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INTERPOLATION OF STATIONARY SEQUENCES OBSERVED WITH A NOISE

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Abstract. The problem of optimal linear estimation of the functional

$$A_s\xi = \sum_{l=0}^{s-1} \sum_{j=M_l}^{M_l+N_{l+1}} a(j)\xi(j), \qquad M_l = \sum_{k=0}^{l}(N_k + K_k), \qquad N_0 = K_0 = 0,$$

which depends on unknown values of a stochastic stationary sequence $\xi(j)$ with the help of observations of the sequence $\xi(j) + \eta(j)$ at points $j \in \mathbb{Z} \setminus S$, where $S = \bigcup_{l=0}^{s-1}\{M_l, \dots, M_l + N_{l+1}\}$, is considered under the assumption that the sequences $\{\xi(j)\}$ and $\{\eta(j)\}$ are mutually uncorrelated. Formulas for calculating the mean-square error and spectral characteristic of the optimal linear estimator of the functional are proposed under the condition of spectral certainty, where both spectral densities of the sequences $\xi(j)$ and $\eta(j)$ are known. The minimax (robust) method of estimation is applied in the case where the spectral densities of the sequences $\xi(j)$ and $\eta(j)$ are not known exactly, but the sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and minimax spectral characteristics are proposed for some special sets of admissible densities.

1. Introduction

The problem of estimation of unknown values of random sequences from known observations of the same sequences or from those of the sequences with a noise constitute an important part of the theory of random sequences and its applications. The setting of the problems of interpolation, extrapolation, and filtration for random sequences is due to A. N. Kolmogorov [15]. Some effective methods for solving these problems are proposed by N. Wiener [29] and A. M. Yaglom [30, 31] for the case of known spectral densities. The theory of stationary random sequences and stochastic processes is presented in the books by Yu. A. Rozanov [25] and E. Hannan [10].

In most applied problems, however, the spectral densities are unknown in practice. In such cases, one may find parametric or non-parametric estimators of the spectral density and then apply the classical theory of estimation. This approach, however, may lead to a substantial increase of the error as shown by K. S. Vastola and H. V. Poor [28].

If a family of admissible spectral densities is known, then one can search for estimators that are optimal for all members of the given set of admissible densities simultaneously. This is the essence of the so-called minimax approach. U. Grenander [9] is the first to use this approach for the problem of extrapolation of stationary processes. The paper by

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S. A. Kassam and H. V. Poor [14] contains a survey of results on minimax methods of data processing. Some models of spectral uncertainty are also described in [14]. J. Franke [5] and J. Franke and H. V. Poor [6] propose a solution to the problem of minimax extrapolation and interpolation for stationary sequences with the help of methods of convex optimization. In the papers by M. P. Moklyachuk [18, 19], the problems of extrapolation, interpolation, and filtration are studied for functionals that depend on unknown values of stationary sequences and processes. M. P. Moklyachuk and I. I. Dubovets'ka [8, 4] investigate the problems of estimation of functionals that depend on unknown values of periodically correlated processes. M. P. Moklyachuk and O. M. Masyutka [20] study the problems of interpolation of multi-dimensional sequences and processes. M. P. Moklyachuk and M. M. Luz [17] have been dealing with the estimation of random sequences with stationary increments. The problems of forecast for stationary processes with missing observations are considered by P. Bondon [1, 2] and M. Pourahmadi, A. Inoue, and Y. Kasahara [23]. Some results on the problem of interpolation of stationary sequences with missing observations are obtained by M. P. Moklyachuk and M. I. Sidei [21].

In the current paper, we study the problem of optimal linear estimation of the functional

$$A_s\xi = \sum_{l=0}^{s-1} \sum_{j=M_l}^{M_l+N_{l+1}} a(j)\xi(j) \tag{1}$$

with

$$M_l = \sum_{k=0}^{l} (N_k + K_k), \quad N_0 = K_0 = 0,$$

that depends on unknown values of a stationary sequence $\xi(j)$, $j \in \mathbb{Z}$, by using the observations of the sequence $\xi(j) + \eta(j)$ at points

$$j \in \mathbb{Z} \setminus S, \qquad S = \bigcup_{l=0}^{s-1} \{M_l, M_l + 1, \dots, M_l + N_{l+1}\},$$

where $\eta(j)$, $j \in \mathbb{Z}$, is a stationary sequence being uncorrelated with $\xi(j)$, $j \in \mathbb{Z}$.

The problem of optimal linear interpolation for stationary sequences is studied for the case of spectral certainty, that is, for the case where the spectral densities of both sequences $\xi(j)$ and $\eta(j)$ are known. If the complete information about the spectral densities is not available but a set of admissible spectral densities is given, we follow the so-called minimax estimation method. Given certain families of spectral densities, we determine the least favorable spectral densities and minimax spectral characteristics for the optimal linear estimator of the functional.

2. The classical projection method in Hilbert spaces

Let $\xi(j)$, $j \in \mathbb{Z}$, and $\eta(j)$, $j \in \mathbb{Z}$, be mutually uncorrelated stationary random sequences with zero means, $\mathsf{E}\,\xi(j) = 0$ and $\mathsf{E}\,\eta(j) = 0$. The covariance functions

$$R_\xi(k) = \mathsf{E}\,\xi(j+k)\overline{\xi(j)} \quad \text{and} \quad R_\eta(k) = \mathsf{E}\,\eta(j+k)\overline{\eta(j)}$$

of the stationary sequences $\xi(j)$, $j \in \mathbb{Z}$, and $\eta(j)$, $j \in \mathbb{Z}$, admit the spectral decompositions [7]

$$R_\xi(k) = \int_{-\pi}^{\pi} e^{ik\lambda}\, F(d\lambda), \qquad R_\eta(k) = \int_{-\pi}^{\pi} e^{ik\lambda}\, G(d\lambda),$$

where $F(d\lambda)$ and $G(d\lambda)$ are spectral measures of the sequences $\{\xi(j)\}$ and $\{\eta(j)\}$, respectively.

We study stationary sequences with absolutely continuous spectral measures $F(d\lambda)$ and $G(d\lambda)$ whose covariance functions are given by

$$R_\xi(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)\, d\lambda, \qquad R_\eta(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\lambda} g(\lambda)\, d\lambda,$$

where $f(\lambda)$ and $g(\lambda)$ are spectral densities of the sequences $\xi(j)$, $j \in \mathbb{Z}$, and $\eta(j)$, $j \in \mathbb{Z}$, respectively. We assume that the functions $f(\lambda)$ and $g(\lambda)$ satisfy the following condition of minimality

$$\int_{-\pi}^{\pi} \frac{1}{f(\lambda)+g(\lambda)}\, d\lambda < \infty. \tag{2}$$

This condition is needed to exclude the case where an error free interpolation of unknown values of the sequences is possible; see [25].

The sequences $\xi(j)$ and $\eta(j)$ admit the spectral decompositions [12] (also see [7])

$$\xi(j) = \int_{-\pi}^{\pi} e^{ij\lambda}\, Z_\xi(d\lambda), \qquad \eta(j) = \int_{-\pi}^{\pi} e^{ij\lambda}\, Z_\eta(d\lambda),$$

where $Z_\xi(d\lambda)$ and $Z_\eta(\lambda)$ are orthogonal random measures in the interval $[-\pi,\pi)$ that are subordinated to the spectral measures $F(d\lambda)$ and $G(d\lambda)$. The measures $Z_\xi(d\lambda)$ and $Z_\eta(\lambda)$ are such that

$$\mathsf{E}\, Z_\xi(\Delta_1)\overline{Z_\xi(\Delta_2)} = F(\Delta_1\cap\Delta_2) = \frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} f(\lambda)\, d\lambda,$$
$$\mathsf{E}\, Z_\eta(\Delta_1)\overline{Z_\eta}(\Delta_2) = G(\Delta_1\cap\Delta_2) = \frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} g(\lambda)\, d\lambda.$$

Consider the problem of the optimal mean square linear estimation of the following functional,

$$A_s\xi = \sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} a(j)\xi(j), \qquad M_l = \sum_{k=0}^{l}(N_k+K_k), \qquad N_0 = K_0 = 0,$$

that depends on unknown values of the stationary sequence $\xi(j)$ with the help of known observations of the sequence $\xi(j)+\eta(j)$ at moments $j \in \mathbb{Z}\setminus S$, where

$$S = \bigcup_{l=0}^{s-1}\{M_l, M_l+1, \dots, M_l+N_{l+1}\}.$$

The functional $A_s\xi$ can be rewritten as

$$A_s\xi = \int_{-\pi}^{\pi} A_s\left(e^{i\lambda}\right)\, Z_\xi(d\lambda),$$

where

$$A_s(e^{i\lambda}) = \sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} a(j)e^{ij\lambda}.$$

Denote by $\hat{A}_s\xi$ the optimal linear estimator of the functional $A_s\xi$ constructed from known observations of the sequence $\xi(j)+\eta(j)$. Let $\Delta(f,g) = \mathsf{E}\left|A_s\xi - \hat{A}_s\xi\right|^2$ be the mean square error of the estimator $\hat{A}_s\xi$. To find the estimator $\hat{A}_s\xi$, we follow the Kolmogorov orthogonal projection method in a Hilbert space.

We treat the random variables $\xi(j)$ and $\eta(j)$ as elements of the Hilbert space $H = L_2(\Omega,\mathcal{F},\mathsf{P})$ of random variables ξ that have zero mean, $\mathsf{E}\,\xi = 0$, and finite second moment

$\mathsf{E}\,|\xi|^2 < \infty$. The space H is equipped with the scalar product $(\xi, \eta) = \mathsf{E}\,\xi\overline{\eta}$. Denote by $H^0(\xi + \eta)$ the closed linear subspace generated by the random variables

$$\{\xi(j) + \eta(j) \colon j \in \mathbb{Z} \setminus S\}$$

in the Hilbert space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$. By $L_2(f + g)$, we denote the Hilbert space of complex valued functions that are defined in the interval $[-\pi, \pi]$ and are square integrable with respect to the measure whose density is $f(\lambda) + g(\lambda)$. Finally, consider the subspace $L_2^s(f + g)$ of the space $L_2(f + g)$ generated by the functions $\{e^{ij\lambda}, j \in \mathbb{Z} \setminus S\}$.

We look for an estimator $\hat{A}_s\xi$ of the functional $A_s\xi$ represented in the following form,

$$\hat{A}_s\xi = \int_{-\pi}^{\pi} h\left(e^{i\lambda}\right) \left(Z_\xi(d\lambda) + Z_\eta(d\lambda)\right),$$

where $h(e^{i\lambda}) \in L_2^s(f)$ is the spectral characteristic of the estimator.

The mean square error $\Delta(h; f)$ of the estimator $\hat{A}_s\xi$ is given by

$$\begin{aligned}\Delta(h; f, g) &= \mathsf{E}\left|A_s\xi - \hat{A}_s\xi\right|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|A_s\left(e^{i\lambda}\right) - h\left(e^{i\lambda}\right)\right|^2 f(\lambda)\,d\lambda + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|h(e^{i\lambda})\right|^2 g(\lambda)\,d\lambda.\end{aligned}$$

According to Kolmogorov's orthogonal projection method in a Hilbert space, the projection of the element $A_s\xi$ onto the subspace $H^0(\xi + \eta)$ is the optimal estimator of the functional $A_s\xi$. The projection is determined from the following set of two conditions:

1) $\hat{A}_s\xi \in H^0(\xi + \eta)$,
2) $A_s\xi - \hat{A}_s\xi \perp H^0(\xi + \eta)$.

Condition 2) implies that the spectral characteristic $h(\lambda)$ satisfies the following condition: For all $j \in \mathbb{Z} \setminus S$,

$$\begin{aligned}&\mathsf{E}\left(A_s\xi - \hat{A}_s\xi\right)\left(\overline{\xi(j)} + \overline{\eta(j)}\right) \\ &\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(A_s\left(e^{i\lambda}\right) - h\left(e^{i\lambda}\right)\right) e^{-ij\lambda} f(\lambda)\,d\lambda - \frac{1}{2\pi}\int_{-\pi}^{\pi} h\left(e^{i\lambda}\right) e^{-ij\lambda} g(\lambda)\,d\lambda = 0.\end{aligned}$$

This means

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left[A_s\left(e^{i\lambda}\right) f(\lambda) - h\left(e^{i\lambda}\right)\left(f(\lambda) + g(\lambda)\right)\right] e^{-ij\lambda}\,d\lambda = 0, \qquad j \in \mathbb{Z} \setminus S.$$

Therefore, the function $\left[A_s(e^{i\lambda})f(\lambda) - h(e^{i\lambda})(f(\lambda) + g(\lambda))\right]$ can be represented in the following form,

$$A_s\left(e^{i\lambda}\right) f(\lambda) - h\left(e^{i\lambda}\right)\left(f(\lambda) + g(\lambda)\right) = C_s\left(e^{i\lambda}\right),$$

$$C_s\left(e^{i\lambda}\right) = \sum_{l=0}^{s-1} \sum_{k=M_l}^{M_l+N_{l+1}} c(k) e^{ik\lambda},$$

where $c(k)$, $k \in S$, are unknown coefficients to be found.

The latter relation yields the following form for the spectral characteristic of the estimator $\hat{A}_s\xi$:

$$h(e^{i\lambda}) = A_s\left(e^{i\lambda}\right) \frac{f(\lambda)}{f(\lambda) + g(\lambda)} - \frac{C_s\left(e^{i\lambda}\right)}{f(\lambda) + g(\lambda)}. \tag{3}$$

Condition 1) above for the optimal estimator of the functional $A_s\xi$, $\hat{A}_s\xi \in H^0(\xi + \eta)$, yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h\left(e^{i\lambda}\right) e^{-ij\lambda}\,d\lambda = 0, \qquad j \in S,$$

that is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A_s\left(e^{i\lambda}\right)\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{C_s\left(e^{i\lambda}\right)}{f(\lambda)+g(\lambda)}\right)e^{-ij\lambda}\,d\lambda=0,\qquad j\in S.$$

The latter equality is used to determine the unknown coefficients $c(k)$, $k\in S$.

Removing the parentheses we obtain

$$\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}a(k)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i(k-j)\lambda}f(\lambda)}{f(\lambda)+g(\lambda)}\,d\lambda-\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}c(k)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i(k-j)\lambda}}{f(\lambda)+g(\lambda)}\,d\lambda=0,\qquad j\in S. \tag{4}$$

Let

$$R^s_{j,k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(j-k)\lambda}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}\,d\lambda,$$
$$B^s_{j,k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(j-k)\lambda}\frac{1}{f(\lambda)+g(\lambda)}\,d\lambda,$$
$$Q^s_{j,k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(j-k)\lambda}\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)}\,d\lambda.$$

Using this notation, relation (4) can be rewritten in the form of a system of equations:

$$\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}R^s_{j,k}a(k)=\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}B^s_{j,k}c(k),\qquad j\in S.$$

An equivalent vector form of the latter system is given by

$$\mathbf{R}_s\vec{\mathbf{a}}_s=\mathbf{B}_s\vec{\mathbf{c}}_s,$$

where $\vec{\mathbf{a}}_s$ is the vector constructed from coefficients of the functional $A_s\xi$ and $\vec{\mathbf{c}}_s$ is the vector constituted from coefficients $c(k)$, $k\in S$. Put $q=N_1+N_2+\cdots+N_s+s$ and let $\mathbf{B}_N$ and $\mathbf{R}_N$ be $q\times q$ matrices with entries $\mathbf{B}_s(j,k)=B^s_{j,k}$ and $\mathbf{R}_s(j,k)=R^s_{j,k}$, respectively, $j,k\in S$. Therefore, the unknown coefficients $c(k)$, $k\in S$, are evaluated in terms of the matrices $\mathbf{B}_N$ and $\mathbf{R}_N$ as follows:

$$c(k)=\left(\mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s\right)_k,$$

where $(\mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s)_k$ is the coordinate k of the vector $\mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s$. Hence, the spectral characteristic $h(e^{i\lambda})$ of the estimator $\hat{A}_s\xi$ is established from the following equality:

$$h\left(e^{i\lambda}\right)=A_s\left(e^{i\lambda}\right)\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}\left(\mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s\right)_k e^{ik\lambda}}{f(\lambda)+g(\lambda)}. \tag{5}$$

Then the mean square error of the estimator $\hat{A}_s\xi$ is given by

$$\begin{aligned}\Delta(h;f,g)&=\mathsf{E}\left|A_s\xi-\hat{A}_s\xi\right|^2\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_s\left(e^{i\lambda}\right)g(\lambda)+\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}(\mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s)_k e^{ik\lambda}\right|^2}{(f(\lambda)+g(\lambda))^2}f(\lambda)\,d\lambda\\&\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_s\left(e^{i\lambda}\right)f(\lambda)-\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}(\mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s)_k e^{ik\lambda}\right|^2}{(f(\lambda)+g(\lambda))^2}g(\lambda)\,d\lambda\\&=\langle\mathbf{R}_s\vec{\mathbf{a}}_s,\mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s\rangle+\langle\mathbf{Q}_s\vec{\mathbf{a}}_s,\vec{\mathbf{a}}_s\rangle,\end{aligned} \tag{6}$$

where $\mathbf{Q}_s$ is the $q\times q$ matrix with entries $\mathbf{Q}_s(j,k)=Q^s_{j,k}$, $j,k\in S$.

Combining the above reasoning we prove the following result.

Theorem 2.1. *Let $\xi(j)$ and $\eta(j)$ be mutually uncorrelated stationary random sequences. Assume that their spectral densities $f(\lambda)$ and $g(\lambda)$ satisfy the condition of minimality* (2).

Let a functional $A_s\xi$ of the form (1) *depend on unknown values of the sequence $\xi(j)$ and let the optimal linear estimator of $A_s\xi$ be constructed from observations of the sequence $\xi(j)+\eta(j)$, $j\in\mathbb{Z}\setminus S$. Then the spectral characteristic $h(e^{i\lambda})$ and mean square error $\Delta(f,g)$ of the estimation are evaluated by equalities* (5) *and* (6)*, respectively.*

Corollary 2.1. *Let $\xi(j)$ be a stationary sequence whose spectral density $f(\lambda)$ is such that the function $f^{-1}(\lambda)$ is integrable. Let the functional $A_s\xi$ depend on unknown values of the sequence $\xi(j)$. Then the spectral characteristic $h(e^{i\lambda})$ and mean square error $\Delta(f)$ of the optimal linear estimator constructed from observations of the sequence $\xi(j)$, $j\in\mathbb{Z}\setminus S$, where $S=\bigcup_{l=0}^{s-1}\{M_l,\dots,,M_l+N_{l+1}\}$, are evaluated from*

$$h\left(e^{i\lambda}\right)=A_s\left(e^{i\lambda}\right)-C_s\left(e^{i\lambda}\right)f^{-1}(\lambda), \tag{7}$$

$$\Delta(h;f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|C_s\left(e^{i\lambda}\right)\right|^2 f^{-1}(\lambda)\,d\lambda=\left\langle B_s^{-1}\vec{\mathbf{a}}_s,\vec{\mathbf{a}}_s\right\rangle, \tag{8}$$

where

$$C_s\left(e^{i\lambda}\right)=\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\left(B_s^{-1}\vec{\mathbf{a}}_s\right)_j e^{ij\lambda},$$

$q=N_1+N_2+\dots+N_s+s$, *and*

$$B_s=\begin{pmatrix} B_{11} & B_{12} & \dots & B_{1s}\\ B_{21} & B_{22} & \dots & B_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ B_{s1} & B_{s2} & \dots & B_{ss}\end{pmatrix}.$$

Here, B_{mn} are $(N_m+1)\times(N_n+1)$ matrices formed by the Fourier coefficients of the function $f^{-1}(\lambda)$:

$$B_{mn}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{-1}(\lambda)e^{-i(k-j)\lambda}\,d\lambda=r_{k-j},$$
$$k=M_{m-1},\dots,M_{m-1}+N_m,\qquad j=M_{n-1},\dots,M_{n-1}+N_n,\qquad m,n=1,\dots,s.$$

Example 2.1. Consider two uncorrelated sequences

$$\{\xi(j)\colon j\in\mathbb{Z}\}\quad\text{and}\quad\{\eta(j)\colon j\in\mathbb{Z}\}$$

whose spectral densities are

$$f(\lambda)=\left|1-ae^{-i\lambda}\right|^2,\qquad |a|<1,$$
$$g(\lambda)=\left|1-be^{-i\lambda}\right|^2,\qquad |b|<1,$$

respectively. We are going to find the mean square optimal linear estimator of the functional $A_2\xi=\xi(0)+\xi(3)$ depending on unknown values $\xi(0)$ and $\xi(3)$. We construct the estimator from observations of the sequence $\xi(j)+\eta(j)$ at points $j\in\mathbb{Z}\setminus\{0,3\}$. The spectral density of the sequence $\xi(j)+\eta(j)$ is given by

$$f(\lambda)+g(\lambda)=\left|1-ae^{-i\lambda}\right|^2+\left|1-be^{-i\lambda}\right|^2=\left|x-ye^{-i\lambda}\right|^2,$$
$$x=\frac{1}{2}\left(\pm\sqrt{(1+a)^2+(1+b)^2}\pm\sqrt{(1-a)^2+(1-b)^2}\right),$$
$$y=\frac{a+b}{x}.$$

Since $|a| < 1$ and $|b| < 1$, we have $|y/x| < 1$. We apply the expansion of the function $\frac{1}{1-t}$ in a power series in order to obtain the factorizations of the functions $(f(\lambda)+g(\lambda))^{-1}$, $f(\lambda)/(f(\lambda)+g(\lambda))$, and $f(\lambda)g(\lambda)/(f(\lambda)+g(\lambda))$:

$$\frac{1}{f(\lambda)+g(\lambda)} = \frac{1}{|x-ye^{-i\lambda}|^2} = \left|\sum_{k=0}^{\infty} \frac{y^k}{x^{k+1}} e^{-ik\lambda}\right|^2,$$

$$\frac{f(\lambda)}{f(\lambda)+g(\lambda)} = \frac{\left|1-ae^{-i\lambda}\right|^2}{|x-ye^{-i\lambda}|^2} = \left|\sum_{k=0}^{\infty} \frac{y^k}{x^{k+1}} e^{-ik\lambda} - \sum_{k=0}^{\infty} \frac{ay^k}{x^{k+1}} e^{-i(k+1)\lambda}\right|^2$$

$$= \left|\frac{1}{x} + \sum_{k=0}^{\infty} \left(\frac{y^{k+1}}{x^{k+2}} - \frac{ay^k}{x^{k+1}}\right) e^{-i(k+1)\lambda}\right|^2,$$

$$\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)} = \frac{\left|1-ae^{-i\lambda}\right|^2 \cdot \left|1-be^{-i\lambda}\right|^2}{|x-ye^{-i\lambda}|^2} = \frac{\left|1-(a+b)e^{-i\lambda}+abe^{-i2\lambda}\right|^2}{|x-ye^{-i\lambda}|^2}$$

$$= \left|\sum_{k=0}^{\infty} \frac{y^k}{x^{k+1}} e^{-ik\lambda} - \sum_{k=0}^{\infty} \frac{(a+b)y^k}{x^{k+1}} e^{-i(k+1)\lambda} + \sum_{k=0}^{\infty} \frac{aby^k}{x^{k+1}} e^{-i(k+2)\lambda}\right|^2$$

$$= \left|\frac{1}{x} + \frac{y-ax-bx}{x^2} e^{-i\lambda} + \sum_{k=0}^{\infty} \left(\frac{y^{k+2}}{x^{k+3}} - \frac{(a+b)y^{k+1}}{x^{k+2}} + \frac{aby^k}{x^{k+1}}\right) e^{-i(k+2)\lambda}\right|^2.$$

The spectral characteristic $h(f,g)$ of the estimator $\hat{A}_2\xi$ is calculated by using formula (5). The spectral characteristic is given by

$$h(i\lambda) = (1+e^{i3\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)} - \frac{\left(\mathbf{B}_2^{-1}\mathbf{R}_2\vec{\mathbf{a}}_2\right)_0 + \left(\mathbf{B}_2^{-1}\mathbf{R}_2\vec{\mathbf{a}}_2\right)_3 \cdot e^{i3\lambda}}{f(\lambda)+g(\lambda)}.$$

The matrices $\mathbf{B}_2$, $\mathbf{R}_2$, and $\mathbf{Q}_2$ are such that

$$\mathbf{B}_2 = \begin{pmatrix} b_0 & b_{-3} \\ b_3 & b_0 \end{pmatrix},$$

$$\mathbf{R}_2 = \begin{pmatrix} r_0 & r_{-3} \\ r_3 & r_0 \end{pmatrix},$$

$$\mathbf{Q}_2 = \begin{pmatrix} q_0 & q_{-3} \\ q_3 & q_0 \end{pmatrix},$$

where b_j, r_j, and q_j, $j \in \{-3, 0, 3\}$, are Fourier coefficients of the functions

$$(f(\lambda)+g(\lambda))^{-1}, \qquad f(\lambda)/(f(\lambda)+g(\lambda)), \quad \text{and} \quad f(\lambda)g(\lambda)/(f(\lambda)+g(\lambda)),$$

respectively. The vector that generates the functional $A_2\xi$ is $\mathbf{a}_2 = (a(0), a(3)) = (1,1)$. Then the unknown coefficients in equality (5) are

$$\left(\mathbf{B}_2^{-1}\mathbf{R}_2\vec{\mathbf{a}}_2\right)_0 = \frac{b_0r_0 - b_{-3}r_3 + b_0r_{-3} - b_{-3}r_0}{b_0^2 - b_{-3}b_3},$$

$$\left(\mathbf{B}_2^{-1}\mathbf{R}_2\vec{\mathbf{a}}_2\right)_3 = \frac{-b_3r_0 + b_0r_3 - b_3r_{-3} + b_0r_0}{b_0^2 - b_{-3}b_3},$$

where

$$b_0 = \sum_{k=0}^{\infty} \left|\frac{y^k}{x^{k+1}}\right|^2,$$

$$b_3 = \overline{b_{-3}} = \sum_{k=0}^{\infty} \left(\frac{y^k}{x^{k+1}}\right) \overline{\left(\frac{y^{k+3}}{x^{k+4}}\right)},$$

$$r_0 = \frac{1}{|x|^2} + \sum_{k=0}^{\infty} \left| \frac{y^{k+1}}{x^{k+2}} - \frac{ay^k}{x^{k+1}} \right|^2,$$

$$r_3 = \overline{r_{-3}} = \frac{1}{x} \overline{\left(\frac{y^3}{x^4} - \frac{ay^2}{x^3} \right)} + \sum_{k=0}^{\infty} \left(\frac{y^{k+1}}{x^{k+2}} - \frac{ay^k}{x^{k+1}} \right) \overline{\left(\frac{y^{k+4}}{x^{k+5}} - \frac{ay^{k+3}}{x^{k+4}} \right)}.$$

The mean square error of the estimator $\hat{A}_2\xi$ is provided by formula (6), that is,

$$\begin{aligned}
\Delta(h; f) &= \mathsf{E} \left| A_2\xi - \hat{A}_2\xi \right|^2 = \left\langle \mathbf{R}_2\vec{\mathbf{a}}_2, \mathbf{B}_2^{-1}\mathbf{R}_2\vec{\mathbf{a}}_2 \right\rangle + \langle \mathbf{Q}_2\vec{\mathbf{a}}_2, \vec{\mathbf{a}}_2 \rangle \\
&= \frac{1}{b_0^2 - b_{-3}b_3} \Big(2b_0r_0^2 - b_{-3}r_0r_3 + 2b_0r_0r_{-3} - b_{-3}r_0^2 - b_{-3}r_{-3}r_3 + b_0r_{-3}^2 \\
&\qquad - b_{-3}r_0r_{-3} - b_3r_0r_3 + b_0r_3^2 - b_3r_3r_{-3} + 2b_0r_0r_3 \\
&\qquad\qquad - b_3r_0^2 - b_3r_0r_{-3} \Big) \\
&\quad + 2q_0 + q_3 + q_{-3},
\end{aligned}$$

where

$$q_0 = \left| \frac{1}{x} \right|^2 + \left| \frac{y - ax - bx}{x^2} \right|^2 + \sum_{k=0}^{\infty} \left| \frac{y^{k+2}}{x^{k+3}} - \frac{(a+b)y^{k+1}}{x^{k+2}} + \frac{aby^k}{x^{k+1}} \right|^2,$$

$$\begin{aligned}
q_3 &= \overline{q_{-3}} \\
&= \frac{1}{x} \overline{\left(\frac{y^3}{x^4} - \frac{(a+b)y^2}{x^3} + \frac{aby}{x^2} \right)} + \frac{y - ax - bx}{x^2} \overline{\left(\frac{y^4}{x^5} - \frac{(a+b)y^3}{x^4} + \frac{aby^2}{x^3} \right)} \\
&\quad + \sum_{k=0}^{\infty} \left(\frac{y^{k+2}}{x^{k+3}} - \frac{(a+b)y^{k+1}}{x^{k+2}} + \frac{aby^k}{x^{k+1}} \right) \overline{\left(\frac{y^{k+5}}{x^{k+6}} - \frac{(a+b)y^{k+4}}{x^{k+5}} + \frac{aby^{k+3}}{x^{k+4}} \right)}.
\end{aligned}$$

3. Minimax interpolation

The method of projection in Hilbert spaces described in Section 2 could be applied in the case of known spectral densities $f(\lambda)$ and $g(\lambda)$. However, the information on spectral densities is not always available in practice. On the other hand, if $f(\lambda)$ and $g(\lambda)$ are known to belong to some class of spectral densities $D = D_f \times D_g$, then one can apply the minimax approach to the problems of estimation. Following this approach, we search for an estimator that minimizes the error for all spectral densities of a given class $D = D_f \times D_g$ simultaneously.

Definition 3.1. Let a family $D = D_f \times D_g$ of spectral densities be given. Spectral densities $f_0(\lambda) \in D_f$ and $g_0(\lambda) \in D_g$ are called the least favorable in D for the optimal linear interpolation of a functional $A_s\xi$ if

$$\Delta(f_0, g_0) = \Delta(h(f_0, g_0); f_0, g_0) = \max_{(f,g) \in D_f \times D_g} \Delta(h(f, g); f, g).$$

Definition 3.2. Let a family $D = D_f \times D_g$ of spectral densities be given. A spectral characteristic $h^0(e^{i\lambda})$ of the optimal estimator of a functional $A_s\xi$ is called minimax (robust) if

$$h^0\left(e^{i\lambda}\right) \in H_D = \bigcap_{(f,g) \in D_f \times D_g} L_2^s(f + g),$$

$$\min_{h \in H_D} \max_{(f,g) \in D} \Delta(h; f, g) = \sup_{(f,g) \in D} \Delta\left(h^0; f, g\right).$$

Lemma 3.1. *Spectral densities $f_0(\lambda)$ and $g_0(\lambda)$ which satisfy the condition of minimality are the least favorable in a class $D = D_f \times D_g$ for the optimal linear interpolation of a functional $A_s\xi$ if the Fourier coefficients of the function*

$$(f_0(\lambda) + g_0(\lambda))^{-1}, \qquad f_0(\lambda)(f_0(\lambda) + g_0(\lambda))^{-1}, \qquad f_0(\lambda)g_0(\lambda)(f_0(\lambda) + g_0(\lambda))^{-1}$$

generate the operators $\mathbf{B}_s^0$, $\mathbf{R}_s^0$, and $\mathbf{Q}_s^0$ that determine a solution of the extremum problem

$$\max_{(f,g)\in D_f\times D_g} \left\langle \mathbf{R}_s\vec{\mathbf{a}}_s, \mathbf{B}_s^{-1}\mathbf{R}_s\vec{\mathbf{a}}_s \right\rangle + \left\langle \mathbf{Q}_s\vec{\mathbf{a}}_s, \vec{\mathbf{a}}_s \right\rangle = \left\langle \mathbf{R}_s^0\vec{\mathbf{a}}_s, \left(\mathbf{B}_s^0\right)^{-1}\mathbf{R}_s^0\vec{\mathbf{a}}_s \right\rangle + \left\langle \mathbf{Q}_s^0\vec{\mathbf{a}}_s, \vec{\mathbf{a}}_s \right\rangle. \tag{9}$$

The minimax spectral characteristic $h^0 = h(f_0, g_0)$ is evaluated according to equality (5) *if $h(f_0, g_0) \in H_D$.*

Lemma 3.1 follows explicitly from Definitions 3.1 and 3.2 of the least favorable densities and minimax spectral characteristic combined together with the results proved in Section 2.

The least favorable spectral densities $f_0(\lambda)$, $g_0(\lambda)$ and minimax spectral characteristic $h^0 = h(f_0, g_0)$ form a saddle point of the function $\Delta(h; f, g)$ in the set $H_D \times D$. The saddle point inequalities

$$\Delta(h; f_0, g_0) \geq \Delta\left(h^0; f_0, g_0\right) \geq \Delta\left(h^0; f, g\right),$$
$$\forall h \in H_D, \qquad \forall f \in D_f, \qquad \forall g \in D_g$$

are satisfied if $h^0 = h(f_0, g_0)$ and $h(f_0, g_0) \in H_D$, where (f_0, g_0) is a solution of the conditional extremum problem

$$\sup_{(f,g)\in D_f\times D_g} \Delta(h(f_0, g_0); f, g) = \Delta(h(f_0, g_0); f_0, g_0), \tag{10}$$

$$\Delta(h(f_0, g_0); f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|A_s\left(e^{i\lambda}\right)g(\lambda) + C_s^0\left(e^{i\lambda}\right)\right|^2}{(f(\lambda) + g(\lambda))^2} f(\lambda)\, d\lambda$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|A_s\left(e^{i\lambda}\right)f(\lambda) - C_s^0\left(e^{i\lambda}\right)\right|^2}{(f(\lambda) + g(\lambda))^2} g(\lambda)\, d\lambda,$$

$$C_s^0\left(e^{i\lambda}\right) = \sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} \left(\left(\mathbf{B}_s^0\right)^{-1}\mathbf{R}_s^0\vec{\mathbf{a}}_s\right)_j e^{ij\lambda}.$$

The conditional extremum problem (10) is equivalent to the following unconditional extremum problem,

$$\Delta_D(f, g) = -\Delta(h(f_0, g_0); f, g) + \delta(f, g \mid D_f \times D_g) \to \inf, \tag{11}$$

where $\delta(f, g \mid D_f \times D_g)$ is the indicator of the set $D = D_f \times D_g$ (see [24]). A solution of problem (11) is determined by the condition $0 \in \partial\Delta_D(f_0, g_0)$, where $\partial\Delta_D(f_0)$ is the sub-differential of the convex functional $\Delta_D(f, g)$ at the point (f_0, g_0).

The form of the functional $\Delta(h(f_0, g_0); f, g)$ allows one to calculate its derivatives and differentials in the space $L_1 \times L_1$. Thus, the complexity of problems (11) is determined by the complexity of calculations of sub-differentials of indicator functions $\delta(f, g \mid D_f \times D_g)$ for the sets $D_f \times D_g$.

Lemma 3.2. *Let (f_0, g_0) be a solution of extremum problem* (11)*. Then a pair $f_0(\lambda)$ and $g_0(\lambda)$ are the least favorable spectral densities in a class $D = D_f \times D_g$ and a spectral characteristic $h^0 = h(f_0, g_0)$ is minimax for the optimal estimation of the functional $A_s\xi$ if $h(f_0, g_0) \in H_D$.*

4. Least favorable spectral densities in the class $D_f^0 \times D_g^0$

Consider the problem of minimax estimation of the functional $A_s\xi$ depending on unknown values of a sequence $\xi(j)$ by using the observations of the sequence $\xi(j)+\eta(j)$ at points $j \in \mathbb{Z} \setminus S$ for the case where the spectral densities $f(\lambda)$ and $g(\lambda)$ belong to a set $D = D_f^0 \times D_g^0$ of admissible spectral densities where

$$D_f^0 = \left\{ f(\lambda) \,\middle|\, \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\,d\lambda \le P_1 \right\}, \qquad D_g^0 = \left\{ g(\lambda) \,\middle|\, \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)\,d\lambda \le P_2 \right\}.$$

Let $f_0(\lambda) \in D_f^0$, $g_0(\lambda) \in D_g^0$ and let the functions

$$h_f(f_0,g_0) = \frac{\left|A_s\left(e^{i\lambda}\right)g_0(\lambda) + C_s^0\left(e^{i\lambda}\right)\right|^2}{(f_0(\lambda)+g_0(\lambda))^2}, \tag{12}$$

$$h_g(f_0,g_0) = \frac{\left|A_s\left(e^{i\lambda}\right)f_0(\lambda) - C_s^0\left(e^{i\lambda}\right)\right|^2}{(f_0(\lambda)+g_0(\lambda))^2} \tag{13}$$

be bounded. In this case,

$$\Delta(h(f_0,g_0);f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_f(f_0,g_0)f(\lambda)\,d\lambda + \frac{1}{2\pi}\int_{-\pi}^{\pi} h_g(f_0,g_0)g(\lambda)\,d\lambda$$

is a continuous linear functional in the space $L_1 \times L_1$ and one can apply the Lagrange multipliers method to solve conditional extremum problem (11) (see [24]). As a result, we obtain the following relations determining the least favorable spectral densities $f^0 \in D_f^0$ and $g^0 \in D_g^0$:

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} h_f(f_0,g_0)\rho(f(\lambda))\,d\lambda - \frac{1}{2\pi}\int_{-\pi}^{\pi} h_g(f_0,g_0)\rho(g(\lambda))\,d\lambda \\ + \alpha_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(f(\lambda))\,d\lambda + \alpha_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(g(\lambda))\,d\lambda = 0,$$

where $\rho(f(\lambda))$ and $\rho(g(\lambda))$ are the variations of the functions $f(\lambda)$ and $g(\lambda)$, and α_1, α_2 are some constants such that $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$. Thus, the least favorable densities $f_0(\lambda) \in D_f^0$, $g_0(\lambda) \in D_g^0$ satisfy the following equations:

$$\left|A_s\left(e^{i\lambda}\right)g_0(\lambda) + C_s^0\left(e^{i\lambda}\right)\right| = \alpha_1(f_0(\lambda)+g_0(\lambda)), \tag{14}$$

$$\left|A_s\left(e^{i\lambda}\right)f_0(\lambda) - C_s^0\left(e^{i\lambda}\right)\right| = \alpha_2(f_0(\lambda)+g_0(\lambda)). \tag{15}$$

Note that $\alpha_1 \ne 0$ if $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_0(\lambda)\,d\lambda = P_1$, and $\alpha_2 \ne 0$ if $\frac{1}{2\pi}\int_{-\pi}^{\pi} g_0(\lambda)\,d\lambda = P_2$. Hence, we proved the following result.

Theorem 4.1. *Let spectral densities $f_0(\lambda) \in D_f^0$ and $g_0(\lambda) \in D_g^0$ satisfy the condition of minimality* (2) *and let the functions $h_f(f_0,g_0)$, $h_g(f_0,g_0)$ defined by* (12), (13) *be bounded. Then the functions $f_0(\lambda)$, $g_0(\lambda)$ being a solution of system of equations* (14), (15) *are the least favorable densities in the class $D = D_f^0 \times D_g^0$ if they determine a solution of extremum problem* (9). *The function defined according to* (3) *is the minimax spectral characteristic of the optimal estimator of the functional $A_s\xi$.*

Theorem 4.2. *Let a spectral density $f(\lambda)$ be known and a spectral density $g_0(\lambda)$ belong to the class D_g^0. Let the functions $f(\lambda)$ and $g_0(\lambda)$ be such that the function $(f(\lambda)+g_0(\lambda))^{-1}$ is integrable and the function $h_g(f,g_0)$ defined by equality* (13) *is bounded. Then the function*

$$g_0(\lambda) = \max\left\{0, \alpha_2^{-1}\left|A_s\left(e^{i\lambda}\right)f_0(\lambda) - C_s^0\left(e^{i\lambda}\right)\right| - f(\lambda)\right\}$$

is the least favorable spectral density in the class D_g^0 for the optimal linear interpolation of the functional $A_s\xi$ and the pair $f(\lambda)$, $g_0(\lambda)$ determines a solution of extremum problem (9). *The function defined by* (3) *is the minimax spectral characteristic of the optimal estimator of the functional $A_s\xi$.*

Theorem 4.3. *Let a spectral density $f_0(\lambda)$ belong to the class D_f^0 and let $f_0^{-1}(\lambda)$ be integrable and the function $h(f_0)$ defined by* (7) *be bounded. Then the function*

$$f_0(\lambda) = \alpha_1 \left|C_s^0\left(e^{i\lambda}\right)\right|$$

is the least favorable spectral density in the class D_f^0 for the optimal linear interpolation of the functional $A_s\xi$ by using observations of the sequence $\xi(j)$, $j \in \mathbb{Z} \setminus S$, if $f_0(\lambda)$ determines a solution of extremum problem (9). *The function $h^0 = h(f_0)$ defined by* (7) *is the minimax spectral characteristic of the optimal estimator of the functional $A_s\xi$.*

5. Least favorable spectral densities in the class $D_v^u \times D_\varepsilon$

Consider the problem of minimax estimation of the functional $A_s\xi$ depending on unknown values of a sequence $\xi(j)$ by using the observations of the sequence $\xi(j)+\eta(j)$ at points $j \in \mathbb{Z} \setminus S$ for the case where the spectral densities $f(\lambda)$ and $g(\lambda)$ belong to the set of admissible spectral densities $D = D_v^u \times D_\varepsilon$, where

$$D_u^v = \left\{ f(\lambda) \;\middle|\; v(\lambda) \le f(\lambda) \le u(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\,d\lambda \le P_1 \right\},$$

$$D_\varepsilon = \left\{ g(\lambda) \;\middle|\; g(\lambda) = (1-\varepsilon)g_1(\lambda) + \varepsilon w(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)\,d\lambda \le P_2 \right\}.$$

Here, the spectral densities $u(\lambda)$, $v(\lambda)$, and $g_1(\lambda)$ are known and fixed and, moreover, the densities $u(\lambda)$ and $v(\lambda)$ are bounded.

Let the spectral densities $f^0(\lambda) \in D_v^u$ and $g^0(\lambda) \in D_\varepsilon$ define two bounded functions $h_f(f_0, g_0)$ and $h_g(f_0, g_0)$ according to equalities (12) and (13), respectively. Then the condition

$$0 \in \partial\Delta_{D_{f,g}}(f_0, g_0)$$

implies the following equations for the least favorable spectral densities:

$$\left|A_s\left(e^{i\lambda}\right)g_0(\lambda) + C_s^0\left(e^{i\lambda}\right)\right| = (f_0(\lambda)+g_0(\lambda))\left(\gamma_1(\lambda)+\gamma_2(\lambda)+\alpha_1^{-1}\right), \tag{16}$$

$$\left|A_s(e^{i\lambda})f_0(\lambda) - C_s^0\left(e^{i\lambda}\right)\right| = (f_0(\lambda)+g_0(\lambda))\left(\varphi(\lambda)+\alpha_2^{-1}\right), \tag{17}$$

where $\gamma_1 \le 0$ and $\gamma_1 = 0$ if $f_0(\lambda) \ge v(\lambda)$; $\gamma_2 \ge 0$ and $\gamma_2 = 0$ if $f_0(\lambda) \le u(\lambda)$; $\varphi(\lambda) \le 0$ and $\varphi(\lambda) = 0$ if $g_0(\lambda) \ge (1-\varepsilon)g_1(\lambda)$.

Therefore, we proved the following result.

Theorem 5.1. *Let $f_0(\lambda) \in D_v^u$ and $g_0(\lambda) \in D_\varepsilon$. Suppose that condition of minimality* (2) *holds. Assume that the functions defined by equalities* (12) *and* (13) *are bounded. Then the functions $f_0(\lambda)$ and $g_0(\lambda)$ defined by equalities* (16) *and* (17) *are the least favorable densities in the class $D_v^u \times D_\varepsilon$ if they determine a solution of extremum problem* (9). *The function $h(f_0, g_0)$ defined by* (3) *is the minimax spectral characteristic of the optimal estimator of the functional $A_s\xi$.*

Theorem 5.2. *Let the spectral density $f(\lambda)$ be known and spectral density $g_0(\lambda)$ belong to the class D_ε. Assume that the function $f(\lambda)+g_0(\lambda)$ satisfies the condition of minimality* (2) *and the function $h_g(f, g_0)$ defined by equality* (13) *is bounded. Then $g_0(\lambda)$ is the least favorable spectral density in the class D_ε for the optimal linear interpolation of the functional $A_s\xi$ if*

$$g_0(\lambda) = \max\left\{(1-\varepsilon)g_1(\lambda), \alpha_2\left|A_s\left(e^{i\lambda}\right)f_0(\lambda) - C_s^0\left(e^{i\lambda}\right)\right| - f(\lambda)\right\}$$

and the pair $f(\lambda)$, $g_0(\lambda)$ determines a solution of extremum problem (9). *The function obtained from relation* (3) *is the minimax spectral characteristic of the optimal estimator of the functional $A_s\xi$.*

Theorem 5.3. *Let a spectral density $f_0(\lambda)$ belong to the class D_v^u, the function $f_0^{-1}(\lambda)$ be integrable, and the function $h(f_0)$ defined by equality* (7) *be bounded. Then the function*

$$f_0(\lambda) = \max\left\{v(\lambda), \min\left\{u(\lambda), \alpha_1 \left|C_s^0\left(e^{i\lambda}\right)\right|\right\}\right\}$$

is the least favorable spectral density in the class D_v^u for the optimal linear interpolation of the functional $A_s\xi$ by using the observations of the sequence $\xi(j)$, $j \in \mathbb{Z} \setminus S$, if the function $f_0(\lambda)$ determines a solution of extremum problem (9). *The minimax spectral characteristic $h^0 = h(f_0)$ of the optimal estimator of the functional $A_s\xi$ is evaluated according to relation* (7).

6. Least favorable spectral densities in the class $D_{2\varepsilon_1} \times D_{1\varepsilon_2}$

Consider the problem of minimax estimation of the functional $A_s\xi$ for the class of spectral densities $D_{2\varepsilon_1} \times D_{1\varepsilon_2}$ corresponding to the "ε-neighborhood" model of random sequences in the space $L_2 \times L_1$. Let

$$D_{2\varepsilon_1} = \left\{ f(\lambda) \;\middle|\; \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\lambda) - f_1(\lambda)|^2 \, d\lambda \le \varepsilon_1 \right\}$$

be the "ε-neighborhood" of a given bounded spectral density $f_1(\lambda)$ in the space L_2, and let

$$D_{1\varepsilon_2} = \left\{ g(\lambda) \;\middle|\; \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(\lambda) - g_1(\lambda)| \, d\lambda \le \varepsilon_2 \right\}$$

be the "ε-neighborhood" of a given bounded spectral density $g_1(\lambda)$ in the space L_2.

Assume that the densities $f_0(\lambda) \in D_{2\varepsilon_1}$ and $g_0(\lambda) \in D_{1\varepsilon_2}$ are such that the functions $h_f(f_0, g_0)$ and $h_g(f_0, g_0)$ defined by equalities (12) and (13) are bounded. If $D = D_{2\varepsilon_1} \times D_{1\varepsilon_2}$, then the condition $0 \in \partial\Delta_{D_{f,g}}(f^0, g^0)$ yields the following equations

$$\left|A_s\left(e^{i\lambda}\right) g_0(\lambda) + C_s^0\left(e^{i\lambda}\right)\right|^2 = (f_0(\lambda) + g_0(\lambda))^2 (f_0(\lambda) - f_1(\lambda))\alpha_1, \tag{18}$$

$$\left|A_s\left(e^{i\lambda}\right) f_0(\lambda) - C_s^0\left(e^{i\lambda}\right)\right|^2 = (f_0(\lambda) + g_0(\lambda))\Psi(\lambda)\alpha_2, \tag{19}$$

where $|\Psi(\lambda)| \le 1$ and $\Psi(\lambda) = \operatorname{sign}(g_0(\lambda) - g_1(\lambda))$ if $g_0(\lambda) \ne g_1(\lambda)$, and α_1, α_2 are some constants. Equations (18) and (19) together with extremum problem (9) and the normalization conditions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\lambda) - f_1(\lambda)|^2 \, d\lambda = \varepsilon_1, \tag{20}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(\lambda) - g_1(\lambda)| \, d\lambda = \varepsilon_2, \tag{21}$$

determine the least favorable spectral densities in the class $D_{2\varepsilon_1} \times D_{1\varepsilon_2}$.

Therefore, the following results hold.

Theorem 6.1. *Let spectral densities $f_0(\lambda) \in D_{2\varepsilon_1}$ and $g_0(\lambda) \in D_{1\varepsilon_2}$ be such that the condition of minimality* (2) *holds and the functions defined by equalities* (12) *and* (13) *are bounded. Then $f_0(\lambda)$ and $g_0(\lambda)$ are the least favorable spectral densities in the class $D_{2\varepsilon_1} \times D_{1\varepsilon_2}$ for the optimal linear interpolation of the functional $A_s\xi$ if they satisfy equations* (18)–(21) *and determine a solution of extremum problem* (9). *The function $h(f_0, g_0)$ defined by relation* (3) *is the minimax spectral characteristic of the optimal estimator of the functional $A_s\xi$.*

Theorem 6.2. *Let a spectral density $f(\lambda)$ be known and spectral density $g_0(\lambda)$ belong to the class $D_{1\varepsilon_2}$. Let the function $f(\lambda)+g_0(\lambda)$ satisfy condition of minimality* (2) *and the function $h_g(f,g_0)$ defined by equality* (13) *be bounded. Then $g_0(\lambda)$ is the least favorable spectral density in the class $D_{1\varepsilon_2}$ for the optimal linear interpolation of the functional $A_s\xi$ if*

$$g_0(\lambda)=\max\left\{g_1(\lambda),\alpha_2\left|A_s\left(e^{i\lambda}\right)f_0(\lambda)-C_s^0\left(e^{i\lambda}\right)\right|-f(\lambda)\right\}$$

and the pair $f(\lambda)$, $g_0(\lambda)$ determines a solution of extremum problem (9). *The function evaluated by relation* (3) *is the minimax spectral characteristic of the optimal estimator of the functional $A_s\xi$.*

Theorem 6.3. *Let a spectral density $f_0(\lambda)$ belong to the class $D_{2\varepsilon_1}$, the function $f_0^{-1}(\lambda)$ be integrable, and the function $h(f_0)$ defined by equation* (7) *be bounded. Then $f_0(\lambda)$ is the least favorable spectral density in the class $D_{2\varepsilon_1}$ for the optimal linear interpolation of the functional $A_s\xi$ by using the observations of the sequence $\xi(j)$, $j\in\mathbb{Z}\setminus S$, if*

$$\left|C_s^0(e^{i\lambda})\right|^2=(f_0(\lambda))^2(f_0(\lambda)-f_1(\lambda))\alpha_1$$

and the function $f_0(\lambda)$ determines a solution of extremum problem (9). *The minimax spectral characteristic $h^0=h(f_0)$ of the optimal estimator of the functional $A_s\xi$ is evaluated according to equality* (7).

7. Concluding remarks

In the paper, we propose a method for solving the optimal linear interpolation problem for the functional

$$A_s\xi=\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}a(j)\xi(j),\qquad M_l=\sum_{k=0}^{l}(N_k+K_k),\quad N_0=K_0=0,$$

that depends on unknown values of a stationary sequence $\xi(j)$ by using the observations of the sequence $\xi(j)+\eta(j)$ for $j\in\mathbb{Z}\setminus S$, where $\eta(j)$ is a stationary sequence being uncorrelated with the sequences $\xi(j)$ and S is the set of missing observations,

$$S=\bigcup_{l=0}^{s-1}\{M_l,\dots,M_l+N_{l+1}\}.$$

If the spectral densities $f(\lambda)$ and $g(\lambda)$ of random sequences $\xi(j)$ and $\eta(j)$ are known, the method of projection in Hilbert spaces is applied to solve this problem.

A closed form formula is also obtained for the spectral characteristic and mean square error of the functional $A_s\xi$. If the spectral densities $\xi(j)$ and $\eta(j)$ are not known but a class of admissible densities is specified, then the minimax method is applied to solve the problem of optimal linear interpolation of the functional $A_s\xi$.

For some classes of spectral densities, we find relations that determine the least favorable spectral densities and explicit expressions for the minimax spectral characteristic of the estimator for the functional $A_s\xi$.

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