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# CROSS-CORRELOGRAM ESTIMATORS OF IMPULSE RESPONSE FUNCTIONS

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Abstract. The integral cross-correlogram estimator of the response function for a linear homogeneous system is considered in this paper. An upper bound for the tail of the distribution of the supremum of the estimation error is found. In the proof, we use some properties of square-Gaussian stochastic processes.

## 1. Introduction

The problem of estimation of characteristics for stochastic linear systems has been intensively studied during last five decades and can be applied in different fields of science such as radiophysics, seismology, meteorology, signal processing, automatic control, filtration theory, financial mathematics, etc.

Some methods of estimation of unknown impulse response functions in linear systems are developed and properties of the corresponding estimators are studied by V. V. Buldygin and his collaborators. Continuous homogeneous systems is an important subclass of linear systems. Estimators in the case of continuous homogeneous systems are constructed from joint periodograms and joint cross-correlograms for input and output processes.

Buldygin and Li [8, 9] studied the asymptotic normality of the correlogram estimator and that of the estimation error in the sense of convergence of finite dimensional distributions in the space of continuous functions.

Buldygin and Kurotschka [7] and Buldygin et al. [10, 11] have studied the cross-correlogram estimator for the discrete time and obtained sufficient conditions for the unbiasedness, asymptotic normality, and mean square consistency as well as for the asymptotic normality in the sense of convergence of finite dimensional distributions and in the sense of convergence of the corresponding distributions in the space of continuous functions.

Buldygin and Blazhievska [1]–[3] proved the asymptotic unbiasedness and mean square consistency of the integral cross-correlogram estimator under milder assumptions than in [8, 9]. The asymptotic normality of the estimator and estimation error are also proved in [8, 9] in the space of continuous functions.

It is worth mentioning that the papers mentioned above deal with the asymptotic normality but do not pay any attention to finding estimates for the distribution of the supremum of the error of estimation.

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In this paper, we study the integral cross-correlogram estimator for the impulse response function and find upper bounds for the distribution of the supremum of the estimation error.

## 2. Cross-correlograms and their properties

Consider a causal homogeneous system with the impulse response function $H(\tau)$, $\tau \in \mathbf{R}$. This means that $H$ is a real-valued function such that $H(\tau) = 0$ for $\tau < 0$ and the response of the system to an admissible input signal $X(t)$, $t \in \mathbf{R}$, is given by

$$Y(t) = \int_0^\infty H(\tau) X(t-\tau)\, d\tau. \tag{1}$$

The problem of estimation of the function $H$ from observations after the input and output signals is an important problem when studying such systems. In this paper, we follow the cross-correlogram method for estimation of the impulse response function $H$ if $H \in L_2(\mathbf{R})$.

Consider a real-valued stationary centered Gaussian stochastic process

$$X = (X(t), t \in \mathbf{R})$$

that perturbs system (1). Let $f = (f(\lambda), \lambda \in \mathbf{R})$ be the spectral density of the process $X$. Assume that $f$ is continuous and such that

$$\sup_{\lambda \in \mathbf{R}} |f(\lambda)| < \infty;$$
$$K_X \in L_1(\mathbf{R}),$$

where

$$K_X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} f(\lambda)\, d\lambda, \qquad t \in \mathbf{R},$$

is the correlation function of the stochastic process $X$.

We introduce an estimator for the value of $H$ at a point $\tau > 0$ as the empirical joint cross-correlogram between input and output processes (see, for example, [3])

$$\hat{H}_T(\tau) = \frac{1}{T} \int_0^T Y(t+\tau) X(t)\, dt, \qquad \tau > 0, \tag{2}$$

where $T$ is the length of the interval for averaging. This estimation method requires that $X(t) = X_\Delta(t)$, that is, input processes from a family depending on a parameter $\Delta > 0$. Another assumption is that the spectral density is of a special form (see Remark 2.2). We omit the parameter $\Delta$ in the rest of the paper.

Assume that $H \in L_2(\mathbf{R})$. By

$$H^*(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda t} H(t)\, dt, \qquad \lambda \in \mathbf{R},$$

we denote the Fourier–Plancherel transformation of the function $H$.

*Remark* 2.1. The integrals in (1) and (2) are defined as square mean Riemann integrals.

The integral on the right-hand side of equality (1) is well defined if and only if the following Riemann integral (see [4])

$$\int_0^\infty \int_0^\infty H(\tau) K_X(s-\tau) H(s)\, ds\, d\tau \tag{3}$$

is well defined. Applying the representation of the correlation function in terms of the spectral density and Fourier–Plancherel transformation for the function $H(\tau)$, we obtain

$$\int_0^\infty \int_0^\infty H(\tau) K(s-\tau) H(s)\, ds\, d\tau = \int_{-\infty}^{+\infty} H^*(-\lambda) \cdot H^*(\lambda) f(\lambda)\, d\lambda.$$

Since $\sup_{\lambda\in\mathbb{R}} |f(\lambda)| < \infty$ and $H \in L_2(\mathbf{R})$, the integral in (3) is well defined and therefore so is the integral in (1).

In what follows we assume that the integral in relation (3) is well defined in the Lebesgue sense, too.

The mean value of $\hat{H}_T(\tau)$ is easy to evaluate:

$$\mathsf{E}\,\hat{H}_T(\tau) = \int_0^\infty H(s) K_X(\tau - s)\,ds. \tag{4}$$

Equality (4) implies in the general case that

$$\mathsf{E}\,\hat{H}_T(\tau) \neq H(\tau), \qquad \tau \in \mathbf{R}.$$

This means that the estimator $\hat{H}_T(\tau)$ is biased.

The results obtained in the papers [1, 3, 8] concern a sequence of covariance functions that depend on a parameter $\Delta$ and contain sufficient conditions for the asymptotic unbiasedness of the estimator $\hat{H}_{T,\Delta}(\tau)$ as $\Delta \to \infty$. It is also shown in [8] that $\hat{H}_{T,\Delta}(\tau) \to H(\tau)$ with probability 1 as $\Delta \to \infty$ and $T \to \infty$ under certain extra conditions.

*Remark* 2.2. It follows from [3] that all assumptions for the asymptotic unbiasedness as well as for the convergence with probability 1 hold for the following sequence of spectral densities:

$$f_\Delta(\lambda) = \frac{c}{2\pi} \exp\left\{-\frac{\lambda^2}{\Delta}\right\}, \qquad \lambda \in \mathbf{R},\ \Delta > 0. \tag{5}$$

Let

$$\hat{Z}_T(\tau) = \hat{H}_T(\tau) - \mathsf{E}\,\hat{H}_T(\tau).$$

It is proved in [2] that the correlation function of $\hat{Z}_T(\tau)$ is given by

$$\begin{aligned} &\mathsf{E}\,\hat{Z}_T(\tau_1)\hat{Z}_T(\tau_2) \\ &\quad = \frac{2\pi}{T} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(e^{i(\tau_1-\tau_2)\lambda_2}|H^*(\lambda_2)|^2 + e^{i(\tau_1\lambda_1+\tau_2\lambda_2)} H^*(\lambda_1)H^*(\lambda_2)\right) \\ &\qquad\qquad \times \Phi_T(\lambda_2 - \lambda_1) f(\lambda_1) f(\lambda_2)\,d\lambda_1\,d\lambda_2, \end{aligned} \tag{6}$$

where $\Phi_T(\lambda)$ is the Fejér kernel,

$$\Phi_T(\lambda) = \frac{1}{2\pi T}\left(\frac{\sin(T\lambda/2)}{\lambda/2}\right)$$

(see [2]).

Let $S$ be a set of parameters. A function $\rho(t,s)$ is called a *pseudometric* on $S$ if it satisfies all conditions imposed on a metric except for the following one:

the set $\{(t,s) \in S \times S\colon \rho(t,s) = 0\}$ coincides with $\{(t,s) \in S \times S\colon t = s\}$.

In other words, the equality $\rho(t,s) = 0$ is valid not only for $s = t$ in the case of a pseudometric.

Consider the function

$$g_H(\tau) = \left[\int_{-\infty}^\infty |H^*(\lambda)|^2 \sin^2\frac{\lambda\tau}{2}\,d\lambda\right]^{1/2}, \qquad \tau > 0 \tag{7}$$

(see [8]). Since $H \in L_2(\mathbf{R})$, the function in (7) is well defined and generates the pseudometric

$$\sqrt{\sigma}(\tau_1, \tau_2) = \sqrt{g_H(|\tau_1 - \tau_2|)}, \qquad \tau_1, \tau_2 \in \mathbf{R}.$$

The proof of the property that $\sqrt{\sigma}(\tau_1,\tau_2)$ is a pseudometric can be found in [8].

**Lemma 2.1.** *Assume that $H \in L_2(\mathbf{R})$ and $\sup_{\lambda \in \mathbf{R}} |f(\lambda)| < \infty$, where $f(t)$ is the spectral density of the process $X(t)$. Then*

$$\left(\mathsf{E}\,|\hat{Z}_T(\tau_1) - \hat{Z}_T(\tau_2)|^2\right)^{1/2} \le \frac{4\sqrt{\pi}}{\sqrt{T}} \cdot \sup_{t \in \mathbf{R}} |f(t)| \cdot \|H^*\|_2^{1/2} \cdot \sqrt{\sigma}(\tau_1, \tau_2), \qquad \tau_1, \tau_2 > 0. \tag{8}$$

*Proof.* The proof repeats the one for an analogous results in [2]. □

The following result is obtained in [5].

**Lemma 2.2.** *Let $\psi(u)$, $u \ge 0$, be a continuous increasing function such that $\psi(0) = 0$ and $\frac{u}{\psi(u)}$ is nondecreasing for $u > u_0$, where the constant $u_0$ is nonnegative. Then*

$$\left|\sin \frac{u}{v}\right| \le \frac{\psi(u + u_0)}{\psi(v + u_0)} \tag{9}$$

*for all $u \ge 0$ and $v > 0$.*

**Example 2.1.** Since assumptions of Lemma 2.2 are valid for $\psi(u) = u^\alpha$, $\alpha \in (0, 1]$, and $u_0 = 0$, we conclude that

$$\left|\sin \frac{u}{v}\right| \le \frac{u^\alpha}{v^\alpha}, \qquad \alpha \in (0, 1]. \tag{10}$$

**Example 2.2.** The assumptions of Lemma 2.2 are valid in the case of $\psi(u) = \ln^\alpha(u+1)$, $\alpha > 0$, and $u_0 = e^\alpha - 1$. Thus, inequality (9) can be rewritten as follows:

$$\left|\sin \frac{u}{v}\right| \le \frac{\ln^\alpha(e^\alpha + u)}{\ln^\alpha(e^\alpha + v)}, \qquad \alpha > 0. \tag{11}$$

**Theorem 2.1.** *Assume that $H \in L_2(\mathbf{R})$ and $\sup_{\lambda \in \mathbf{R}} |f(\lambda)| < \infty$, where $f(t)$ is the spectral density of the process $X(t)$. Let a function $\psi(u)$, $u \ge 0$, satisfy all the assumptions of Lemma 2.2 and let the integral*

$$\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \psi^2 \left(\frac{\lambda}{2} + u_0\right) d\lambda$$

*converge. Then*

$$\left(\mathsf{E}\,|\hat{Z}_T(\tau_1) - \hat{Z}_T(\tau_2)|^2\right)^{1/2} \le K \cdot \psi^{-1/2} \left(\frac{1}{|\tau_1 - \tau_2|} + u_0\right), \qquad \tau_1, \tau_2 \in \mathbf{R}, \tag{12}$$

*where $u_0$ is the constant defined in Lemma 2.2 and*

$$K = K(T) = \frac{4\sqrt{\pi}}{\sqrt{T}} \cdot \sup_{t \in \mathbf{R}} |f(t)| \cdot \|H^*\|_2^{1/2} \cdot \left(\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \psi^2 \left(\frac{\lambda}{2} + u_0\right) d\lambda\right)^{1/4}. \tag{13}$$

*Proof.* The proof of the theorem follows from relation (7) and Lemmas 2.1 and 2.2. Indeed, Lemma 2.2 implies that

$$\sin^2 \frac{\lambda\tau}{2} \le \frac{\psi^2\left(\frac{\lambda}{2} + u_0\right)}{\psi^2\left(\frac{1}{\tau} + u_0\right)}.$$

Then relation (7) together with the latter inequality yield

$$\begin{aligned} \sqrt{\sigma}(\tau_1, \tau_2) &= \sqrt{g_H(|\tau_1 - \tau_2|)} \\ &\le \psi^{-1/2}\left(\frac{1}{|\tau_1 - \tau_2|} + u_0\right) \left(\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \psi^2 \left(\frac{\lambda}{2} + u_0\right) d\lambda\right)^{1/4}. \end{aligned}$$

Substituting the right-hand of the latter inequality in (8) we complete the proof of the theorem. □

**Corollary 2.1.** *Assume that* $H \in L_2(\mathbf{R})$ *and* $\sup_{\lambda\in\mathbf{R}} |f(\lambda)| < \infty$*, where* $f(t)$ *is the spectral density of the process* $X(t)$*, and let*

$$\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \lambda^{2\alpha}\, d\lambda < \infty, \qquad \alpha \in (0,1].$$

*Then*

$$\left(\mathsf{E}\,|\hat{Z}_T(\tau_1) - \hat{Z}_T(\tau_2)|^2\right)^{1/2} \le K_\alpha \cdot |\tau_1 - \tau_2|^{\alpha/2}, \qquad \alpha \in (0,1], \tag{14}$$

*where the constant* $K_\alpha$ *is given by*

$$K_\alpha = K_\alpha(T) = \frac{2^{2-\alpha/2}\sqrt{\pi}}{\sqrt{T}} \cdot \sup_{t\in\mathbf{R}} |f(t)| \cdot \|H^*\|_2^{1/2} \cdot \left(\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \lambda^{2\alpha}\, d\lambda\right)^{1/4}. \tag{15}$$

*Proof.* Corollary 2.1 follows from Theorem 2.1 and Example 2.1. □

**Corollary 2.2.** *Assume that* $H \in L_2(\mathbf{R})$ *and* $\sup_{\lambda\in\mathbf{R}} |f(\lambda)| < \infty$*, where* $f(t)$ *is the spectral density of the process* $X(t)$*. Let the integral*

$$\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \ln^{2\alpha}\left(\frac{\lambda}{2} + e^\alpha\right) d\lambda < \infty, \qquad \alpha > 0,$$

*converge. Then*

$$\left(\mathsf{E}\,|\hat{Z}_T(\tau_1) - \hat{Z}_T(\tau_2)|^2\right)^{1/2} \le K_{\ln} \cdot \ln^{-\alpha/2}\left(\frac{1}{|\tau_1 - \tau_2|} + e^\alpha\right), \qquad \tau_1, \tau_2 \in \mathbf{R},\ \alpha > 0, \tag{16}$$

*where the constant* $K_{\ln}$ *is given by*

$$K_{\ln} = K_{\ln}(T) = \frac{4\sqrt{\pi}}{\sqrt{T}} \cdot \sup_{t\in\mathbf{R}} |f(t)| \cdot \|H^*\|_2^{1/2} \cdot \left(\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \ln^{2\alpha}\left(\frac{\lambda}{2} + e^\alpha\right) d\lambda\right)^{1/4}. \tag{17}$$

*Proof.* Corollary 2.2 follows from Theorem 2.1 and Example 2.2. □

## 3. Square-Gaussian stochastic processes

In this section, we provide necessary definitions and recall some properties of square-Gaussian random variables and stochastic processes.

Let $(\Omega, F, \mathsf{P})$ be a probability space and $(T, \rho)$ be a compact metric space with respect to a metric $\rho$.

The definition below can also be found in the book [5].

**Definition 3.1** ([5])**.** Let $\Xi = \{\xi_t, t \in \mathbf{T}\}$ be a family of jointly Gaussian random variables such that $\mathsf{E}\,\xi_t = 0$. For example, $\xi_t$, $t \in \mathbf{T}$, could be a centered Gaussian stochastic process.

Let $SG_\Xi(\Omega)$ be the *space of square-Gaussian random variables*, that is, a collection of random variables whose elements $\eta \in SG_\Xi(\Omega)$ can either be represented as

$$\eta = \bar{\xi}^T A \bar{\xi} - \mathsf{E}\,\bar{\xi}^T A \bar{\xi}, \tag{18}$$

where $\bar{\xi}^T = (\xi_1, \xi_2, \dots, \xi_n)$, $\xi_k \in \Xi$, $k = 1, \dots, n$, and $A$ is a real matrix, or are the mean square limits of sequences of random variables of the form (18), that is,

$$\eta = \operatorname*{l.\,i.\,m.}_{n\to\infty} \left(\bar{\xi}_n^T A \bar{\xi}_n - \mathsf{E}\,\bar{\xi}_n^T A \bar{\xi}_n\right).$$

**Definition 3.2** ([5])**.** A stochastic process $\xi(t) = \{\xi(t), t \in \mathbf{T}\}$ is called *square-Gaussian* if, for every $t \in \mathbf{T}$, the random variable $\xi(t)$ belongs to the space $SG_\Xi(\Omega)$.

It is shown in [6] that

– $SG_\Xi(\Omega)$ is a Banach space with respect to the norm $\|\zeta\| = \sqrt{\mathsf{E}\,\zeta^2}$;

– $SG_\Xi(\Omega)$ is a subspace of the Orlicz space generated by the function

$$U(x) = \exp|x| - 1;$$

– the norm $\|\zeta\|_{L_U(\Omega)}$ is equivalent to the norm $\sqrt{\mathsf{E}\,\zeta^2}$ in $SG_\Xi(\Omega)$.

**Example 3.1.** Consider a family of Gaussian centered stochastic processes

$$\xi_1(t), \xi_2(t), \dots, \xi_n(t), \qquad t \in \mathbf{T}.$$

Let a matrix $A(t)$ be symmetric. Then

$$X(t) = \bar{\xi}^T(t)A(t)\bar{\xi}(t) - \mathsf{E}\,\bar{\xi}^T(t)A(t)\bar{\xi}(t)$$

is a square-Gaussian stochastic process, where $\bar{\xi}^T(t) = (\xi_1(t), \xi_2(t), \dots, \xi_n(t))$.

Some general results about and properties of square-Gaussian stochastic processes can be found in the papers [5, 6, 12].

The minimal number of closed balls of radius $u$ (with respect to a metric $\rho$) that cover the set $\mathbf{T}$ is denoted by $N(u)$.

Let $\xi(t) = \{\xi(t), t \in \mathbf{T}\}$ be a square-Gaussian stochastic process. Assume that there exists an increasing continuous function $\sigma(h)$, $h > 0$, such that $\sigma(h) \to 0$ as $h \to 0$ and

$$\sup_{\rho(t,s)\le h} \big(\operatorname{Var}(\xi(t) - \xi(s))\big)^{1/2} \le \sigma(h).$$

We introduce the following constants:

$$\varepsilon_0 = \inf_{t\in\mathbf{T}} \sup_{s\in\mathbf{T}} \rho(t,s), \qquad t_0 = \sigma(\varepsilon_0),$$

$$\gamma_0 = \sup_{t\in\mathbf{T}} (\operatorname{Var}\xi(t))^{1/2}.$$

**Theorem 3.1** ([5]). *Let $\xi(t) = \{\xi(t), t \in \boldsymbol{T}\}$ be a separable square-Gaussian stochastic process. Let an increasing function $r(u) \ge 0$, $u \ge 1$, be such that $r(u) \to \infty$ as $u \to \infty$ and let the function $r(\exp\{t\})$ be convex. If*

$$\int_0^{t_0} r\left(N\left(\sigma^{(-1)}(u)\right)\right) du < \infty,$$

*then*

$$\mathsf{P}\left\{\sup_{t\in\boldsymbol{T}} |\xi(t)| > x\right\} \le W(p,x,u) \tag{19}$$

*for all integers $M = 1, 2, \dots$, $0 < p < 1$, and $u$ such that*

$$0 < u < \frac{1-p}{\sqrt{2}} \min\left\{\frac{1}{\gamma_0}, \frac{1}{t_0 p^{M-1}}\right\}, \tag{20}$$

*where*

$$W(p,x,u) = 2\left(R\left(\frac{u\sqrt{2}\gamma_0}{1-p}\right)\right)^{1-p} A(p)\left(1 - \frac{p^{M-1}u\sqrt{2}t_0}{1-p}\right)^{-p/2} \times \exp\left\{-\frac{p^M u\sqrt{2}t_0}{2(1-p)} - ux\right\},$$

$R(s) = (1-|s|)^{-1/2}\exp\{-|s|/2\}$, *and*

$$A(p) = r^{(-1)}\left(\frac{1}{t_0 p^M}\int_0^{t_0 p^M} r\left(N\left(\sigma^{(-1)}(v)\right)\right) dv\right).$$

Put $C = \max\{t_0, \gamma_0\}$.

**Corollary 3.1.** *Let all the assumptions of Theorem* 3.1 *hold. If the integral*

$$\int_0^{t_0} r\left(N\left(\sigma^{(-1)}(u)\right)\right) du$$

*exists, then*

$$\begin{aligned}\mathsf{P}\left\{\sup_{t\in\mathbf{T}}|\xi(t)|>x\right\} \le 2\inf_{0<p<1}\Bigg\{ & r^{(-1)}\left(\frac{1}{t_0 p}\int_0^{t_0 p} r\left(N\left(\sigma^{(-1)}(\nu)\right)\right)d\nu\right) \\ & \times\left(1+\frac{\sqrt{2}x(1-p)}{C}\right)^{\frac12}\exp\left\{-\frac{x(1-p)}{\sqrt{2}\gamma_0}\right\}\Bigg\}\end{aligned} \tag{21}$$

*for all* $x>0$.

*Proof.* Corollary 3.1 follows from Theorem 3.1. Indeed, let $M=1$. Since the left-hand side of inequality (19) does not depend on $p$ and $u$, bound (19) is still valid after the right-hand side of (19) is minimized with respect to $p$ and $u$:

$$\mathsf{P}\left\{\sup_{t\in\mathbf{T}}|\xi(t)|>x\right\} \le \inf_{0<p<1}\ \inf_{u\in\left(0,\frac{1-p}{\sqrt{2}C}\right)} W(p,x,u). \tag{22}$$

It is obvious that $C\ge t_0$ and $C\ge\gamma_0$. Since the function

$$R(s)=(1-|s|)^{-1/2}\exp\{-|s|/2\}$$

increases for $0\le s\le 1$, we obtain

$$\begin{aligned}&\left(R\left(\frac{u\sqrt{2}\gamma_0}{1-p}\right)\right)^{1-p}\left(1-\frac{u\sqrt{2}t_0}{1-p}\right)^{-p/2}\exp\left\{-\frac{up\sqrt{2}t_0}{2(1-p)}-ux\right\}\\ &=\left[\left(R\left(\frac{u\sqrt{2}\gamma_0}{1-p}\right)\right)^{1-p}\left(R\left(\frac{u\sqrt{2}t_0}{1-p}\right)\right)^{p}\exp\{-ux\}\right]\\ &\le\left[\left(1-\frac{u\sqrt{2}C}{1-p}\right)^{-1/2}\exp\left\{-\frac{u\sqrt{2}C}{2(1-p)}\right\}\exp\{-ux\}\right].\end{aligned}$$

Minimizing the right-hand side of the latter inequality with respect to $u\in\left(0,\frac{1-p}{\sqrt{2}C}\right)$, we conclude that

$$\begin{aligned}&\inf_{u\in\left(0,\frac{1-p}{\sqrt{2}C}\right)}\left[\left(1-\frac{u\sqrt{2}C}{1-p}\right)^{-1/2}\exp\left\{-\frac{u\sqrt{2}C}{2(1-p)}\right\}\exp\{-ux\}\right]\\ &\le\left(\frac{C}{C+\sqrt{2}x(1-p)}\right)^{-1/2}\exp\left\{-\frac{x(1-p)}{\sqrt{2}\gamma_0}\right\}.\end{aligned}$$

Corollary 3.1 is proved. □

## 4. The rate of convergence of cross-correlograms in the space of continuous functions

This section is devoted to finding the rate of convergence of integral cross-correlogram estimators of unknown impulse response functions of linear systems in the space of continuous functions. More precisely, we find a bound for the distribution of the supremum of the estimation error in an interval $[0,A]$.

Assume that $X = (X(t), t \in \mathbb{R})$ is a measurable real-valued stationary centered Gaussian process perturbing system (1).

Consider the correlogram

$$\hat{H}_T(\tau) = \frac{1}{T}\int_0^T Y(t+\tau)X(t)\,dt, \qquad \tau > 0,$$

which is an estimator of the response function $H$. The stochastic process $Y(t)$ is defined by equality (1).

First we prove an auxiliary result.

**Lemma 4.1.** *The stochastic process* $\hat{Z}_T(\tau) = \hat{H}_T(\tau) - \mathsf{E}\,\hat{H}_T(\tau)$, $\tau > 0$, *is square-Gaussian.*

*Proof.* The process $\hat{Z}_T(\tau)$, $\tau > 0$, admits the representation

$$\hat{Z}_T(\tau) = \frac{1}{T}\int_0^T \left(Y(t+\tau)X(t) - \mathsf{E}\,Y(t+\tau)X(t)\right)dt. \tag{23}$$

Since all integral sums

$$\sum_k (Y(t_k+\tau)X(t_k) - \mathsf{E}\,Y(t_k+\tau)X(t_k)) \bigtriangleup t_k$$

for the integral (23) belong to the space $SG_\Xi(\Omega)$ and the process $\hat{Z}_T(\tau)$ itself is a mean square limit of these sums, $\hat{Z}_T(\tau)$ is a square-Gaussian process. Lemma 4.1 is proved. □

Consider the difference between the estimator $\hat{H}_T(\tau)$ and true impulse response function $H(\tau)$,

$$\hat{H}_T(\tau) - H(\tau), \qquad \tau > 0.$$

Any functional of the difference $\hat{H}_T(\tau) - H(\tau)$ is viewed as an index of the accuracy of estimation. We are going to estimate the supremum of the estimation error in an interval $[0, A]$, where $A$ is a fixed positive number. We have

$$\mathsf{P}\left\{\sup_{\tau\in[0,A]} |\hat{H}_T(\tau) - H(\tau)| \ge \varepsilon\right\}, \qquad \varepsilon > 0.$$

Let

$$h(\tau) = \mathsf{E}\,\hat{H}_T(\tau) - H(\tau), \qquad \tau \in [0, A].$$

Assume that the function $h(\tau)$ is bounded in the interval $[0, A]$.

*Remark* 4.1. The latter condition holds if, for example, the functions $\mathsf{E}\,\hat{H}_T(\tau)$ and $H(\tau)$ are continuous in the interval $[0, A]$.

Put

$$h_- = \min_{\tau\in[0,A]} h(\tau), \qquad h_+ = \max_{\tau\in[0,A]} h(\tau),$$
$$h^* = \max_{\tau\in[0,A]} |h(\tau)| = \max\{h_+, -h_-\}.$$

Relation (6) implies

$$\begin{aligned}\gamma_0 = \gamma_0(T) &= \sup_{\tau\in[0,A]} \left(\operatorname{Var}\hat{Z}_T(\tau)\right)^{1/2} \\ &= \sup_{\tau\in[0,A]} \Bigg(\frac{2\pi}{T}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(|H^*(\lambda_2)|^2 + e^{i\tau(\lambda_1+\lambda_2)}H^*(\lambda_1)H^*(\lambda_2)\right) \\ &\qquad\qquad \times \Phi_T(\lambda_2-\lambda_1) f(\lambda_1) f(\lambda_2)\,d\lambda_1\,d\lambda_2\Bigg)^{1/2}.\end{aligned}$$

Set

$$C = C(T) = \max\left\{\gamma_0, K_\alpha \left(\frac{A}{2}\right)^{\alpha/2}\right\},$$

where the constant $K_\alpha$ is defined by equality (15).

Let

$$M_\alpha = 2^{2-2/\alpha} e^{2/\alpha} \gamma_0^{-1/2-2/\alpha} \alpha^{2/\alpha-1/2}, \qquad \alpha \in (0,1].$$

**Theorem 4.1.** *Let $X = (X(t), t \in \mathbb{R})$ be a separable real-valued stationary Gaussian process perturbing system* (1). *Assume that $H \in L_2(\mathbf{R})$ and*

$$\int_{-\infty}^{\infty} |H^*(\lambda)|^2 \lambda^{2\alpha}\, d\lambda < \infty$$

*for some $\alpha \in (0,1]$.*

*Further, assume that the spectral density $f(t)$ of the stochastic process $X(t)$ is such that $\sup_{\lambda \in \mathbf{R}} |f(\lambda)| < \infty$. Then*

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{\tau \in [0,A]} |\hat{H}_T(\tau) - H(\tau)| > \varepsilon\right\} \\
&\le M_\alpha (\varepsilon - h^*)^{-\frac{1}{2}-\frac{2}{\alpha}} \left(C + \sqrt{2}\alpha(\varepsilon - h^*)^2 - 2\gamma_0(\varepsilon - h^*)\right)^{1/2} \\
&\quad \times \exp\left\{-\frac{\varepsilon - h^*}{\sqrt{2}\gamma_0} + \frac{1}{\alpha}\right\}
\end{aligned} \tag{24}$$

*for*

$$\varepsilon > \frac{\sqrt{2}\gamma_0}{\alpha} + h^*, \qquad \alpha \in (0,1].$$

*Proof.* The difference $\hat{H}_T(\tau) - H(\tau)$ admits the following representation:

$$\hat{H}_T(\tau) - H(\tau) = \hat{H}_T(\tau) - \mathsf{E}\,\hat{H}_T(\tau) + h(\tau) = \hat{Z}_T(\tau) + h(\tau).$$

Then

$$\begin{aligned}
\hat{H}_T(\tau) - H(\tau) \ge \varepsilon \quad &\Leftrightarrow \quad \hat{Z}_T(\tau) \ge \varepsilon - h(\tau), \\
\hat{H}_T(\tau) - H(\tau) \le -\varepsilon \quad &\Leftrightarrow \quad \hat{Z}_T(\tau) \le -\varepsilon - h(\tau).
\end{aligned}$$

Thus,

$$\left\{|\hat{H}_T(\tau) - H(\tau)| \ge \varepsilon\right\} \subset \left\{|\hat{Z}_T(\tau)| \ge \min\{\varepsilon - h_+, \varepsilon + h_-\}\right\}$$

and

$$\mathsf{P}\left\{\sup_{\tau \in [0,A]} |\hat{H}_T(\tau) - H(\tau)| \ge \varepsilon\right\} \le \mathsf{P}\left\{\sup_{\tau \in [0,A]} |\hat{Z}_T(\tau)| \ge \min\{\varepsilon - h_+, \varepsilon + h_-\}\right\} \tag{25}$$

for $\varepsilon > h^*$.

It is clear that $\min\{\varepsilon - h_+, \varepsilon + h_-\} = \varepsilon - h^*$. Denote

$$x = \varepsilon - h^*.$$

Lemma 4.1 implies that $\hat{Z}_T(\tau)$ is a square-Gaussian process and thus Corollary 3.1 can be applied. Then inequality (14) allows us to choose $K_\alpha \cdot h^{\alpha/2}$ as $\sigma(h)$, where the constant $K_\alpha$ is defined by equality (15). Therefore, $\sigma^{(-1)}(h) = (h/K_\alpha)^{2/\alpha}$.

The metric massiveness of the interval $[0, A]$ with the metric $\rho(t,s) = |t-s|$ is bounded as follows:

$$N(u) \le \frac{A}{2u} + 1.$$

Hence,

$$N(\sigma^{(-1)}(u)) \le \left(\frac{A}{2\sigma^{(-1)}(u)} + 1\right) = \left(\frac{A}{2}\left(\frac{K_\alpha}{u}\right)^{2/\alpha} + 1\right).$$

Consider the function $r(u) = u^\beta - 1$ with $\beta \in (0, \frac{\alpha}{2})$ and note that assumptions of Corollary 3.1 are valid for this function. Since $0 < p < 1$ and $t_0 = K_\alpha (A/2)^{\alpha/2}$, we get

$$\frac{1}{2} A (K_\alpha/(pt_0))^{2/\alpha} > 1.$$

This yields

$$N\left(\sigma^{(-1)}(u)\right) \le A\left(\frac{K_\alpha}{u}\right)^{2/\alpha}$$

if $0 < u < t_0 p$. The inverse function to $r(u)$ is equal to $r^{(-1)}(u) = (u+1)^{1/\beta}$, whence

$$\begin{aligned}
r^{(-1)}\left(\frac{1}{t_0 p}\int_0^{t_0 p} r\left(N\left(\sigma^{(-1)}(\nu)\right)\right) d\nu\right) &= \left(\frac{1}{t_0 p}\int_0^{t_0 p}\left[\left(\frac{A}{2}\left(\frac{K_\alpha}{u}\right)^{2/\alpha} + 1\right)^\beta\right] du\right)^{1/\beta} \\
&\le \left(\frac{1}{t_0 p}\int_0^{t_0 p}\left[A\left(\frac{K_\alpha}{u}\right)^{2/\alpha}\right]^\beta du\right)^{1/\beta} \\
&= 2\left(\frac{\alpha}{\alpha - 2\beta}\right)^{1/\beta} p^{-2/\alpha}.
\end{aligned}$$

Now we find the minimal value of the right-hand side of this inequality with respect to the parameter $\beta$:

$$\inf_{\beta \in (0, \frac{\alpha}{2})}\left(\frac{\alpha}{\alpha - 2\beta}\right)^{1/\beta} = \lim_{\beta \to 0}\left(\frac{1}{1 - 2\beta/\alpha}\right)^{1/\beta} = e^{2/\alpha}.$$

Inequality (21) together with the latter relation implies

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{\tau \in [0,A]} |\hat{Z}_T(\tau)| > x\right\} \\
&\le 4 e^{2/\alpha} C^{-1/2} \inf_{0<p<1}\left\{\frac{\sqrt{C + \sqrt{2}x(1-p)}}{p^{2/\alpha}} \exp\left\{-\frac{x(1-p)}{\sqrt{2}\gamma_0}\right\}\right\} \qquad (26)
\end{aligned}$$

for $x > 0$. Now we determine the point of minimum of the right-hand side of inequality (26). We have

$$p_{\min} = \frac{\sqrt{2}\gamma_0}{\alpha x}.$$

By assumptions of the theorem, $p \in (0,1)$ and one can use $p_{\min}$ on the right-hand side of inequality (26) if

$$x > \frac{\sqrt{2}\gamma_0}{\alpha}.$$

Then we get

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{\tau \in [0,A]} |\hat{Z}_T(\tau)| > x\right\} \qquad (27) \\
&\le M_\alpha x^{-1/2 - 2/\alpha}\sqrt{C + \sqrt{2}\alpha x^2 - 2\gamma_0 x}\, \exp\left\{-\frac{x}{\sqrt{2}\gamma_0} + \frac{1}{\alpha}\right\},
\end{aligned}$$

where the constant $M_\alpha$ is such that

$$M_\alpha = 2^{2-2/\alpha} e^{2/\alpha} \gamma_0^{-1/2-2/\alpha} \alpha^{2/\alpha-1/2}.$$

Taking into account $x = \varepsilon - h^*$, we complete the proof of the theorem in view of inequality (27). □

**Example 4.1.** We find a bound for $h^*$ in inequality (24) of Theorem 4.1 for a particular case.

Consider a family of spectral densities of the form (5) with $c = 2$. In other words, the functions $f_\Delta(\lambda)$ are given by

$$f_\Delta(\lambda) = \frac{1}{\pi} \exp\left\{-\frac{\lambda^2}{\Delta}\right\}, \qquad \lambda \in \mathbf{R},\ \Delta > 0. \tag{28}$$

Then the correlation function of the process $X$ equals

$$K_X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} f_\Delta(\lambda)\, d\lambda = \sqrt{\frac{\Delta}{\pi}} \exp\left\{-\frac{t^2\Delta}{4}\right\}.$$

Equality (4) yields

$$\mathsf{E}\,\hat{H}_T(\tau) = \int_0^\infty H(s) K_X(\tau - s)\, ds = \sqrt{\frac{\Delta}{\pi}} \int_0^\infty H(s) \exp\left\{-\frac{\Delta(s-\tau)^2}{4}\right\} ds.$$

Using the latter relation and

$$\sqrt{\frac{\Delta}{\pi}} \int_0^\infty \exp\left\{-\frac{\Delta(s-\tau)^2}{4}\right\} ds = 1$$

one can represent the function $h(\tau)$ as follows:

$$\begin{aligned} h(\tau) &= \mathsf{E}\,\hat{H}_T(\tau) - H(\tau) \\ &= \sqrt{\frac{\Delta}{\pi}} \int_0^\infty H(s) \exp\left\{-\frac{\Delta(s-\tau)^2}{4}\right\} ds \\ &\quad - H(\tau)\sqrt{\frac{\Delta}{\pi}} \int_0^\infty \exp\left\{-\frac{\Delta(s-\tau)^2}{4}\right\} ds \\ &= \sqrt{\frac{\Delta}{\pi}} \int_0^\infty (H(s) - H(\tau)) \exp\left\{-\frac{\Delta(s-\tau)^2}{4}\right\} ds. \end{aligned} \tag{29}$$

Assume that the impulse response function $H$ satisfies the Lipschitz condition of order $\kappa$ in the interval $[0, A]$. This means that there exists a constant $C_{\mathrm{Lip}} > 0$ such that

$$|H(s) - H(\tau)| \le C_{\mathrm{Lip}} |s - \tau|^\kappa, \qquad \kappa > 0,\ s, \tau \in [0, A]. \tag{30}$$

Then (29) and (30) yield

$$\begin{aligned} |h(\tau)| &\le \sqrt{\frac{\Delta}{\pi}} \int_0^\infty |H(s) - H(\tau)| \exp\left\{-\frac{\Delta(s-\tau)^2}{4}\right\} ds \\ &\le C_{\mathrm{Lip}} \sqrt{\frac{\Delta}{\pi}} \int_0^\infty |s-\tau|^\kappa \exp\left\{-\frac{\Delta(s-\tau)^2}{4}\right\} ds \\ &= \frac{C_{\mathrm{Lip}} 4^{\kappa/2}}{\Delta^{\kappa/2}\sqrt{\pi}} \Gamma\left(\frac{\kappa+1}{2}\right), \end{aligned} \tag{31}$$

where $\Gamma(\cdot)$ is the Gamma function.

Therefore, $h^*$ in Theorem 4.1 is such that

$$h^* = \frac{C_{\text{Lip}} 4^{\kappa/2}}{\Delta^{\kappa/2}\sqrt{\pi}} \Gamma\left(\frac{\kappa+1}{2}\right) = o\left(\Delta^{-\kappa/2}\right), \qquad \Delta \to \infty,$$

for the case under consideration.

## 5. Concluding remarks

The integral cross-correlogram estimator of the impulse response function of a linear homogeneous system is considered in this paper. A bound for the rate of convergence of this estimator is found in the space of continuous functions. The rate of convergence is expressed in terms of the distribution of the supremum of the error of estimation.

In a forthcoming paper, we plan to establish some bounds for the closeness of the true value $H(\tau)$ and mean value of the correlogram $\mathsf{E}\,\hat{H}_T(\tau)$ and to apply the corresponding bounds for real-valued processes. We also plan to construct statistical procedures to test the hypothesis concerning $h(\tau)$ in a finite interval.

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