpha\right\}\Big|_0^\infty + \int_0^\infty (1-F(u))\exp\left\{\left(\frac{u}{a}\right)^\alpha\right\} d\left(\frac{u}{a}\right)^\alpha.$$

The first term equals zero, since $a > D$ and

$$\begin{aligned}(1-F(u))\exp\left\{\left(\frac{u}{a}\right)^\alpha\right\} &\le C\exp\left\{-\left(\frac{u}{D}\right)^\alpha\right\}\exp\left\{\left(\frac{u}{a}\right)^\alpha\right\}\\ &= C\exp\left\{-\left(\frac{u}{D}\right)^\alpha + \left(\frac{u}{a}\right)^\alpha\right\} \to 0\end{aligned}$$

as $u \to \infty$ and since its value at zero equals 1 in view of $1-F(0) \le 1$. Then inequality (3) implies that

$$\begin{aligned}\mathsf{E}\exp\left\{\frac{|\xi|^\alpha}{a^\alpha}\right\} &\le 1 + C\int_0^\infty \exp\left\{\left(\frac{x}{a}\right)^\alpha\right\}\exp\left\{-\left(\frac{x}{D}\right)^\alpha\right\} d\left(\frac{x}{a}\right)^\alpha\\ &= 1 + C\int_0^\infty \exp\left\{\left(\frac{x}{a}\right)^\alpha - \left(\frac{x}{D}\right)^\alpha\right\} d\left(\frac{x}{a}\right)^\alpha\\ &= 1 + C\int_0^\infty \exp\left\{x^\alpha\left(\frac{1}{a^\alpha} - \frac{1}{D^\alpha}\right)\right\} d\left(\frac{x}{a}\right)^\alpha\\ &= 1 + \frac{C}{a^\alpha}\frac{1}{\frac{1}{a^\alpha}-\frac{1}{D^\alpha}}\exp\left\{x^\alpha\left(\frac{1}{a^\alpha} - \frac{1}{D^\alpha}\right)\right\}\Big|_0^\infty = 1 + \frac{C}{a^\alpha}\frac{a^\alpha D^\alpha}{a^\alpha - D^\alpha}\\ &= 1 + \frac{CD^\alpha}{a^\alpha - D^\alpha}.\end{aligned}$$

If $a = D(C+1)^{1/\alpha}$, then $\mathsf{E}\exp\{|\xi|^\alpha/a^\alpha\} \le 2$, whence $\xi \in S_{U_{1\alpha}}(\Omega)$. □

**Lemma 3.2.** *Let $\xi$ be a symmetric random variable such that*

$$\mathsf{P}\{|\xi| > x\} \le R\exp\left\{-\frac{x^\alpha}{D}\right\}, \qquad 0 < \alpha < 1,$$

*for $D > 0$ and $x > 0$, where $R$ is a constant. Put $\eta = \xi^2 - \mathsf{E}\,\xi^2$. Then*

$$\mathsf{P}\{|\eta| > x\} \le a\exp\left\{-\frac{x^{\frac{\alpha}{2}}}{D}\right\}, \tag{5}$$

*where*

$$a = R\left(1 + \exp\left\{\left(\frac{8R}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\right)^{\alpha/2}\right\}\right).$$

*Proof.* For $x > 0$, we have

$$\begin{aligned}\mathsf{P}\{|\eta| > x\} &= \mathsf{P}\left\{\left|\xi^2 - \mathsf{E}\,\xi^2\right| > x\right\} = \mathsf{P}\left\{\xi^2 - \mathsf{E}\,\xi^2 > x\right\} + \mathsf{P}\left\{\xi^2 - \mathsf{E}\,\xi^2 < -x\right\}\\ &= \mathsf{P}\left\{\xi^2 > x + \mathsf{E}\,\xi^2\right\} + \mathsf{P}\left\{\xi^2 < \mathsf{E}\,\xi^2 - x\right\} = \mathsf{P}\left\{|\xi| > \sqrt{x + \mathsf{E}\,\xi^2}\right\} + Z(x)\\ &\le R\exp\left\{-D^{-1}\left(x + \mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\} + Z(x),\end{aligned}$$

where

$$Z(x) = \begin{cases}0, & x \ge \mathsf{E}\,\xi^2,\\ \mathsf{P}\{|\xi| < \sqrt{\mathsf{E}\,\xi^2 - x}\}, & x < \mathsf{E}\,\xi^2.\end{cases}$$

Let $F(x)$ be the distribution function of the random variable $\xi$. Then

$$\begin{aligned}\mathsf{E}\,\xi^2 &= \int_{-\infty}^{\infty} x^2\,dF(x) = 2\int_0^{\infty} x^2\,dF(x) = -2\int_0^{\infty} x^2\,d(1-F(x))\\ &= -2x^2(1-F(x))\big|_0^{\infty} + 4\int_0^{\infty} x(1-F(x))\,dx \le 4R\int_0^{\infty} x\exp\left\{-D^{-1}x^{\alpha}\right\}\,dx\\ &= \frac{4RD^{\frac{2}{\alpha}}}{\alpha}\int_0^{\infty} u^{\frac{2}{\alpha}-1}\exp\{-u\}\,du = \frac{4R}{\alpha}D^{\frac{2}{\alpha}}\Gamma\left(\frac{2}{\alpha}\right).\end{aligned}$$

Thus if $x < \mathsf{E}\,\xi^2$, then

$$\begin{aligned}\mathsf{P}\{|\eta| > x\} &\le R\exp\left\{-D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\\ &\qquad + R\exp\left\{-D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\exp\left\{D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\\ &\qquad \times \mathsf{P}\left\{|\xi| < \sqrt{\mathsf{E}\,\xi^2 - x}\right\}\\ &= R\exp\left\{-D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\\ &\qquad \times\left(1+\mathsf{P}\left\{|\xi| < \sqrt{\mathsf{E}\,\xi^2 - x}\right\}\exp\left\{D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\right)\\ &\le R\exp\left\{-D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\\ &\qquad \times \sup_{0<x\le \mathsf{E}\,\xi^2}\left(1+\mathsf{P}\left\{|\xi| < \sqrt{\mathsf{E}\,\xi^2 - x}\right\}\exp\left\{D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\right)\\ &\le R\exp\left\{-D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\sup_{0<x\le \mathsf{E}\,\xi^2}\left(1+\exp\left\{D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\right)\\ &\le \left(1+\exp\left\{D^{-1}\left(2\,\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\right)R\exp\left\{-D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}\\ &\le \left(1+\exp\left\{\left(\frac{8R}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\right)^{\frac{\alpha}{2}}\right\}\right)R\exp\left\{-D^{(-1)}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\}.\end{aligned}$$

This implies that

$$\mathsf{P}\{|\eta| > x\} \le a\exp\left\{-D^{-1}\left(x+\mathsf{E}\,\xi^2\right)^{\frac{\alpha}{2}}\right\} \le a\exp\left\{-D^{-1}x^{\frac{\alpha}{2}}\right\}$$

for an arbitrary $x > 0$, where

$$a = R\left(1+\exp\left\{\left(\frac{8R}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\right)^{\frac{\alpha}{2}}\right\}\right). \qquad \square$$

**Lemma 3.3.** *The family $S_{U_{1\alpha}}(\Omega)$ is a linear space and the functional $\langle\!\langle\xi\rangle\!\rangle_{U_{1\alpha}}$ is well defined on it by equality (2). Moreover, if $\xi \in S_{U_{1\alpha}}(\Omega)$ and $\eta = \xi^2 - \mathsf{E}\,\xi^2$, then $\eta \in S_{U_{1\frac{\alpha}{2}}}(\Omega)$ and condition* **H** *holds with the constant*

$$C_{\psi,\frac{\alpha}{2}} = 4\cdot 9^{\frac{2}{\alpha}}\left(\frac{e^{4/\alpha+2}\left(1+\frac{e^{1/12}}{\sqrt{2\pi}}\right)^{2/\alpha}}{\frac{1}{2^{4/\alpha}}\left(e^{4/\alpha}+1\right)^{-2/\alpha}\alpha^{2/\alpha}}\right)^2,$$

*where $\eta_i$, $i = 1,\dots,n$, are independent copies of the random variable $\eta$.*

*Proof.* Let $\lambda \ne 0$ be a constant and $\xi \in S_{U_{1\alpha}}$. Then

$$\begin{aligned}\langle\!\langle\lambda\xi\rangle\!\rangle_{U_{1\alpha}} &= \inf\left\{r > 0; \mathsf{E}\,U_{1\alpha}\left(\frac{\lambda\xi}{r}\right) \le 2\right\} = |\lambda|\inf\left\{\frac{|r|}{\lambda} > 0; \mathsf{E}\,U_{1\alpha}\left(\frac{\xi}{r/|\lambda|}\right) \le 2\right\}\\ &= |\lambda|\langle\!\langle\xi\rangle\!\rangle_{U_{1\alpha}}.\end{aligned}$$

We show that if $\xi_1, \xi_2 \in S_{U_{1\alpha}}(\Omega)$, then $\xi_1+\xi_2 \in S_{U_{1\alpha}}(\Omega)$. To prove this property, consider

$$\mathsf{E}\left(U_{1\alpha}\left(\frac{\xi_1+\xi_2}{r}\right)\right) = \mathsf{E}\exp\left\{\frac{(\xi_1+\xi_2)^\alpha}{r}\right\}.$$

The Hölder inequality with $\alpha < 1$ implies that

$$\mathsf{E}\left(U_{1\alpha}\left(\frac{\xi_1+\xi_2}{r}\right)\right) \le \mathsf{E}\exp\left\{\frac{\xi_1^\alpha}{r}\frac{\xi_2^\alpha}{r}\right\} \le \left(\mathsf{E}\exp\left\{\frac{p\xi_1^\alpha}{r}\right\}\right)^{\frac{1}{p}} \times \left(\mathsf{E}\exp\left\{\frac{q\xi_2^\alpha}{r}\right\}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Putting $r = \max\{p\langle\!\langle\xi_1\rangle\!\rangle_{U_{1\alpha}}, q\langle\!\langle\xi_2\rangle\!\rangle_{U_{1\alpha}}\}$, we obtain

$$\mathsf{E}\left(U_{1\alpha}\left(\frac{\xi_1+\xi_2}{r}\right)\right) \le \left(\mathsf{E}\exp\left\{\frac{\xi_1^\alpha}{\langle\!\langle\xi_1\rangle\!\rangle_{U_{1\alpha}}}\right\}\right)^{\frac{1}{p}} \left(\mathsf{E}\exp\left\{\frac{\xi_2^\alpha}{\langle\!\langle\xi_2\rangle\!\rangle_{U_{1\alpha}}}\right\}\right)^{\frac{1}{q}} \le 2^{1/p}\cdot 2^{1/q} = 2.$$

This proves the inclusion $\xi_1+\xi_2 \in S_{U_{1\alpha}}(\Omega)$.

Since $\xi \in S_{U_{1\alpha}}(\Omega)$, inequality (4) implies

$$\mathsf{P}\{|\xi| \ge x\} \le 2\exp\left\{-\left(\frac{x}{\langle\!\langle\xi\rangle\!\rangle_{U_{1\alpha}}}\right)^\alpha\right\}.$$

Then Lemma 3.2 yields

$$\mathsf{P}\{|\eta| \ge x\} \le a\exp\left\{-\frac{x^{\frac{\alpha}{2}}}{\langle\!\langle\xi\rangle\!\rangle^\alpha_{U_{1\alpha}}}\right\},$$

that is, $\eta \in S_{U_{1\frac{\alpha}{2}}}(\Omega)$. We conclude from Lemma 2.2 that condition **H** holds for random variables belonging to the space $L_{U_\alpha}(\Omega)$. Hence

$$\left\|\sum_{i=1}^n \eta_i\right\|^2_{U_{\alpha/2}} \le C_{\psi,\frac{\alpha}{2}} \sum_{i=1}^n \|\eta_i\|^2_{U_{\frac{\alpha}{2}}}. \qquad \square$$

**Corollary 3.1.** *Let $\eta_i$, $i=1,\dots,n$, be independent identically distributed random variables, $\eta \in S_{U_{1\frac{\alpha}{2}}}(\Omega)$. Then*

$$\left\langle\!\!\left\langle\sum_{i=1}^n \eta_i\right\rangle\!\!\right\rangle^2_{U_{1\frac{\alpha}{2}}} \le C_{S,\frac{\alpha}{2}} \sum_{i=1}^n \langle\!\langle\eta_i\rangle\!\rangle^2_{U_{1\frac{\alpha}{2}}},$$

*where* $C_{S,\frac{\alpha}{2}} = C_{\psi,\frac{\alpha}{2}}\left(e^{2/\alpha+2}\right)^2\left(e^{2/\alpha}+1\right)^{2/\alpha}$.

*Proof.* Using Lemmas 2.1 and 3.3 we obtain

$$\begin{aligned}\left\langle\!\!\left\langle\sum_{i=1}^n \eta_i\right\rangle\!\!\right\rangle^2_{U_{1\frac{\alpha}{2}}} &\le \left(e^{2/\alpha}+1\right)^{2/\alpha}\left\|\sum_{i=1}^n \eta_i\right\|^2_{U_{\frac{\alpha}{2}}} \le \left(e^{2/\alpha}+1\right)^{2/\alpha} C_{\psi,\frac{\alpha}{2}}\sum_{i=1}^n \|\eta_i\|^2_{U_{\frac{\alpha}{2}}} \\ &\le \left(e^{2/\alpha}+1\right)^{2/\alpha}\left(e^{2/\alpha+2}\right)^2 C_{\psi,\frac{\alpha}{2}}\sum_{i=1}^n \langle\!\langle\eta_i\rangle\!\rangle^2_{U_{1\frac{\alpha}{2}}}. \qquad \square\end{aligned}$$

*Remark* 3.1. It is easy to show that if $\xi_i$, $i=1,\dots,n$, are independent identically distributed random variables, $\xi_i \in S_{U_{1\alpha}}(\Omega)$, then

$$\left\langle\!\!\left\langle\sum_{i=1}^n \xi_i\right\rangle\!\!\right\rangle^2_{U_{1\alpha}} \le \tilde{C_S}\sum_{i=1}^n \langle\!\langle\xi_i\rangle\!\rangle^2_{U_{1\alpha}},$$

where $\tilde{C_S} = C_{\psi,\alpha}\left(e^{2/\alpha+2}\right)^2\left(e^{2/\alpha}+1\right)^{2/\alpha}$.

**Theorem 3.1.** *Let $A$ be a matrix constituted from $N$ columns and $n$ rows, $n \le N$. Let $a_{ij} = n^{-1/2}\xi_{ij}$, where $\xi_{ij} \in S_{U_{1\alpha}}(\Omega)$ and $\xi_{ij}$ are independent symmetric identically distributed random variables. Let $x \in \boldsymbol{R}^N$ and $\mathsf{E}\,\xi_{ij}^2 = b$ for all $i$ and $j$. Then*

$$\mathsf{P}\left\{\left|\|Ax\|_2^2 - b\|x\|_2^2\right| > \varepsilon\|x\|_2^2\right\} \le 2\exp\left\{-\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}$$

*for an arbitrary $\varepsilon > 0$, where*

$$a = R\left(1 + \exp\left\{\left(\frac{8R}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\right)^{\frac{\alpha}{2}}\right\}\right)$$

*is the constant defined in Lemma 3.2, $C_{S,\frac{\alpha}{2}}$ and $\tilde{C}_S$ are defined in Corollary 3.1 and Remark 3.1, respectively, $g = \langle\!\langle \xi_{ij}\rangle\!\rangle_{U_{1\alpha}}^2$.*

*Proof.* Without loss of generality, we may assume that $\|x\|_2^2 = 1$. Then

$$\begin{aligned}\mathsf{P}\left\{\left|\|Ax\|_2^2 - b\|x\|_2^2\right| > \varepsilon\|x\|_2^2\right\} &= \mathsf{P}\left\{\left|\sum_{i=1}^n\left(\sum_{j=1}^N x_j a_{ij}\right)^2 - b\sum_{j=1}^N x_j^2\right| > \varepsilon\right\}\\ &= \mathsf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n\left(\sum_{j=1}^N x_j \xi_{ij}\right)^2 - b\sum_{j=1}^N x_j^2\right| > \varepsilon\right\}\\ &= \mathsf{P}\left\{\left|\sum_{i=1}^n\left(\sum_{j=1}^N x_j \xi_{ij}\right)^2 - nb\sum_{j=1}^N x_j^2\right| > n\varepsilon\right\}\end{aligned}$$

for an arbitrary $\varepsilon > 0$.

Since $b = \mathsf{E}\,\xi_{ij}^2$, we get

$$\begin{aligned}\mathsf{P}\left\{\left|\|Ax\|_2^2 - b\|x\|_2^2\right| > \varepsilon\right\} &\le \mathsf{P}\left\{\left|\sum_{i=1}^n\left(\sum_{j=1}^N x_j \xi_{ij}\right)^2 - \sum_{i=1}^n \mathsf{E}\left(\sum_{j=1}^N x_j^2\xi_{ij}^2\right)\right| > n\varepsilon\right\}\\ &= \mathsf{P}\left\{\left|\sum_{i=1}^n\left(\left(\sum_{j=1}^N x_j \xi_{ij}\right)^2 - \mathsf{E}\left(\sum_{j=1}^N x_j \xi_{ij}\right)^2\right)\right| > n\varepsilon\right\}.\end{aligned}$$

Let $\theta_i = \sum_{j=1}^N x_j\xi_{ij}$. Then $\theta_i \in S_{U_{1\alpha}}$, $i = 1,\dots,n$, by Lemma 3.3. As in the proof of Lemma 3.1, we obtain

$$\mathsf{P}\{|\theta_i| > \varepsilon\} \le 2\exp\left\{-\left(\frac{\varepsilon}{\langle\!\langle\theta_i\rangle\!\rangle_{U_{1\alpha}}}\right)^{\alpha}\right\}.$$

Now

$$\mathsf{P}\left\{\left|\theta_i^2 - \mathsf{E}\,\theta_i^2\right| > \varepsilon\right\} \le a\exp\left\{-\frac{\varepsilon^{\frac{\alpha}{2}}}{\langle\!\langle\theta_i\rangle\!\rangle_{U_{1\alpha}}^{\alpha}}\right\}$$

according to Lemma 3.2.

Let $\zeta = \sum_{i=1}^n\left(\theta_i^2 - \mathsf{E}\,\theta_i^2\right)$. It follows from Lemma 3.3 that $\zeta \in S_{U_{1\frac{\alpha}{2}}}$, while Lemma 3.1 implies that

$$\mathsf{P}\left\{\left|\|Ax\|_2^2 - b\|x\|_2^2\right| > \varepsilon\right\} = \mathsf{P}\{|\zeta| > n\varepsilon\} \le 2\exp\left\{-\left(\frac{n\varepsilon}{\langle\!\langle\zeta\rangle\!\rangle_{U_{1\frac{\alpha}{2}}}}\right)^{\frac{\alpha}{2}}\right\}.$$

Since condition **H** with the constant $C_{S,\frac{\alpha}{2}}$ holds for the space $S_{U_{1\frac{\alpha}{2}}}$,

$$\langle\!\langle\zeta\rangle\!\rangle^2_{U_{1\frac{\alpha}{2}}} = \left\langle\!\!\left\langle \sum_{i=1}^n \left(\theta_i^2 - \mathsf{E}\,\theta_i^2\right) \right\rangle\!\!\right\rangle^2_{U_{1\frac{\alpha}{2}}} \le C_{S,\frac{\alpha}{2}} \sum_{i=1}^n \langle\!\langle\theta_i^2 - \mathsf{E}\,\theta_i^2\rangle\!\rangle^2_{U_{1\frac{\alpha}{2}}}.$$

Then Lemma 3.1 implies that $\langle\!\langle\theta_i^2 - \mathsf{E}\,\theta_i^2\rangle\!\rangle_{U_{1\frac{\alpha}{2}}} \le (1+a)^{2/\alpha}\langle\!\langle\theta_i\rangle\!\rangle_{U_{1\alpha}}$.

Considering Remark 3.1, we conclude that condition **H** holds with the constant $\tilde{C}_S$ for random variables belonging to the space $S_{U_{1\alpha}}$. Therefore

$$\langle\!\langle\theta_i\rangle\!\rangle^2_{U_{1\alpha}} = \left\langle\!\!\left\langle \sum_{j=1}^N x_j\xi_{ij} \right\rangle\!\!\right\rangle^2_{U_{1\alpha}} \le \tilde{C}_S \sum_{j=1}^N x_j^2 \langle\!\langle\xi_{ij}\rangle\!\rangle^2_{U_{1\alpha}} = g\cdot\tilde{C}_S,$$

where $g = \langle\!\langle\xi_{ij}\rangle\!\rangle^2_{U_{1\alpha}}$.

Then

$$\langle\!\langle\theta_i^2 - \mathsf{E}\,\theta_i^2\rangle\!\rangle_{U_{1\frac{\alpha}{2}}} \le (1+a)^{\frac{4}{\alpha}}\sqrt{g\cdot\tilde{C}_S},$$

whence $\langle\!\langle\zeta\rangle\!\rangle^2_{U_{1\frac{\alpha}{2}}} \le C_{S,\frac{\alpha}{2}} n(1+a)^{\frac{4}{\alpha}}\tilde{C}_S g$ and $\langle\!\langle\zeta\rangle\!\rangle_{U_{1\frac{\alpha}{2}}} \le \left(C_{S,\frac{\alpha}{2}} n(1+a)^{\frac{4}{\alpha}}\tilde{C}_S g\right)^{1/2}$.

Now the inequality

$$\mathsf{P}\{|\zeta| > n\varepsilon\} \le 2\exp\left\{-\frac{(n\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}} n(1+a)^{\frac{4}{\alpha}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}}\right\} = 2\exp\left\{-\frac{n^{\frac{\alpha}{4}}\varepsilon^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}$$

completes the proof. □

The following result shows how Theorem 3.1 can be used to prove the restricted isometry property for a fixed $K$-dimensional subspace. Let $T$ denote the set that contains indices of nonzero elements in the vector $x \in \mathbf{R}^N$, and let $\#(T) = K$. Denote by $X_T$ the set of all vectors in $\mathbf{R}^N$ for which all elements are zero except those elements whose indices belong to $T$.

**Lemma 3.4.** *Let $A$ be a matrix constituted from $N$ rows and $n$ columns, $n \le N$. Assume that $a_{ij} = n^{-1/2}\xi_{ij}$, where $\xi_{ij} \in S_{U_{1\alpha}}(\Omega)$ and $\xi_{ij}$ are independent symmetric identically distributed random variables. Let $x \in \boldsymbol{R}^N$ and $\mathsf{E}\,\xi_{ij}^2 = 1$ for all $i$ and $j$. Then*

$$\mathsf{P}\left\{\left|\|Ax\|_2^2 - \|x\|_2^2\right| \ge \varepsilon\|x\|_2^2\right\} \le 2\left(\frac{12}{\varepsilon}\right)^K \exp\left\{-\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}$$

*for an arbitrary set of indices $T$ of cardinality $K < n$, $X_T \subset R^N$, and for all $\varepsilon \in (0;1)$ and $x \in X_T$, where*

$$a = R\left(1 + \exp\left\{\left(\frac{8R}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\right)^{\alpha/2}\right\}\right),$$

*$C_{S,\frac{\alpha}{2}}$ and $\tilde{C}_S$ are defined in Corollary 3.1 and Remark 3.1, respectively, $g = \langle\!\langle\xi_{ij}\rangle\!\rangle^2_{U_{1\alpha}}$.*

*Proof.* Without loss of generality we assume that $\|x\|_2^2 = 1$. This assumption does not restrict the generality, indeed, since $Ax$ is a linear operator. Let $A_q = \left|\|Aq\|_2^2 - \|q\|_2^2\right|$, $q \in \mathbf{R}^N$. Lemma 2.3 implies that one can choose a finite set of points $Q_T \subseteq X_T$, $\#(Q_T) \le (12/\varepsilon)^K$, such that $\|x-q\|_2 \le \frac{\varepsilon}{4}$ for all $q \in Q_T$ and $x \in X_T$. Theorem 3.1

together with the property of $\sigma$-additivity of probability implies that

$$\mathsf{P}\left\{\overline{\bigcap_{q\in Q_T}\{q\colon A_q<\varepsilon\}}\right\}=\mathsf{P}\left\{\bigcup_{q\in Q_T}\overline{\{q\colon A_q<\varepsilon\}}\right\}=\mathsf{P}\left\{\bigcup_{q\in Q_T}\{q\colon A_q>\varepsilon\}\right\}$$
$$\leq\sum_{q\in Q_T}\mathsf{P}\{\{q\colon A_q<\varepsilon\}\}$$
$$\leq 2\left(\frac{12}{\varepsilon}\right)^K\exp\left\{-\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}$$

for all $q\in Q_T$ and $0<\varepsilon<1$. Then

$$\mathsf{P}\left\{\bigcap_{q\in Q_T}\{q\colon A_q<\varepsilon\}\right\}=1-\mathsf{P}\left\{\bigcup_{q\in Q_T}\overline{\{q\colon A_q<\varepsilon\}}\right\}$$
$$\geq 1-2\left(\frac{12}{\varepsilon}\right)^K\exp\left\{-\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}.$$

Now we choose $q$ such that $\min_{q\in Q_T}\|x-q\|_2\leq\frac{\varepsilon}{4}$. Since $\|Aq\|_2^2\leq(1+\varepsilon)\|q\|_2^2$, we get

$$\|Aq\|_2\leq\sqrt{1+\varepsilon}\|q\|_2. \tag{6}$$

Let $\varepsilon_1$ be the minimal number such that

$$\|Ax\|_2\leq(1+\varepsilon_1)\|x\|_2. \tag{7}$$

Note that $\|Ax\|_2\leq\|Aq\|_2+\|A(x-q)\|_2$. Inequality (6) with $\|q\|_2=1$ implies that $\|Aq\|_2\leq\sqrt{1+\varepsilon}$. Then we deduce from (7) that

$$\|A(x-q)\|_2\leq(1+\varepsilon_1)\|x-q\|_2\leq(1+\varepsilon_1)\frac{\varepsilon}{4}.$$

Hence

$$\|Ax\|_2\leq\sqrt{1+\varepsilon}+(1+\varepsilon_1)\frac{\varepsilon}{4}\leq 1+\varepsilon+(1+\varepsilon_1)\frac{\varepsilon}{4}.$$

Since $\varepsilon_1$ is the minimal number for which (7) holds,

$$\varepsilon_1\leq\varepsilon+(1+\varepsilon_1)\frac{\varepsilon}{4},$$

whence $\varepsilon_1\leq\frac{3\varepsilon}{4-\varepsilon}$. Taking into account the inequalities $0\leq\varepsilon\leq1$ we conclude that $\frac{3}{4-\varepsilon}\leq1$ and thus $\varepsilon_1\leq\varepsilon$.

Similarly one can show that $\|Ax\|_2\geq1-\varepsilon$ for all $x\in X_T$. □

**Theorem 3.2.** *Let $A$ be a matrix constituted from $N$ columns and $n$ rows, $n\leq N$, and let $a_{ij}=n^{-1/2}\xi_{ij}$, where $\xi_{ij}\in S_{U_{1\alpha}}(\Omega)$ are independent symmetric identically distributed random variables. Let $x\in\boldsymbol{R}^N$ and $\mathsf{E}\,\xi_{ij}^2=1$ for all $i$ and $j$. Then*

$$\mathsf{P}\left\{\left|\|Ax\|_2^2-\|x\|_2^2\right|\geq\varepsilon\|x\|_2^2\right\}\leq 2\exp\left\{K\ln\left(\frac{12eN}{\varepsilon K}\right)-\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}$$

*for all $x\in\Sigma_K$ if $K$ is such that*

$$1\leq K<\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)\ln\left(\frac{12eN}{\varepsilon K}\right)}$$

*for* $0 < \varepsilon < 1$, *where* $C_{S,\frac{\alpha}{2}}$ *and* $\tilde{C}_S$ *are defined in Corollary* 3.1 *and Remark* 3.1, *respectively,* $g = \langle\langle \xi_{ij} \rangle\rangle^2_{U_{1\alpha}}$,

$$a = R\left(1 + \exp\left\{\left(\frac{8R}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\right)^{\frac{\alpha}{2}}\right\}\right).$$

*Proof.* Lemma 3.4 implies that

$$\mathsf{P}\left\{\left|\|Ax\|_2^2 - \|x\|_2^2\right| \geq \varepsilon\|x\|_2^2\right\} \leq 2\left(\frac{12}{\varepsilon}\right)^K \exp\left\{-\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}$$

for an arbitrary $x \in X_T$, where $T$ is the set of indices of cardinality $K < n$, $X_T \subset \mathbf{R}^N$. Taking into account $C_N^K \leq (eN/K)^K$ we conclude that

$$\begin{aligned}\mathsf{P}\left\{\left|\|Ax\|_2^2 - \|x\|_2^2\right| \geq \varepsilon\|x\|_2^2\right\} &\leq 2\left(\frac{eN}{K}\right)^K\left(\frac{12}{\varepsilon}\right)^K \exp\left\{-\frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}\\ &= 2\exp\left\{K\ln\left(\frac{12eN}{\varepsilon K}\right) - \frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}\end{aligned}$$

for an arbitrary $x \in \Sigma_K$.

This implies with probability exceeding

$$p_n = 1 - 2\exp\left\{K\ln\left(\frac{12eN}{\varepsilon K}\right) - \frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)}\right\}$$

that the matrix $A$ satisfies the restricted isometry property if

$$1 \leq K < \frac{(\sqrt{n}\varepsilon)^{\frac{\alpha}{2}}}{\left(C_{S,\frac{\alpha}{2}}\tilde{C}_S g\right)^{\frac{\alpha}{4}}(1+a)\ln\left(\frac{12eN}{\varepsilon K}\right)}$$

and $0 < \varepsilon < 1$.

Note that $p_n \to 1$ as $n \to \infty$. □

## 4. Concluding remarks

It is shown in the paper that matrices constituted from random variables satisfy the restricted isometry property if their entries belong to the Orlicz spaces $L_{U_\alpha}(\Omega)$, where the function $U_\alpha(x)$ is defined by equality (1), or to the equivalent spaces $S_{U_{1\alpha}}(\Omega)$.

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Department of Probability Theory and Mathematical Analysis, Faculty of Mathematics, Uzhgorod National University, Universytets'ka Street, 14, Uzhgorod 88000, Ukraine
*E-mail address*: btroshki@ukr.net

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