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A PROBABILISTIC APPROACH TO STUDIES OF DP-TRANSFORMATIONS AND FAITHFULLNESS OF COVERING SYSTEMS TO EVALUATE THE HAUSDORFF–BESICOVITCH DIMENSION

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This paper is dedicated to the 90th anniversary
of Academician Volodymyr Semenovych Korolyuk

Abstract. This paper is devoted to the development of a probabilistic approach to transformations preserving the Hausdorff-Besicovitch dimension. New relations between fractal faithfulness of fine covering systems and DP-properties of related probability distribution functions are found. Necessary and sufficient conditions for the probability distribution functions of random variables with independent Q^*-symbols to be DP-functions are obtained.

1. Introduction

The Hausdorff measure and Hausdorff–Besicovitch dimension are the main tools for investigating the theory of fractals, where there still exist many open problems. The problem of evaluating or at least estimating the Hausdorff–Besicovitch dimension is an unsolved problem so far, even for the class of two-dimensional self-affine fractals satisfying the open set condition. Therefore improving methods to evaluate the fractal dimension is important from both points of view, namely for the development of a general theory and for the study of fractal properties of specific families of fractal sets. One of the approaches that has intensively been developed over the last 5 years or so is the study of the property of faithfulness of covering families to evaluate the Hausdorff–Besicovitch dimension and the investigation of the comparability of generated measures with the classical Hausdorff measure.

Recall that a family Φ_M of subsets of a metric space (M, ρ) is called a *family of locally fine coverings* of the set M if, for all $\varepsilon > 0$, there exists $\{E_j\}_{j\in\mathbb{N}}$ such that $E_j \in \Phi_M$, $d(E_j) \le \varepsilon$ for all $j \in \mathbb{N}$ and

$$M \subset \bigcup_j E_j.$$

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We also recall that the α-dimensional Hausdorff measure of a subset $E \subset M$ with respect to a given family of locally fine coverings Φ_M is given by

$$H^{\alpha}(E, \Phi_M) = \lim_{\varepsilon \to 0} \left[\inf_{d(E_j) \leq \varepsilon} \left\{ \sum_j d\,(E_j)^{\alpha} \right\} \right] = \lim_{\varepsilon \to 0} H^{\alpha}_{\varepsilon}(E, \Phi_M),$$

where the infimum is evaluated over all at most countable ε-coverings $\{E_j\}_{j \in \mathbb{N}}$ of the subset E such that $E_j \in \Phi_M$ for all $j \in \mathbb{N}$. If $(M, \rho) = \mathbb{R}^n$, then the family of all open (closed) subsets generates the classical α-dimensional Hausdorff measure $H^{\alpha}(\cdot)$.

Definition 1.1. The nonnegative number

$$\dim_H(E, \Phi_M) = \inf\{\alpha \colon H^{\alpha}(E, \Phi_M) = 0\}$$

is called the Hausdorff–Besicovitch dimension of a subset $E \subset M$ with respect to a family of locally fine coverings Φ_M.

Definition 1.2. A family of locally fine coverings Φ_M is called a *faithful family of coverings* for the Hausdorff–Besicovitch dimension in M if

$$\dim_H(E, \Phi) = \dim_H(E), \quad \forall E \subseteq M.$$

Definition 1.2 implies that an arbitrary family Φ_M of comparable coverings is faithful for the Hausdorff–Besicovitch dimension. Sufficient conditions for the faithfulness of a family of locally fine coverings are studied by many authors. The first result is due to A. Besicovitch who proved that the family of cylinders related to the binary expansion is faithful. This result is generalized by P. Billingsley for the families of s-adic cylinders. M. Pratsevytyĭ extended earlier results to the family of Q-S-cylinders ([16]). S. Albeverio and G. Torbin obtained these results for the case of Q^*-cylinders if the matrices Q^* whose elements p_{0k}, $p_{(s-1)k}$, are separated from zero ([2]). Some general sufficient conditions are found in the papers [7, 12] for faithfulness and for the comparability of families of coverings. All these results are obtained by using the following approach.

Lemma 1.1. *Let a family of locally fine coverings Φ be given. Assume that there are positive constants $\beta \in \mathbb{R}$ and $N^* \in \mathbb{N}$ such that, for every ball B, at most N^* sets $B_j \in \Phi$ exist with the two properties that they cover B and $d(B_j) \leq \beta \cdot d(B)$. Then the family Φ is faithful.*

We study the problem of comparability of families of locally fine coverings generated by various expansions of real numbers in order to evaluate the Hausdorff–Besicovitch dimension. Such systems of coverings consist of cylinders $\Delta^{\varphi}_{c_1 c_2 \ldots c_n}$ of all possible ranks of the corresponding generalized φ-mapping, that is, of a representation of real numbers constructed as follows. Let $\{\Omega_k\}$ be a sequence of finite or infinite subsets of the set N_0 of nonnegative integer numbers. Let $\mathcal{B}_k = 2^{\Omega_k}$ and $(\Omega, \mathcal{B}) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{B}_k)$ be a measurable space. Let φ be a measurable mapping $\Omega \to [0, 1]$ such that

(1) the images of Ω-cylinders of rank k, $\varphi(\omega_1 \omega_2 \ldots \omega_k)$ are segments (also called φ-cylinders) $\Delta^{\varphi}_{\omega_1 \omega_2 \ldots \omega_k}$, $\omega_j \in \Omega_k$;
(2) $\Delta^{\varphi}_{\omega_1 \omega_2 \ldots \omega_k i} \subset \Delta^{\varphi}_{\omega_1 \omega_2 \ldots \omega_k}$ and

$$\Delta^{\varphi}_{\omega_1 \omega_2 \ldots \omega_k} = \bigcup_{i \in \Omega_{k+1}} \Delta^{\varphi}_{\omega_1 \omega_2 \ldots \omega_k i}, \qquad [0, 1] = \bigcup_{i \in \Omega_1} \Delta^{\varphi}_{i};$$

(3) if Ω-cylinders are disjoint, then their images do not have common inner points;
(4) $\lim_{k \to \infty} |\Delta^{\varphi}_{c_1 c_2 \ldots c_k}| = 0$ for an arbitrary sequence $\{c_k\}$, $c_k \in \Omega_k$.

If all the conditions listed above hold, then for every real number x of the unit interval (except at most a countable subset of real numbers) there exists a unique sequence $\{\alpha_k(x)\}$ such that $\alpha_k(x) \in \Omega_k$ and

$$x = \bigcap_{k=1}^{\infty} \Delta^{\varphi}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)} =: \Delta^{\varphi}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)\ldots}. \tag{1}$$

Expansion (1) is called the generalized φ-representation of a real number x. Every mapping φ generates its own geometry and metric properties.

If $A_k = \{0, 1, \ldots, s-1\}$ and $\varphi(\omega_1\omega_2\ldots\omega_k\ldots) = \sum_{k=1}^{\infty} \omega_k/s^k =: \Delta^s_{\omega_1\omega_2\ldots\omega_k\ldots}$, then (1) is the classical s-adic expansion of a real number x.

If $A_k = N$ and

$$\varphi(\omega_1\omega_2\ldots\omega_k\ldots) = \cfrac{1}{\omega_1 + \cfrac{1}{\omega_2 + \cfrac{1}{\omega_3 + \ldots}}} =: \Delta^{c.f.}_{\omega_1\omega_2\ldots\omega_k\ldots},$$

then (1) is the classical continued fraction expansion.

Let s be an arbitrary positive integer number, $s \geq 2$, and let

$$Q^* = \|q_{ij}\|, \qquad i \in \{0, 1, \ldots, s-1\},\ j \in N,$$

be a stochastic matrix such that

$$1)\ q_{ij} > 0; \qquad 2)\ \sum_{i=0}^{s-1} q_{ij} = 1; \qquad 3)\ \prod_{j=1}^{\infty} \max_i q_{ij} = 0. \tag{2}$$

Given a matrix Q^* let

$$\varphi(\omega_1\omega_2\ldots\omega_k\ldots) = \beta_1 + \sum_{k=2}^{\infty} \beta_k \prod_{i=1}^{k-1} q_{\omega_k,k} =: \Delta^{Q^*}_{\omega_1\omega_2\ldots\omega_k\ldots},$$

where $\beta_k := \sum_{i=0}^{\omega_k - 1} q_{ik}$ and $\sum_{i=0}^{-1} q_{ik} := 0$. Such a representation is called the Q^*-expansion of real numbers and is a convenient tool for constructing irregular fractal sets and singular continuous probability measures of various topological and metric types (see, for example, [2, 3]). If $q_{ik} = q_i$, $k \in \mathbf{N}$, then the Q^*-expansion coincides with the self-similar Q-expansion. Moreover, if $q_{ik} = s^{-1}$ for some $s > 1$, then (1) is the s-adic expansion.

It is well known that, when calculating the Hausdorff–Besicovitch dimension of sets defined in terms of a certain φ-expansion, the problem becomes essentially simpler if one restricts the consideration to coverings that contain only cylinders of the corresponding expansion. The main problem then is to check whether or not the corresponding system of coverings Φ^{φ} is faithfull for the Hausdorff–Besicovitch dimension.

2. DP-transformations and probability measures with independent Q^*-symbols

A transformation F of the space $\mathbb{R}^n$ (in the sense of bijective mappings of $\mathbb{R}^n$ to itself) is called a transformation preserving the Hausdorff–Besicovitch dimension (DP-transformation) in the set $L \subset \mathbb{R}^n$ if

$$\dim_H(E) = \dim_H(F(E))$$

for an arbitrary set $E \subset L$.

It is shown in [4] that fractal geometry can naturally be viewed as a part of mathematics that studies the invariants of the group of DP-transformations. Note that the latter group is essentially wider than the group of bi-Lipschitz transformations. As shown in [5], the problem of investigating the one-dimensional DP-transformations is equivalent to the problem of investigating the continuous increasing distribution functions in the unit interval. In this section, we essentially generalize the results of the papers [4, 5] and find some necessary and sufficient conditions for preserving the Hausdorff–Besicovitch dimension by the distribution functions of random variables with independent Q^*-symbols. We also establish a relationship between the property of faithfulness of coverings and DP-properties of the corresponding distribution function.

Let Q^* be a stochastic matrix satisfying conditions (2). Let $\{\eta_k\}$ be a sequence of independent discrete random variables assuming values $0, 1, \dots, s-1$ with probabilities $p_{0k}, p_{1k}, \dots, p_{(s-1)k}$, respectively. Further, let η be a random variable with independent Q^*-symbols, that is,

$$\eta = \Delta^{Q^*}_{\eta_1\eta_2\dots\eta_k\dots}. \tag{3}$$

Denote by μ_η the corresponding probability measure. It is known (see [2]) that

(1) the distribution of μ_η is discrete if and only if

$$\prod_{k=1}^{\infty} \max_i p_{ik} > 0; \tag{4}$$

(2) the measure μ_η is absolutely continuous with respect to the Lebesgue measure if and only if

$$\prod_{k=1}^{\infty} \left(\sqrt{p_{0k}q_{0k}} + \sqrt{p_{1k}q_{1k}} + \dots + \sqrt{p_{(s-1)k}q_{(s-1)k}}\right) > 0; \tag{5}$$

(3) the measure μ_η is singularly continuous if and only if the infinite products in (4) and (5) diverge to zero.

If $\inf_{ik} q_{ik} > 0$, then the Hausdorff–Besicovitch dimension of the measure μ_η, that is, the infimum of the set of all Hausdorff–Besicovitch dimensions of all supports (not necessarily closed supports) of the measure μ_η, is equal to

$$\dim_H \mu = \varliminf_{n\to\infty} \frac{h_1 + h_2 + \dots + h_n}{b_1 + b_2 + \dots + b_n} \tag{6}$$

(see [2]), where $h_k = -\sum_{i=0}^{s-1} p_{ik} \ln p_{ik}$ and $b_k = -\sum_{i=0}^{s-1} p_{ik} \ln q_{ik}$.

Since any one-dimensional continuous DP-transformation is increasing, we restrict the consideration to the case where the stochastic matrix $P = \|p_{ik}\|$ does not contain zeros.

The following result provides a relationship between the property of faithfulness of the image of a family of coverings

$$\Phi' := \{F_\xi(E) \colon E \in \Phi\}$$

for the Hausdorff–Besicovitch dimension in $[0, 1]$ and the property of a distribution function F_ξ to be a DP-transformation.

Let F_ξ be a distribution function of a random variable with independent s-adic symbols and $P^* = \|p_{ik}\|$ be the corresponding matrix whose entries are such that $p_{ik} > 0$ and

$$\prod_{k=1}^{\infty} \max_i p_{ik} = 0.$$

Let Φ be the family of cylinders of the s-expansion and let

$$\Phi' := \{F_\xi(E) \colon E \in \Phi\}$$

be the family of cylinders of the Q^*-expansion generated by the matrix $Q^* = P^*$.

Lemma 2.1. *Let $x \in [0,1]$ be an arbitrary number and let the condition*

$$\lim_{k\to\infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\dots\alpha_k(x)})}{\ln \lambda(\Delta_{\alpha_1(x)\dots\alpha_k(x)})} = 1 \tag{7}$$

hold. Then

(1) Φ' *is a faithful family for the Hausdorff–Besicovitch dimension in* $[0,1]$;
(2) F_ξ *is a DP-transformation of the interval* $[0,1]$;
(3) *statements* (1) *and* (2) *are equivalent.*

Proof. Statement (3). Condition (7) together with Billingsley's theorem implies that

$$\dim_\lambda(E,\Phi) = 1 \cdot \dim_{\mu_\xi}(E,\Phi)$$

for all $E \subset [0,1]$, where $\dim_\lambda(E,\Phi)$ and $\dim_{\mu_\xi}(E,\Phi)$ are the Hausdorff–Billingsley dimensions with respect to the measures λ and μ_ξ, respectively. Thus

$$\dim_H(E) = \dim_H(E,\Phi) = \dim_\lambda(E,\Phi),$$

$$\dim_{\mu_\xi}(E,\Phi) = \dim_H\left(F_\xi(E),\Phi'\right), \quad \forall E \subset [0,1].$$

Then the latter remark implies that

$$\dim_H(E) = \dim_H\left(F_\xi(E),\Phi'\right), \quad \forall E \subset [0,1]. \tag{8}$$

If Φ' is a faithful family for the Hausdorff–Besicovitch dimension in the interval [0,1], then

$$\dim_H\left(E',\Phi'\right) = \dim_H\left(E'\right), \qquad \forall E' \subset [0,1].$$

Thus (8) yields

$$\dim_H(E) = \dim_H\left(F_\xi(E)\right), \qquad \forall E \in [0,1],$$

since Φ' is faithful for the Hausdorff–Besicovitch dimension in $[0,1]$, that is, F_ξ is a DP-transformation of the interval $[0,1]$.

If F_ξ is a DP-transformation of the interval $[0,1]$, then

$$\dim_H\left(E'\right) = \dim_H\left(F_\xi^{-1}(E')\right), \quad \forall E' \subset [0,1].$$

Relation (8) implies that

$$\dim_H\left(E',\Phi'\right) = \dim_H(F_\xi^{-1}(E')), \quad \forall E' \subset [0,1],$$

whence

$$\dim_H\left(E'\right) = \dim_H\left(E',\Phi'\right), \quad \forall E' \subset [0,1].$$

Therefore Φ' is faithful for the Hausdorff–Besicovitch dimension in $[0,1]$.

Statement (1). We prove that Φ' is faithful (whence statement (2) follows, as well). Let E' be an arbitrary subset of $[0,1]$ and $E := F_\xi^{-1}(E')$. Let $x \in E$ be an arbitrary point. According to condition (7), given an arbitrary $\delta > 0$ there exists a minimal number $n_0 := n_0(\delta,x)$ such that

$$\left|\Delta^{\Phi}_{\alpha_1(x)\dots\alpha_n(x)}\right|^{1+\delta} \le \left|\Delta^{\Phi'}_{\alpha_1(x')\dots\alpha_n(x')}\right| \le \left|\Delta^{\Phi}_{\alpha_1(x)\dots\alpha_n(x)}\right|^{1-\delta} \tag{9}$$

for all $n > n_0$, where $x' := F_\xi(x)$.

For the sake of convenience we state

$$\Delta_n(x) := \Delta^{\Phi}_{\alpha_1(x)\dots\alpha_n(x)} \quad \text{and} \quad \Delta'_n(x') := \Delta^{\Phi'}_{\alpha_1(x')\dots\alpha_n(x')}.$$

Using this notation, inequality (9) is rewritten as follows:

$$\left|\Delta_n(x)\right|^{1+\delta} \le \left|\Delta'_n(x')\right| \le \left|\Delta_n(x)\right|^{1-\delta}. \tag{10}$$

Let $m \in \mathbb{N}$ and $\delta > 0$ be fixed and state

$$W_{m,\delta} := \left\{x\colon x \in E \wedge |\Delta_n(x)|^{1+\delta} \le |\Delta'_n(x')| \le |\Delta_n(x)|^{1-\delta}, \forall n > m\right\}$$

and

$$W'_{m,\delta} := F_\xi\left(W_{m,\delta}\right).$$

Then

$$W_{1,\delta} \subset W_{2,\delta} \subset \cdots \subset W_{m,\delta} \subset \dots .$$

Moreover,

$$E := \bigcup_{m=1}^{\infty} W_{m,\delta}, \quad \forall \delta > 0.$$

We are going to show that one can restrict the consideration to coverings of the family Φ' when evaluating the Hausdorff–Besicovitch dimension of a subset $E' \subset [0,1]$.

Since F_ξ is continuous in $[0,1]$, we conclude that both functions F_ξ and F_ξ^{-1} are uniformly continuous in the interval $[0,1]$. This implies that, given an arbitrary number $\varepsilon > 0$, one can find

$$\varepsilon'(\varepsilon) > 0 \tag{11}$$

such that $|F_\xi^{-1}(I')| \le \varepsilon$ for all $I' \subset [0,1]$ with $|I'| \le \varepsilon'(\varepsilon)$.

Since m and $\delta > 0$ are fixed, we choose ε such that $(1/s)^m = \varepsilon$. Consider an arbitrary ε'-covering $\left\{E'_j\right\}_{j\in\mathbb{N}}$ of the set $W'_{m,\delta}$ by intervals

$$E'_j := [a'_j, b'_j], \quad \forall j \in \mathbb{N},$$

where $\varepsilon' \le \varepsilon'(\varepsilon)$ (see (11)). Without loss of generality, one can assume that

$$E'_j \cap W'_{m,\delta} \neq \varnothing.$$

Let $E_j := F_\xi^{-1}(E'_j) = [a_j, b_j]$, where $a_j = F_\xi^{-1}(a'_j)$ and $b_j = F_\xi^{-1}(b'_j)$. Then $\{E_j\}_{j\in\mathbb{N}}$ is an ε-covering of the set $W_{m,\delta}$. For any given $j \in \mathbb{N}$, there exists an s-adic cylinder Δ_{n_j} of a minimal rank n_j that belongs to E_j. Then the corresponding cylinder $\Delta'_{n_j} := F_\xi\left(\Delta_{n_j}\right) \in \Phi'$ belongs to E'_j. Since $\Delta_{n_j} \subset E_j$, we get $|\Delta_{n_j}| \le \varepsilon$, that is, $n_j \ge m$.

The set $E_j \cap W_{m,\delta}$ can be covered by at most $2s$ cylinders of rank n_j that contain at least one point of $E_j \cap W_{m,\delta}$. These cylinders are denoted by

$$\Delta^0_{n_j}, \Delta^1_{n_j}, \dots, \Delta^{l_j}_{n_j}.$$

Thus $\left|\Delta^0_{n_j}\right| = \left|\Delta^1_{n_j}\right| = \cdots = \left|\Delta^{l_j}_{n_j}\right| = (1/s)^{n_j}$. Since $\Delta^i_{n_j} \cap W_{m,\delta} \neq \varnothing$, $\forall i \in \{0, \dots, l_j\}$, and $\Delta'_{n_j E_j} \subset E'_j$, we conclude that

$$\left|\Delta'^i_{n_j}\right| \le \left|\Delta^i_{n_j}\right|^{1-\delta} \le \left|\Delta'^i_{n_j}\right|^{\frac{1-\delta}{1+\delta}} \le \left|E'_j\right|^{\frac{1-\delta}{1+\delta}}, \qquad \forall i \in \{0, \dots, l_j\},$$

where $\Delta'^i_{n_j} := F_\xi\bigl(\Delta^i_{n_j}\bigr)$, $\forall i \in \{0, \dots, l_j\}$.

Hence

$$\left|\Delta'^i_{n_j}\right| \le \left|E'_j\right|^{\frac{1-\delta}{1+\delta}} \le (\varepsilon')^{\frac{1-\delta}{1+\delta}}, \quad \forall i \in \{0, \dots, l_j\}.$$

Therefore

$$\sum_{i=0}^{l_j} \left|\Delta'^i_{n_j}\right|^\alpha \le 2s \cdot \left|E'_j\right|^{\alpha\cdot\frac{1-\delta}{1+\delta}}, \qquad \alpha > 0,$$

whence

$$\sum_j \sum_{i=0}^{l_j} \left|\Delta'^i_{n_j}\right|^\alpha \le 2s \cdot \sum_j \left|E'_j\right|^{\alpha\cdot\frac{1-\delta}{1+\delta}}, \qquad \alpha > 0. \tag{12}$$

Now, for an arbitrary $\varepsilon > 0$ and for every ε-covering $\left\{E'_j\right\}_{j\in\mathbb{N}}$ of the set $W'_{m,\delta}$ by intervals $E'_j := [a'_j, b'_j]$, $\forall j \in \mathbb{N}$, where $\varepsilon' \leq \varepsilon'(\varepsilon)$, there exists a family of cylinders $\Delta^i_{n_j}$, $\forall j \in \mathbb{N}$, $i \in \{0, \dots, l_j\}$, such that:

(1) $\left|\Delta^i_{n_j}\right| \leq (\varepsilon')^{\frac{1-\delta}{1+\delta}}$;

(2) $\sum_j \sum_{i=0}^{l_j} \left|\Delta'^i_{n_j}\right|^\alpha \leq 2s \cdot \sum_j \left|E'_j\right|^{\alpha\cdot\frac{1-\delta}{1+\delta}}$, $\alpha > 0$.

Hence

$$H^{\alpha\frac{1-\delta}{1+\delta}}_{\varepsilon'}\left(W'_{m,\delta}, \Phi'\right) \leq 2s \cdot \sum_j \left|E'_j\right|^{\alpha\cdot\frac{1-\delta}{1+\delta}}, \qquad \alpha > 0,$$

for an arbitrary ε'-covering $\left\{E'_j\right\}_{j\in\mathbb{N}}$. This means that

$$H^{\alpha\frac{1-\delta}{1+\delta}}_{\varepsilon'}\left(W'_{m,\delta}, \Phi'\right) \leq 2s \cdot H^{\alpha\cdot\frac{1-\delta}{1+\delta}}_{\varepsilon'}\left(W'_{m,\delta}\right), \qquad \alpha > 0.$$

Passing to the limit as $\varepsilon' \to 0$, we obtain

$$H^{\alpha}\left(W'_{m,\delta}, \Phi'\right) \leq 2s \cdot H^{\alpha\cdot\frac{1-\delta}{1+\delta}}\left(W'_{m,\delta}\right), \qquad \alpha > 0. \tag{13}$$

Let

$$\alpha_0 = \inf\left\{\alpha\colon H^{\alpha\cdot\frac{1-\delta}{1+\delta}}\left(W'_{m,\delta}\right) = 0\right\},$$

that is, $\alpha_0 \cdot \frac{1-\delta}{1+\delta} = \dim_H\left(W'_{m,\delta}\right)$. Then $H^\beta\left(W'_{m,\delta}, \Phi'\right) = 0$ for an arbitrary $\beta > \alpha_0$. Hence

$$\dim_H\left(W'_{m,\delta}, \Phi'\right) \leq \frac{1+\delta}{1-\delta} \cdot \dim_H\left(W'_{m,\delta}\right)$$

and thus

$$\begin{aligned}\dim_H\left(E', \Phi'\right) &= \dim_H\left(\bigcup_{m=1}^{\infty} W'_{m,\delta}, \Phi'\right) = \sup_m \dim_H\left(W'_{m,\delta}, \Phi'\right)\\ &\leq \frac{1+\delta}{1-\delta}\sup_m \dim_H\left(W'_{m,\delta}\right) = \frac{1+\delta}{1-\delta}\dim_H\left(E'\right), \quad \forall\delta > 0.\end{aligned}$$

Therefore

$$\dim_H\left(E', \Phi'\right) \leq \frac{1+\delta}{1-\delta}\dim_H\left(E'\right), \quad \forall\delta > 0.$$

The latter result implies

$$\dim_H\left(E', \Phi'\right) \leq \dim_H\left(E'\right).$$

This proves that $\dim_H\left(E', \Phi'\right) = \dim_H\left(E'\right)$ for all $E' \subset [0, 1]$. □

Remark 2.1. Lemma 2.1 plays an important role in the proof of the main result of this paper. Note however that it has its own value and admits a generalization to other representations (in particular, to Q-representations and Q^*-representations such that $\inf_{ik} q_{ik} > 0$) and to the corresponding transformations.

Now we turn to the main result of the paper. Assume that $\inf_{ik} q_{ik} > 0$. Let $p_k := \min_i p_{ik}$, $q_{\min} := \min_{ik} q_{ik}$, and $q_{\max} := \max_{ik} q_{ik}$. Put

$$T^{(1)} := \left\{k\colon k \in N, p_k < \frac{1}{2}q_{\min}\right\},$$

$$T^{(1)}_k := T^{(1)} \cap \{1, 2, \dots, k\}.$$

Let

$$A := \varlimsup_{k\to\infty} \frac{\sum_{j\in T^{(1)}_k} \ln\frac{1}{p_j}}{k}.$$

Theorem 2.1. *Let* $\inf_{ik} q_{ik} > 0$*. Then the distribution function* F_μ *preserves the Hausdorff–Besicovitch dimension of any subset of the unit interval if and only if*

$$\begin{cases} \dim_H \mu_\eta = 1; \\ A = 0. \end{cases} \tag{14}$$

Proof. Sufficiency. Let $\dim_H \mu = 1$ and $A = 0$.

Recall that the classical Gibbs inequality is equivalent to the statement that the Kullback–Leibler distance is nonnegative (see [10]). This yields

$$h_k = -\ln\left(p_{0k}^{p_{0k}} \cdot p_{1k}^{p_{1k}} \cdot \dots \cdot p_{(s-1)k}^{p_{(s-1)k}}\right) \le b_k = -\ln\left(q_{0k}^{p_{0k}} \cdot q_{1k}^{p_{1k}} \cdot \dots \cdot q_{(s-1)k}^{p_{(s-1)k}}\right). \tag{15}$$

Then (6) implies that the condition $\dim_H \mu = 1$ is equivalent to the existence of the limit

$$\lim_{n\to\infty} \frac{h_1 + h_2 + \dots + h_n}{b_1 + b_2 + \dots + b_n} = 1. \tag{16}$$

Let ε be an arbitrary positive number such that $\varepsilon < \frac{1}{2}q_{\min}$. Consider the sets

$$T_{\varepsilon,k}^{+} = \left\{j \colon j \in N, j \le k, |p_{ij} - q_{ij}| \le \varepsilon, \forall i \in N_{s-1}^{0} := \{0, 1, \dots, s-1\}\right\}$$

and

$$T_{\varepsilon,k}^{-} = \left\{j \colon j \in N, j \le k, |p_{ij} - q_{ij}| > \varepsilon \text{ for some } i \in N_0^{s-1}\right\}.$$

It follows from the equality $\dim_H \mu = 1$ that

$$\lim_{k\to\infty} |T_{\varepsilon,k}^{+}|/k = 1,$$

where $|E|$ denotes the number of elements in a set E.

The set $T_{\varepsilon,k}^{-}$ can be represented as follows $T_{\varepsilon,k}^{-} = T_k^{(1)} \cup T_{\varepsilon,k}$, where $T_k^{(1)}$ is defined above and

$$T_{\varepsilon,k} = \left\{j \colon j \in N, j \le k; p_j \ge \frac{1}{2}q_{\min}, |p_{ij} - q_{ij}| > \varepsilon \text{ for some } i \in N_0^{s-1}\right\}.$$

It is clear that

$$\lim_{k\to\infty} \frac{|T_{\varepsilon,k}^{-}|}{k} = \lim_{k\to\infty} \frac{\left|T_k^{(1)}\right|}{k} = \lim_{k\to\infty} \frac{|T_{\varepsilon,k}|}{k} = 0.$$

Let $\Delta^{Q^*}_{\alpha_1(x)\dots\alpha_k(x)}$ be a cylinder of the Q^*-representation that contains a point x, $\mu = \mu_\xi$, and let λ denote the Lebesgue measure. If $x \in [0, 1]$ is an arbitrary point, then

$$-\ln\mu\left(\Delta^{Q^*}_{\alpha_1(x)\dots\alpha_k(x)}\right) = -\left(\sum_{j\in T_k^{(1)}} \ln p_{\alpha_j(x)j} + \sum_{j\in T_{\varepsilon,k}} \ln p_{\alpha_j(x)j} + \sum_{j\in T_{\varepsilon,k}^{+}} \ln p_{\alpha_j(x)j}\right)$$

for all positive integer numbers k and all positive numbers $\varepsilon < \frac{1}{2}q_{\min}$.

Since

$$\sum_{j\in T_{\varepsilon,k}} \ln \frac{1}{p_{\alpha_j(x)j}} \le |T_{\varepsilon,k}| \ln \frac{2}{q_{\min}},$$

we obtain

$$\sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{p_{\alpha_j(x)j}} \le \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}-\varepsilon} = \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + \ln\left(1+\frac{\varepsilon}{q_{\alpha_j(x)j}-\varepsilon}\right)$$
$$\le \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + \sum_{j\in T^+_{\varepsilon,k}} \frac{\varepsilon}{q_{\alpha_j(x)j}-\varepsilon} \le \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + \sum_{j\in T^+_{\varepsilon,k}} \frac{2\varepsilon}{q_{\alpha_j(x)j}}$$
$$\le \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + \sum_{j\in T^+_{\varepsilon,k}} \frac{2\varepsilon}{q_{\min}} \le \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + \left|T^+_{\varepsilon,k}\right| \frac{2\varepsilon}{q_{\min}}.$$

Thus

$$\varlimsup_{k\to\infty} \frac{\ln\mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}{\ln\lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}$$
$$\le \varlimsup_{k\to\infty} \frac{\sum_{j\in T^{(1)}_k} \ln\frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}|\ln\frac{2}{q_{\min}} + \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + |T^+_{\varepsilon,k}|\frac{2\varepsilon}{q_{\min}}}{-\ln[\prod_{j=1}^k q_{\alpha_j(x)j}]}$$
$$= \varlimsup_{k\to\infty} \frac{\sum_{j\in T^{(1)}_k} \ln\frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}|\ln\frac{2}{q_{\min}} + \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + |T^+_{\varepsilon,k}|\frac{2\varepsilon}{q_{\min}}}{\sum_{j=1}^k \ln\frac{1}{q_{\alpha_j(x)j}}}$$
$$\le 1 + \varlimsup_{k\to\infty} \frac{\sum_{j\in T^{(1)}_k} \ln\frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}|\ln\frac{2}{q_{\min}} + |T^+_{\varepsilon,k}|\frac{2\varepsilon}{q_{\min}}}{\sum_{j=1}^k \ln\frac{1}{q_{\alpha_j(x)j}}}$$
$$\le 1 + \varlimsup_{k\to\infty} \frac{\sum_{j\in T^{(1)}_k} \ln\frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}|\ln\frac{2}{q_{\min}} + |T^+_{\varepsilon,k}|\frac{2\varepsilon}{q_{\min}}}{k\ln\frac{1}{q_{\max}}} = 1 + \frac{2\varepsilon}{q_{\min}\cdot\ln\frac{1}{q_{\max}}}$$

for all $x\in[0,1]$ and all positive $\varepsilon < \frac{1}{2}q_{\min}$.

On the other hand,

$$\sum_{j\in T_{\varepsilon,k}} \ln\frac{1}{p_{\alpha_j(x)j}} \ge |T_{\varepsilon,k}|\ln\frac{2}{2-q_{\min}}$$

and

$$\sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{p_{\alpha_j(x)j}} \ge \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}+\varepsilon} = \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + \ln\frac{q_{\alpha_j(x)j}}{q_{\alpha_j(x)j}+\varepsilon}$$
$$= \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} + \sum_{j\in T^+_{\varepsilon,k}} \ln\left(1-\frac{\varepsilon}{q_{\alpha_j(x)j}+\varepsilon}\right)$$
$$\ge \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} - \sum_{j\in T^+_{\varepsilon,k}} \frac{2\varepsilon}{q_{\alpha_j(x)j}+\varepsilon}$$
$$\ge \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} - \sum_{j\in T^+_{\varepsilon,k}} \frac{2\varepsilon}{q_{\min}} = \sum_{j\in T^+_{\varepsilon,k}} \ln\frac{1}{q_{\alpha_j(x)j}} - \left|T^+_{\varepsilon,k}\right| \frac{2\varepsilon}{q_{\min}}.$$

Hence

$$\varliminf_{k\to\infty} \frac{\ln \mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}{\ln \lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}$$

$$\geq \varliminf_{k\to\infty} \frac{\sum_{j\in T_k^{(1)}} \ln \frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}| \ln \frac{2-q_{\min}}{2} + \sum_{j\in T^+_{\varepsilon,k}} \ln \frac{1}{q_{\alpha_j(x)j}} - |T^+_{\varepsilon,k}| \frac{2\varepsilon}{q_{\min}}}{-\ln[\prod_{j=1}^k q_{\alpha_j(x)j}]}$$

$$= \varliminf_{k\to\infty} \frac{\sum_{j\in T_k^{(1)}} \ln \frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}| \ln \frac{2-q_{\min}}{2} + \sum_{j\in T^+_{\varepsilon,k}} \ln \frac{1}{q_{\alpha_j(x)j}} - |T^+_{\varepsilon,k}| \frac{2\varepsilon}{q_{\min}}}{\sum_{j=1}^k \ln \frac{1}{q_{\alpha_j(x)j}}}$$

$$= 1 + \varliminf_{k\to\infty} \frac{\sum_{j\in T_k^{(1)}} \ln \frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}| \ln \frac{2-q_{\min}}{2} - |T^+_{\varepsilon,k}| \frac{2\varepsilon}{q_{\min}}}{\sum_{j=1}^k \ln \frac{1}{q_{\alpha_j(x)j}}}$$

$$\geq 1 + \varliminf_{k\to\infty} \frac{\sum_{j\in T_k^{(1)}} \ln \frac{1}{p_{\alpha_j(x)j}} + |T_{\varepsilon,k}| \ln \frac{2-q_{\min}}{2} - |T^+_{\varepsilon,k}| \frac{2\varepsilon}{q_{\min}}}{k \ln \frac{1}{q_{\min}}} = 1 - \frac{2\varepsilon}{q_{\min} \cdot \ln \frac{1}{q_{\min}}}$$

for all $x \in [0,1]$ and all positive $\varepsilon < \frac{1}{2} q_{\min}$.

Combining all results above,

$$1 - \frac{2\varepsilon}{q_{\min} \ln \frac{1}{q_{\min}}} \leq \varliminf_{k\to\infty} \frac{\ln \mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}{\ln \lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)} \leq \varlimsup_{k\to\infty} \frac{\ln \mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}{\ln \lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}$$

$$\leq 1 + \frac{2\varepsilon}{q_{\min} \ln \frac{1}{q_{\max}}}.$$

Since these bounds hold for an arbitrary number $x \in [0,1]$ and all positive numbers $\varepsilon < \frac{1}{2} q_{\min}$, we get

$$\lim_{k\to\infty} \frac{\ln \mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)}{\ln \lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}\right)} = 1, \quad \forall x \in [0,1]. \tag{17}$$

Then relation (17) together with Billingsley's theorem (see [6]) implies that

$$\dim_H(E, \lambda, \Phi(Q^*)) = 1 \cdot \dim_H(E, \mu_\eta, \Phi(Q^*))$$

for an arbitrary subset $E \subset [0,1]$, where $\dim_H(E, \lambda, \Phi(Q^*))$ is the Hausdorff–Billingsley dimension of the set E with respect to the Lebesgue measure λ and family of cylinders of the Q^*-representation and where $\dim_H(E, \mu_\eta, \Phi(Q^*))$ is the Hausdorff–Billingsley dimension of the set E with respect to the measure μ_η and family of cylinders of the Q^*-representation.

Since $\inf_{ik} q_{ik} > 0$, the family of cylinders of the Q^*-representation is faithful for the Hausdorff–Besicovitch dimension in the unit interval (see [2]). Thus

$$\dim_H(E, \lambda, \Phi(Q^*)) = \dim_H(E).$$

Now the definition of the Hausdorff–Billingsley dimension implies that

$$\dim_H(E, \mu_\eta, \Phi(Q^*)) = \dim_H\big(E, F_\mu(\Phi(Q^*))\big) = \dim_H(E, \Phi(P^*)).$$

Finally, Lemma 2.1 yields that the family $\Phi(P^*)$ is faithful for the Hausdorff–Besicovitch dimension in the unit interval. Therefore F_μ is a DP-transformation of the unit interval.

Necessity. Let F_μ be a DP-transformation of the unit interval. We prove that both conditions $\dim_H \mu = 1$ and $A = 0$ hold.

If $\dim_H \mu < 1$, then there exists a support E of the measure μ such that

$$\dim_H \mu \le \dim_H(E) < 1.$$

Since $\mu(E) = 1$, we get $\dim_H(F_\mu(E)) = 1 \ne \dim_H(E)$, and this contradicts the assumption. Thus the property of the measure μ to be superfractal is a necessary condition for the distribution function F_μ to preserve the dimension.

Now we prove that $A = 0$. Assume that $A > 0$. In order to show that F_μ does not preserve the fractal dimension we consider the set

$$L = \left\{ x\colon x = \Delta^{Q^*}_{\alpha_1\ldots\alpha_k\ldots} \right\},$$

where

$$\alpha_k \in N_0^{s-1} \quad \text{if } k \notin T^{(1)},$$
$$\alpha_k = n_k \quad \text{if } k \in T^{(1)},$$

and $p_{n_k k} = \min_i p_{ik}$.

It is clear that $\lambda(L) = 0$. On the other hand, $\dim_H(L) = 1$, since $\lim_{k\to\infty} \left|T_k^{(1)}\right|/k = 0$ and $\inf_{ik} q_{ik} > 0$. Then

$$\overline{\lim_{k\to\infty}} \frac{\sum_{j\in T_k^{(1)}} \ln p_j}{-k} = A$$

implies that there exists a subsequence $\{k_m\}$ such that the limit

$$\lim_{m\to\infty} \frac{\sum_{j\in T_{k_m}^{(1)}} \ln p_j}{-k_m}$$

exists and equals A.

Thus

$$\begin{aligned}
&\underline{\lim}_{m\to\infty} \frac{\ln \mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)}{\ln \lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)} \\
&\quad = \underline{\lim}_{m\to\infty} \frac{\sum_{j\in T_{k_m}^{(1)}} \ln \frac{1}{p_{\alpha_j(x)j}} + \sum_{j\in T_{\varepsilon,k_m}} \ln \frac{1}{p_{\alpha_j(x)j}} + \sum_{j\in T^+_{\varepsilon,k_m}} \ln \frac{1}{p_{\alpha_j(x)j}}}{\sum_{j=1}^{k_m} \ln \frac{1}{q_{\alpha_j(x)j}}} \\
&\quad = 1 + \underline{\lim}_{m\to\infty} \frac{\sum_{j\in T_{k_m}^{(1)}} \ln \frac{1}{p_{\alpha_j(x)j}}}{\sum_{j=1}^{k_m} \ln \frac{1}{q_{\alpha_j(x)j}}} \\
&\quad = 1 + \underline{\lim}_{m\to\infty} \frac{\sum_{j\in T_{k_m}^{(1)}} \ln \frac{1}{p_j}}{\sum_{j=1}^{k_m} \ln \frac{1}{q_{\alpha_j(x)j}}} \ge 1 + \underline{\lim}_{m\to\infty} \frac{\sum_{j\in T_{k_m}^{(1)}} \ln \frac{1}{p_j}}{k_m \ln \frac{1}{q_{\min}}} = 1 + c \cdot A
\end{aligned}$$

for an arbitrary $x \in L$, where $c = \frac{-1}{\ln q_{\min}}$.

Then, given an arbitrary $\delta > 0$, one can find $m(\delta)$ such that

$$1 + c\cdot A - \delta \le \frac{\ln \mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)}{\ln \lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)}$$

for all $m > m(\delta)$ and any $x \in L$. The latter condition is equivalent to the inequality

$$\mu\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right) \le \lambda\left(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)^{1+c\cdot A-\delta}.$$

Therefore,

$$\left|\Delta'_{\alpha_1(x)\dots\alpha_{k_m}(x)}\right|^{\frac{1}{1+c\cdot A-\delta}} \le \left|\Delta^{Q^*}_{\alpha_1(x)\dots\alpha_{k_m}(x)}\right| \tag{18}$$

for arbitrary $x \in L$, $\delta > 0$, and $m > m(\delta)$, where $\Delta'_{\alpha_1(x)\dots\alpha_{k_m}(x)} = F_\mu\big(\Delta^{Q^*}_{\alpha_1(x)\dots\alpha_{k_m}(x)}\big)$.

Now we choose $\delta \in (0, c\cdot A)$.

Since $\lambda(L) = 0$, the Hausdorff premeasure is such that $H^1_\varepsilon(L) = 0$ for any positive ε. Thus if $\varepsilon > 0$ and $t > 0$ are given, then there exists an ε-covering $\{E_i\}$ of the set L by cylinders of the Q^*-representation of rank k_m (m depends on ε and t) such that $\sum_i |E_i| < t$. The family of sets $\{E'_i\} = \{F_\mu(E_i)\}$ constitutes an ε'-covering of the set $L' = F_\mu(L)$. It is clear that $\varepsilon' \to 0$ if and only if $\varepsilon \to 0$, since F_μ is uniformly continuous in the unit interval.

Without loss of generality one can restrict the consideration to the case of coverings such that every set E_i has a nonempty intersection with L. Then

$$\sum_i |E'_i|^{\frac{1}{1+c\cdot A-\delta}} \le \sum_i |E_i| < t$$

follows from inequality (18).

Since ε and t are arbitrary, we conclude that

$$H_{\varepsilon'}^{\frac{1}{1+c\cdot A-\delta}}(L') = 0, \quad \forall \varepsilon' > 0.$$

Thus the Hausdorff measure $H^{\frac{1}{1+c\cdot A-\delta}}(L')$ of the set L' equals zero, whence

$$\dim_H(L') \le \frac{1}{1+c\cdot A-\delta} < 1.$$

This means that if $A > 0$, then the distribution function F_μ does not belong to the class DP. □

Corollary 2.1. *If the distribution function F_μ preserves the Hausdorff–Besicovitch dimension of an arbitrary subset of the unit interval, then the probability measure μ is of full Hausdorff dimension.*

Corollary 2.2. *If the entries of the matrix P are separated from zero, that is, if*

$$\inf_{ij} p_{ij} > 0,$$

then F_μ is a DP-transformation of the unit interval if and only if the corresponding probability measure μ has full Hausdorff dimension.

Corollary 2.3. *If*

$$\lim_{k\to\infty} p_{ik} = q_i$$

for all $i \in N_0^{s-1}$, then F_μ preserves the Hausdorff–Besicovitch dimension in the unit interval.

Corollary 2.4. *Let F_μ be an increasing absolutely continuous distribution function of a random variable with independent Q-symbols. Then F_μ preserves the Hausdorff–Besicovitch dimension in the unit interval.*

Corollary 2.5. *Whatever the stochastic vector Q is, there are singularly continuous distribution functions of random variables with independent Q-symbols that preserve the Hausdorff–Besicovitch dimension in the unit interval.*

Proof. To construct such a distribution function we choose the matrix P such that

$$p_{ik} \to q_i \quad \text{as } k \to \infty$$

and

$$\sum_{k=1}^{\infty} \left(\sum_{i=0}^{s-1} \left(1 - \frac{p_{ik}}{q_i} \right)^2 \right) = \infty.$$

Then the corresponding distribution function F_μ is a singularly continuous DP-function. □

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